Holography and Renormalization in the AdS/CFT correspondence

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based on
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The Issues

Holography states that a $(d+1)$-dimensional *gravitational* theory should have a description in terms of a d-dimensional field theory (without gravity).

Q: How is spacetime encoded in the dual field theory?

The AdS/CFT correspondence provides an explicit realization of holography.

We will address the question within this framework.

The answer to the question is:

• *Spacetime metric* is encoded in correlation functions of the *stress energy tensor* of the dual field theory.

• Physics of *matter fields* in this (bulk) spacetime is encoded in correlation functions of various *composite operators* of the dual field theory.

It is the purpose of this work to make these statements precise.

Correlation functions of composite operators suffer from *UV divergences*. One needs to *renormalize* the theory.

Q: What is the holographic image of the field theory UV divergences?

They correspond to *IR divergences* of the $(d+1)$-dimensional gravitational theory.

UV/IR connection

Q: How does one deal with them?

As in field theory, one *regulates* the theory, adds *counterterms* to cancel the infinities, and then removes the regulator.

AdS/CFT duality

[Maldacena
Gubser, Klebanov, Polyakov
Witten]

String theory on AdS_{d+1} backgrounds is *dual* to certain CFT_d.

In low energies:

$\boxed{AdS_{d+1}\text{ SUGRA} \Leftrightarrow \text{strongly coupled, large } N\ CFT_d}$

$$
\begin{aligned}
Z_{\boldsymbol{SUGRA}}[\boldsymbol{\phi_{(0)}}, \ldots] &= \int_{\phi|_{\partial M} \sim \boldsymbol{\phi_{(0)}}} \mathcal{D}\phi \exp[-S_{\boldsymbol{SUGRA}}] \\
&= \langle \exp \int_{\partial M} \boldsymbol{\phi_{(0)}} \mathcal{O} \rangle_{\boldsymbol{CFT}}
\end{aligned}
$$

$\mathcal{O}$: CFT composite operator.

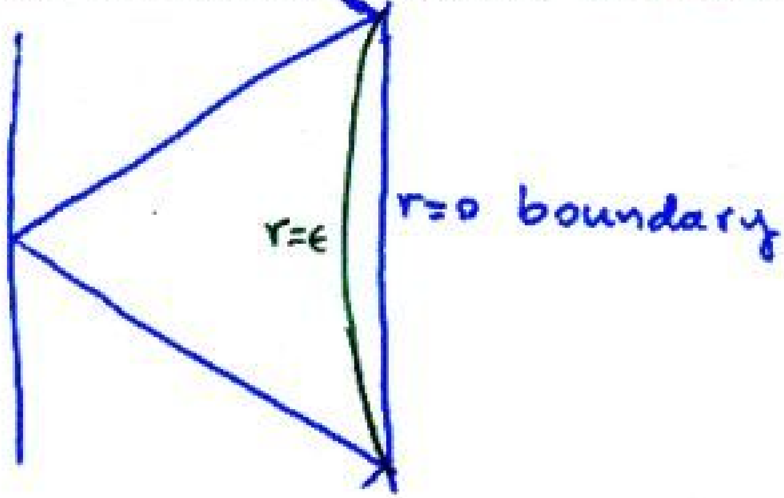
To leading order

$$
\boxed{S_{\boldsymbol{SUGRA}}[\boldsymbol{\phi_{(0)}}, \ldots] = W_{\boldsymbol{CFT}}[\boldsymbol{\phi_{(0)}}, \ldots]}
$$

on-shell SUGRA action = Generating functional of CFT connected graphs

Regularization

To regulate the on-shell value of the SUGRA action we restrict the radial integration, $r \geq \epsilon$, and evaluate the boundary terms at $r = \epsilon$.



$$
\begin{aligned}
&S^{reg}_{SUGRA}[\phi_{(0)}, \dots; \epsilon] \equiv \\
&\equiv \int_{r \geq \epsilon} dr d^d x L^{\text{bulk}}_{SUGRA} + \int_{r=\epsilon} d^d x L^{\text{boundary}}_{SUGRA} \\
&= \sum_k a_{(k)}(\phi_{(0)}) \frac{1}{\epsilon^k} + \underbrace{a_{(d)}(\phi_{(0)})}_{\text{holographic conformal anomaly}} \log \epsilon + \mathcal{O}(\epsilon^0)
\end{aligned}
$$

[Henningson, Skenderis]

Renormalization

Add counterterms to cancel infinities

$$S^{ren}_{SUGRA}[\phi_{(0)}, ...] \equiv \lim_{\epsilon \to 0} \left(S^{reg}_{SUGRA}[\phi_{(0)}, ...; \epsilon] + S_{ct}[\phi_{(0)}, ...; \epsilon] \right)$$

where

$$S_{ct}[\phi_{(0)}, ...; \epsilon] = - \left(\sum_k a_{(k)}(\phi_{(0)}) \frac{1}{\epsilon^k} + a_{(d)}(\phi_{(0)}) \log \epsilon \right)$$

One can now obtain finite correlation functions by differentiating S^{ren}_{SUGRA} w.r.t. the sources $\phi_{(0)}$.

For example,

$$\langle T_{ij} \rangle = \frac{2}{\sqrt{g_{(0)}}} \frac{\delta S^{ren}_{SUGRA}}{\delta g^{ij}_{(0)}}$$

Asymptotic solutions

To obtain the counterterms we need to find asymptotic solutions of the supergravity field equations with Dirichlet boundary conditions.

- **Gravity**

$$ds^2 = \frac{dr^2}{r^2} + \frac{1}{r^2} g_{ij}(x,r) dx^i dx^j$$

where

$$\begin{aligned} g_{ij}(x,r) &= \mathbf{g_{(0)ij}} + r^2 \mathbf{g_{(2)ij}} + \ldots + \\ &\quad + r^d \left(\log r^2 \mathbf{h_{(d)ij}} + \mathbf{g_{(d)}} \right) + \ldots \end{aligned}$$

- Scalar fields of mass $m^2 = \Delta(\Delta - d)$

$$\Phi(x,r) = r^{d-\Delta}\phi(x,r)$$

where

$$\begin{aligned}\phi(x,r) &= \phi_{(0)} + r^2\phi_{(2)} + \ldots \\ &+ r^{2\Delta-d}\left(\log r^2 \psi_{(2\Delta-d)} + \phi_{(2\Delta-d)}\right) + \ldots\end{aligned}$$

Similarly for other fields.

- Field equations are solved perturbatively in r.

- $g_{(0)}$ and $\phi_{(0)}$ are the input in the Dirichlet boundary problem.

- $g_{(2)}, \cdots, g_{(d-2)}, h_{(d)}$ and $\phi_{(2)}, \cdots, \phi_{(2\Delta-d-2)}, \psi_{(2\Delta-d)}$ are uniquely determined by the field equations.

- $h_{(d)}$ and $\psi_{(2\Delta-d)}$ are directly related with conformal anomalies of the CFT.

- $\phi_{(2\Delta-d)}$ is *not determined* by the bulk field equations. Only the trace, $\mathrm{Tr}\, g_{(d)}$, and the divergence, $\nabla^i g_{(d)ij}$, in d dimensions are determined by the bulk field equations. The undetermined parts are determined *holographically* from the 1-point function of the dual operators.

$$g_{(2)ij} = \frac{1}{d-2}\left(R_{ij} - \frac{1}{2(d-1)} R\, g_{(0)ij}\right)$$

$$\begin{aligned}
g_{(4)ij} \;=\; & \frac{1}{d-4}\left(-\frac{1}{8(d-1)} D_i D_j R\right. \\
& + \frac{1}{4(d-2)} D_k D^k R_{ij} - \frac{1}{8(d-1)(d-2)} D_k D^k R g_{(0)ij} \\
& - \frac{1}{2(d-2)} R^{kl} R_{ikjl} + \frac{d-4}{2(d-2)^2} R_i{}^k R_{kj} \\
& + \frac{1}{(d-1)(d-2)^2} R R_{ij} + \frac{1}{4(d-2)^2} R^{kl} R_{kl} g_{(0)ij} \\
& \left. - \frac{3d}{16(d-1)^2(d-2)^2} R^2 g_{(0)ij}\right)
\end{aligned}$$

Holographic scalar fields

$$\langle \mathcal{O}(x) \rangle = -\frac{1}{\sqrt{g_{(0)}}} \frac{\delta S^{ren}}{\delta \phi_{(0)}} = (2\Delta - d)\phi_{(2\Delta-d)}$$

- The proportionality coefficient, $(2\Delta - d)$, is crucial in order 2-point functions to be normalized correctly. Without a careful regularization/renormalization one obtains d as a proportionality coefficient. Only the value $(2\Delta - d)$ is consistent with Ward identities.

- The coefficient $\phi_{(2\Delta-d)}$ left undetermined by the field equations is determined by the 1-point function, $\langle \mathcal{O}(x) \rangle$, of the dual operator.

Holographic stress energy tensors

$$\langle T_{ij} \rangle = \frac{dl}{16\pi G_N} (g_{(d)ij} + X_{ij}^{(d)}).$$

- $\langle T_{ij} \rangle$ is conserved when the bulk field equations are satisfied and its trace is equal to the holographic trace anomaly.

- The part of $g_{(d)}$ left undetermined by the bulk field equation is *holographically* determined by the 1-point function of the dual stress energy tensor.

$\underline{d = 2k+1}$

$$X_{ij}^{(2k+1)} = 0$$

no conformal anomaly in odd dimensions

$\underline{d = 2}$

$$X_{ij}^{(2)} = -g_{(0)ij} \, \mathrm{Tr}\, g_{(2)}$$

$\underline{d = 4}$

$$\begin{aligned} X_{ij}^{(4)} &= -\frac{1}{8} g_{(0)ij} [(\mathrm{Tr}\, g_{(2)})^2 - \mathrm{Tr}\, g_{(2)}^2] \\ &\quad -\frac{1}{2} (g_{(2)}^2)_{ij} + \frac{1}{4} g_{(2)ij} \mathrm{Tr}\, g_{(2)} \end{aligned}$$

$\underline{d=6}$

$$X^{(6)}_{ij} = -A_{(6)ij} + \frac{1}{24} S_{ij}$$

where

$$\begin{aligned}
A_{(6)ij} \;=\; & \frac{1}{3}\Big(2(g_{(2)}g_{(4)})_{ij} + (g_{(4)}g_{(2)})_{ij} - (g^3_{(2)})_{ij} \\
& + \frac{1}{8}[\mathrm{Tr}\; g^2_{(2)} - (\mathrm{Tr}\; g_{(2)})^2]\, g_{(2)ij} \\
& - \mathrm{Tr}\; g_{(2)}\, [g_{(4)ij} - \frac{1}{2}(g^2_{(2)})_{ij}] \\
& - [\frac{1}{8}\mathrm{Tr}\; g^2_{(2)}\, \mathrm{Tr}\; g_{(2)} - \frac{1}{24}(\mathrm{Tr}\; g_{(2)})^3 \\
& - \frac{1}{6}\mathrm{Tr}\; g^3_{(2)} + \frac{1}{2}\mathrm{Tr}\; (g_{(2)}g_{(4)})]\, g_{(0)ij}\Big) \,.
\end{aligned}$$

$$
\begin{aligned}
S_{ij} &= \nabla^2 C_{ij} - 2R^{k}{}_{i}{}^{l}{}_{j} C_{kl} + 4(g_{(2)}g_{(4)} - g_{(4)}g_{(2)})_{ij} \\
&+ \frac{1}{10}(\nabla_i \nabla_j B - g_{(0)ij}\nabla^2 B) + \frac{2}{5} g_{(2)ij} B \\
&+ g_{(0)ij}(-\frac{2}{3}\mathrm{Tr}\ g_{(2)}^3 - \frac{4}{15}(\mathrm{Tr}\ g_{(2)})^3 \\
&+ \frac{3}{5}\mathrm{Tr}\ g_{(2)}\mathrm{Tr}\ g_{(2)}^2)
\end{aligned}
$$

where

$$
\begin{aligned}
C_{ij} &= (g_{(4)} - \frac{1}{2} g_{(2)}^2 + \frac{1}{4} g_{(2)}\mathrm{Tr}\ g_{(2)})_{ij} + \frac{1}{8} g_{(0)ij} B \\
B &= \mathrm{Tr}\ g_{(2)}^2 - (\mathrm{Tr}\ g_{(2)})^2
\end{aligned}
$$

• The expectation value of the dual stress energy tensor can also be interpreted as the the gravitational quasi-local stress energy tensor of Brown and York. [Balasubramanian, Kraus]

Given a solution of Einstein's equation with negative cosmological constant (for example, the Kerr-AdS solution) in $d \leq 6$ one can immediately obtain the corresponding gravitational stress energy tensor by

1 reaching the coordinate system

$$\begin{aligned} ds^2 &= \frac{dr^2}{r^2} + \frac{1}{r^2} g_{ij}(x,r) dx^i dx^j \\ g_{ij}(x,r) &= g_{(0)ij} + r^2 g_{(2)ij} + \dots \end{aligned}$$

2 using the formulae of T_{ij} in terms of $g_{(m)ij}$

- Conformal transformation properties of T_{ij}

It is well-known that the conformal transformation properties of T_{ij} are modified due to the conformal anomaly. The holographic description provides a straightforward manner of obtaining this transformation rule.

A conformal transformation in the boundary theory

$$g_{(0)} \to e^{2\sigma(x)} g_{(0)}$$

can be achieved by a bulk diffeomorphism

$$\begin{aligned} r &\to r e^{-\sigma(x)} \\ x^i &\to x^i + a^i(x,r) \end{aligned}$$

that preserves the form of the metric. The latter requirement determines $a^i(x,r)$ perturbatively in r.

[Imbimbo, Schwimmer, Theisen, Yankielowicz]

It is straightforward to obtain the transformation properties of the coefficients $g_{(m)}(x)$ under this diffeomorphism. Since T_{ij} is given in terms of $g_{(m)}$ one can immediately obtain how T_{ij} transforms.

- $d = 2$

$$\delta\langle T_{ij}\rangle = \frac{1}{12\pi}\left(\underbrace{\frac{3l}{2G_N}}_{\text{Brown-Henneaux central charge}}\right)(\nabla_i\nabla_j\sigma - g_{(0)ij}\nabla^2\sigma)$$

- $d = 2k + 1$

$$\delta\langle T_{ij}\rangle = -(d-2)\sigma\langle T_{ij}\rangle$$

Classical transformation rule

- $d = 4$

$$
\begin{aligned}
\delta\langle T_{ij}\rangle \;=\; & -2\sigma\langle T_{ij}\rangle + \frac{l^3}{4\pi G_N}\Big(-2\sigma h_{(4)} \\
& +\frac{1}{4}\nabla^k\sigma[\nabla_k R_{ij} - \frac{1}{2}(\nabla_i R_{jk} + \nabla_j R_{ik}) - \frac{1}{6}\nabla_k R g_{(0)ij}] \\
& +\frac{1}{48}(\nabla_i\sigma\nabla_j R + \nabla_i\sigma\nabla_j R) \\
& +\frac{1}{12}R(\nabla_i\nabla_j\sigma - g_{(0)ij}\nabla^2\sigma) \\
& +\frac{1}{8}[R_{ij}\nabla^2\sigma - (R_{ik}\nabla^k\nabla_j\sigma + R_{jk}\nabla^k\nabla_i\sigma) \\
& +g_{(0)ij}R_{kl}\nabla^k\nabla^l\sigma]\Big)
\end{aligned}
$$

Agrees with previous results **[Cappeli, Coste]** when restricted to conformally flat $g_{(0)}$.

Conclusions/Future directions

• We have provided a systematic method to renormalize the on-shell SUGRA action. We explicitly worked out the case of scalars and gravitons. The method generalizes straightforwardly to other cases.

• We have explicitly seen how the bulk spacetime is encoded holographically (to leading order) in correlation functions of the dual CFT.

It would be interesting to

• Extend the results to the holographic discussion of RG-flows.

• Apply the results to Brane-World scenarios. Counterterms are now re-interpreted as providing the action for the bulk modes localized in the brane.

Counterterms

Graviton

$$S_{ct} = -\frac{1}{16\pi G_N}\int_{r=\epsilon} d^dx\sqrt{\gamma}\left[2(1-d)+\frac{1}{d-2}R+\ldots\right]$$

$-\log\epsilon\, a_{(d)}$

Bulk massive scalar $m^2=(\Delta-d)\Delta$

$$S_{ct} = \int_{r=\epsilon} d^dx\sqrt{\gamma}\left[\frac{1}{2(2\Delta-d-2)}\Phi(x,\epsilon)\Box_\gamma\Phi(x,\epsilon)\right.$$
$$\left.+\frac{(d-\Delta)}{2}\left(1+\frac{1}{2(d-1)(2\Delta-d-2)}R[\gamma]\right)\Phi^2(x,\epsilon)+\ldots\right.$$