Worldsheet Derivation
of a Large $N$ Duality

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Based on a work with Cumrun Vafa

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String Theory from a Large $N$ Limit of a Gauge Theory

\[ S = \frac{1}{g_{YM}^2} \int L(A), \quad A_\mu = U(N) \text{ gauge field} \]

**Ribbon Graph**

Each \( \bigcirc \) gives a factor of \( N \)

Fill \( \bigcirc \) by a disk \( \bigcirc \)

\[ \Rightarrow \text{ a closed Riemann surface} \]

**t'Hooft Counting**

\[ (g_{YM}^2)^{2g-2} \cdot (g_{YM}^2 N)^h \]

\( g \) = genus of the surface

\( h \) = # loops = # holes

\[ F = \sum_{g=0}^{\infty} (g_{YM}^2)^{2g-2} F_g(t), \quad t = g_{YM}^2 N \]

**Conjecture**

There is a closed string theory whose $g$-loop amplitude is given by $F_g(t)$.

\( \lambda_s = g_{YM}^2 \) : string coupling constant

\( t = g_{YM}^2 N \) : parameter of worldsheet theory
The 't Hooft Conjecture is the low energy limit of a more general conjecture about the equivalence of:

\[ D \text{ Branes} \leftrightarrow \text{Closed String Background} \]
Several examples of large $N$ dualities have been discovered assuming:

- D branes $\leftrightarrow$ Closed string background
- M theory dualities

(1) \textbf{AdS/CFT correspondence} \hspace{1cm} \text{MALDACENA}

(2) \textbf{Topological string dualities}

- Chern-Simons gauge theory $\leftarrow$ Open string on A branes on deformed $\text{CY}_3$.
- Closed string with A twist on resolved $\text{CY}_3$.
  \hspace{1cm} \text{GOPAKUMAR + VAFA}

- Holomorphic matrix model $\leftarrow$ Open string on B branes on resolved $\text{CY}_3$.
- Closed string with B twist on deformed $\text{CY}_3$.
  \text{"Kodaira-Spencer theory of gravity"}
  \hspace{1cm} \text{DIJKGRAAF + VAFA}

It is desirable to prove these conjectures without appealing to M theory dualities.
STATEMENT OF THE CONJECTURE

\[ U(N) \text{ Chern-Simons Theory on } S^3 \]

\[ S = \frac{k}{4\pi} \int \text{tr} (\text{Ad}A + \frac{2}{3} A^3), \quad g_{YM} = \frac{i}{k+N} \]

\[ \Downarrow \]

Topological closed string on conifold

\[ \lambda_5 = g_{YM}^2 = \frac{i}{k+N} : \text{string coupling constant} \]

\[ t = g_{YM}^2 N = \frac{iN}{k+N} : \text{NS-NS 2-form } B \]

\[ i \int B = t \]

UPLIFTING TO M THEORY

Chern-Simons theory on \( S^3 \) \Rightarrow Low energy effective action terms on \( N \) D\(_6\) branes on \( S^3 \times \mathbb{R}^4 \)

Topological closed string on conifold \Rightarrow Corresponding terms in IIA string on conifold with RR flux

These configurations in IIA string theory are smoothly interpolated when uplifted to M theory on G\(_2\) holonomy manifolds.

Atiyah, Maldacena + Vafa / 0011256
The linear sigma-model provides a good description of the closed string world sheet for $t = g_{YM}^2 N \to 0$.

$V$: vector multiplet

$A_i, B_i \ (i=1,2)$: chiral multiplets of opposite charges

Potential = $10^{-12} \cdot (1a_1^2 + 1a_2^2 + 1b_1^2 + 1b_2^2)$

$+ e^2 \cdot (1a_1^2 + 1a_2^2 - 1b_1^2 - 1b_2^2)$

$\sigma \in V, \ a_i \in A_i, \ b_i \in B_i$

$e$: electric charge $\to \infty$ in the infrared.

$t = g_{YM}^2 N$ appears in the theta term:

$$\frac{t}{2\pi} F_{12} = \int d\theta \ W$$

$W = t \Sigma$: superpotential

$$\Sigma = \overline{D}_+ D_- V = \sigma + ...$$: twisted chiral superfield

The Higgs branch theory ($|\sigma|^2 \ll e^2$)

flows to the sigma-model on conifold.

When $t \to 0$, the theory develops two phases:

- $H$ phase: $\sigma = 0; a_i, b_i \neq 0$

  $\Rightarrow$ target space geometry

- $C$ phase: $\sigma \neq 0; a_i, b_i = 0$ New phase at $t = 0$
For each configuration of the $\sigma$ field, we can separate the worldsheet into the two phases.

$C$ phase: $|\sigma| > \sigma_*$  
$H$ phase: $|\sigma| < \sigma_*$

A part of the functional integral over $\sigma$ can be turned into a sum over shapes and sizes of the $C$ domain.

Collective coordinates of $\sigma$

We need to show:

- Contributions from $C$ domains such as:
  - Vanish.

Namely, every $C$ domain has the topology of the disk.

- Each disk in $C$ phase gives the factor of $t$.

$$ t = g_{YM}^2 N $$
Our task is simplified by the localization of the functional integral over $\sigma$.

Because of the topological twisting, $\sigma = \text{const on a fixed Riemann surface}$. 

$\Rightarrow$ The worldsheet is either in pure $C$ phase or in pure $H$ phase. 

$C$ and $H$ do not co-exist.

In the topological string, we integrate over $M_g$. 

$\sigma$ can be non-constant if there is a long cylinder.

$\begin{align*}
\sigma(\gamma_0) &= \sigma_0, \\
|\sigma_0| &= \sigma^* \quad (\gamma_0) = l \\
\end{align*}$

$\Rightarrow$ Reduction to a one-dimensional quantum mechanics.

Change of functional integral variables:

$\sigma(\gamma_0), \bar{\sigma}(\gamma_0) \rightarrow \gamma_0$ and the phase of $\sigma_0$

$\gamma_0$ becomes a part of the moduli of the worldsheet with the boundary.

- Each $C$ domain contributes as

$\sigma = \sigma_0 \quad \int d\sigma_0 \, \frac{2}{\sigma_0} \mathcal{F}^{(C)}(\sigma_0)$
By the topological BRST symmetry, the amplitude $\mathcal{F}^{(c)}(\sigma_0)$ of each C domain is a holomorphic function of $\sigma_0$.

$$\Rightarrow \oint d\sigma_0 \frac{2}{\partial \sigma_0} \mathcal{F}^{(c)}(\sigma_0) = 0 \text{ if } \mathcal{F}^{(c)}(\sigma_0) \text{ is single-valued.}$$

This is the case for

[Drawings of various shapes, including a disk and a cylinder.]

The only non-zero contribution comes from the case when the C domain is a disk. The amplitude can be computed exactly.

$$\oint_{\text{disk}} d\sigma_0 \frac{2}{\partial \sigma_0} \mathcal{F}^{(c)} = \oint \frac{d\sigma_0}{\sigma_0^2} e^{t\sigma_0}$$

$$= t = 8\pi^2 N$$

This is what we wanted to show.
Pure C phase

\[ \sum_{Mg} = \chi(Mg) \cdot \frac{1}{t^{2g-2}} \]

Singular as \( t \to 0 \).

This does not correspond to any term in 't'Hooft expansion.

What is it in the gauge theory?

\[ \sum_{g} \left( g_{YM}^2 \right)^{2g-2} \cdot \chi(Mg) \cdot \frac{1}{(g_{YM}^2 N)^{2g-2}} \]

\[ = \sum_{g} \frac{B_{2g}}{2g(2g-2) N^{2g-2}} \approx -\log \text{Vol}(U(N)) \]

\[ \text{Vol}(U(N)) = \frac{(2\pi)^{\frac{1}{2}N(N+1)}}{(N-1)! (N-2)! \cdots 3! 2! 1!} \]

(\because U(N) \sim S^1 \times S^3 \times \cdots \times S^{2N-1})

\[ \text{Vol}(U(N)) \text{ is the measure factor in} \]

\[ \int [\text{IdA}] e^{\frac{i}{4\pi} \int_M (A \wedge A + \frac{2}{3} A^3)} \]

The closed string theory knows about the gauge theory beyond the 't'Hooft expansion.