Statistics of flux vacua

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Abstract

We give an introduction to the statistical approach to studying vacua of string/M theory, and discuss recent results of Ashok and Douglas, hep-th/0307049 counting supersymmetric flux vacua in type IIb Calabi-Yau compactification, using techniques of ensembles of effective field theories, inspired by work on zeroes and critical points of random sections of line bundles (Bleher, Shiffman and Zelditch, math-ph/0002039; Zelditch, math-ph/0010012; Douglas, Shiffman and Zelditch, to appear).

1. Statistics of vacua

Why can't we test string theory against experiment ?

1983: We have a consistent theory of quantum gravity, but we don't know how to get a realistic spectrum of chiral fermions.

1993: We can get Standard Model-like spectrum, but we don't understand nonperturbative effects, needed to stabilize the vacuum and break supersymmetry.

2003: We understand nonperturbative effects, and we can find vacua which realize any single feature, or a few features of the Standard Model. But there are too many vacua, and we don't know which is the right one.

Even if we were to find the "complete formulation of M theory," we would still be in this position.

We might make analogies to a group of condensed matter physicists who know the Schrödinger equation, and want to understand a crystal X, but do not know what molecules make it up, and are not allowed to break it up to find out.

To get a more amusing analogy, imagine that these physicists have only one sample to work with, and have discovered the Schrödinger equation by purely theoretical considerations. After much work, they establish the existence of atoms, and are very happy – perhaps their sample is a periodic array of atoms! Of course, they don't know which atom, but since there are only 80 stable atoms, the problem seems manageable.

But, then, somebody works harder, and discovers the H_2 molecule. After much more work, they discover carbon chains, and realize that they can be of arbitrary length.

Now they are in trouble – they face the "molecule selection problem."

String theorists have made three classes of proposals to solve their own problems:

1. Hope that something "truly stringy," *i.e.* not describable by $d = 4$ effective field theory, is discovered experimentally.

2. Look for an *a priori* principle which selects the right vacuum.

3. Look for generic predictions of large subsets of string vacua.

While (1) would be the most exciting outcome, of course it is not clear whether any such effect is experimentally accessible. This hope would be well motivated if one could identify the scale of this new physics, with the $1\ \mathrm{TeV}$ scale at which we believe new physics should appear. However, the Standard Model (in particular small FCNC) works too well for this, making such hopes seem rather optimistic.

As an example to illustrate "Vacuum Selection Principles," let me give you my best idea along these lines.

One might imagine that our vacuum is in some sense "the most symmetric" among the various possibilities. Besides esthetics, many other candidate principles – for example, maximizing some natural wave function of the universe – seem likely to prefer such vacua.

On the face of it, this proposal is absurd. Higher dimensional models with more supersymmetry are obviously more symmetric. By rights, the "most symmetric" four dimensional theory, is the one with the most gauge symmetry. F theory examples are known with rank 10^5 gauge groups (Candelas and Skarke, 9706226). The Standard Model is not even in the running.

However, one might imagine that there exists some unstable nonsupersymmetric vacuum with a huge gauge group, which is preferred by Planck scale cosmology. It would then roll down to our physical vacuum. Then, the "right vacuum" would be near this preferred symmetric point.

This principle could conceivably be right and deserves study, but this is somewhat beyond what we can do at present. I mention it more to illustrate some general properties of candidate "Vacuum Selection Principles" –

- Using them requires some fairly detailed knowledge about the set of possible vacua, and the configuration space which contains the vacua.
- One can imagine many equally well motivated proposals.
- If this principle turned out to be false, we would not be able to claim that string/M theory was falsified.

To falsify string/M theory, we really need to show that **no** vacuum reproduces all the data.

Indeed, from a scientific point of view, there is no reason there has to be **any** *a priori* principle which picks out our vacuum. Following Copernicus and Heisenberg, we may be better off "un-asking" this question.

This leaves option (3), of finding generic predictions of large sets of string vacua. But what does "generic" mean? Is the Standard Model gauge group "generic"?

Let us try to ask this question in a more precise way: Out of **all** the four dimensional vacua obtained by string/M theory compactification, **how many** of them have $SU(3) \times SU(2) \times U(1)$ gauge symmetry unbroken at low energy? If we define our terms, and if string/M theory has a precise definition, **and** if there are finitely many physically distinct vacua, then this question has a definite answer.

The only major assumptions we are making are,

- Real world physics can be described by an effective gauge theory. Of course, the Standard Model is an effective gauge theory. If we take this position, we can ignore vacua whose low energy physics is not describable by effective field theory.

- The number of vacua, possibly after making some precise "cut," is finite. As an example of a cut, we would not care if there were infinite series of vacua which run off to a decompactification limit, as long as we can consistently exclude them from our count.

One can just as easily generalize the question to, out of all vacua, how many have low energy gauge group G ? Let us denote this number by the function

$$d\mu[G].$$

While finding this function exactly is hard, perhaps it can be approximated in some simple and useful way.

For example, could it be that the rank $r = \mathrm{rk}\, G$ of the unbroken gauge group, roughly satisfies a power law distribution,

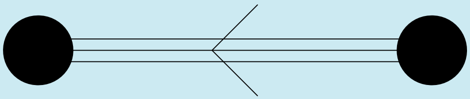
$$d\mu[r] \sim N \times r^{-\alpha}.$$

If so, and if we could estimate N and α, we could get a rough estimate for how many vacua have a rank 4 gauge group, without much effort.

Although this may sound ambitious, given that the function $d\mu[r]$ is well defined, why shouldn't it have a simple approximate description ?

An example of such an argument is given in (MRD 0303194). We consider quiver gauge theories arising from type II string on Calabi-Yau manifolds. The gauge group of such a theory is $\prod_i U(N_i)$, and the chiral matter spectrum is described by a matrix I_{ij} counting the number of bifundamentals in the $(\bar{N}_i, N_j)$, say for $U(N_1) \times U(N_2)$ one has

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

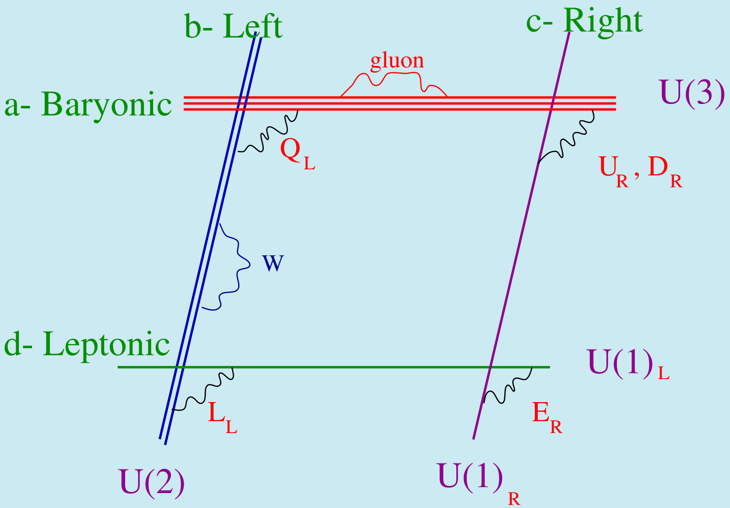
The quiver n2  n1 has $I = 3$.

This is a very simple gauge theory, and one can work out the complete list of symmetry breaking patterns in this theory, and the resulting low energy spectra. These are quiver gauge theories of the same form, but with a list of intersection numbers

$$I \sim I, I^2 - 1, I^3 - I, \ldots$$

Statistics of vacua
Flux vacua
New results
Conclusions
Home Page
Title Page
Page 9 of 28
Go Back
Full Screen
Close
Quit

In work of many authors, quiver theories have been used to realize the Standard Model on branes. An example:



From (Cremades, Ibañez and Marchesano, hep-ph/0212048).

The quiver on the previous transparency, as a subquiver of this one, would describe 3 generations of quarks in the $(2, 3)$ of $SU(2) \times SU(3)$.

Let us grant that the totality of type II compactifications on CY, with subsequent choices, leads to a finite set of vacua, each with a quiver gauge theory realized on D-brane world-volumes. If this set is finite, it defines a distribution $d\mu[N_i; I_{ij}]$, the number of theories realizing each possible choice of the N_i and I_{ij}.

In (MRD 0303194), we give arguments that as a matrix element I_{ij} becomes large (but not too large), this distribution goes as

$$d\mu[N_i; I_{ij}] \sim \frac{dI_{ij}}{|I_{ij}|},$$

generalizing the result for $U(N_1) \times U(N_2)$ we just described.

This motivates the ansatz

$$d\mu[N_i; I_{ij}] \sim \prod_{i<j} \frac{dI_{ij}}{|I_{ij}|}.$$

Given an'ensemble of quiver theories, any specific choice of matter content, for example that of the Standard Model, appears as some definite fraction of the models.

In the ensemble we just described, the fraction of brane models which realizes three generations of quarks and leptons with Standard Model quantum numbers, is 3×10^{-6}.

While this ensemble is a bit oversimplified, we believe that a description of the true ensemble of brane gauge theories, which suffices for this purpose, need not be too much more complicated. One can refine our estimate by formulating more detailed ensembles, and comparing them with actual string theory constructions. We suspect this will lead to similar results, say

$$10^{-16} < \frac{N_{SM}}{N_{\mathrm{all}\ G,R}} < 1.$$

Thus, we can formulate a precise sense in which the Standard Model matter content is "generic." But is a fraction 10^{-6} generic? Is 10^{-16} generic?

Obviously, this depends on what other numbers enter the problem. If the total number of vacua is $O(10^{16})$, clearly not. But if it is $O(10^{100})$, then obviously this is not the hard part of the problem – many models realize the Standard Model spectrum. We must look for other criteria to guide the search for the "right vacuum."

So, how many vacua are there ?

2. Flux vacua

In a series of works (Strominger, Polchinski, Becker[2], ...) it has been shown that turning on p-form gauge field strengths in a compactification space, leads to a nontrivial potential which can stabilize moduli.

A particularly computable example is type IIb on a Calabi-Yau M (Giddings, Kachru, Polchinski, Trivedi, Kallosh, Linde,...).

The potential can be computed exactly at large volume, using special geometry and the superpotential (Gukov, Vafa, Witten):

$$\begin{aligned} W(z) &= \int_M (F^{(3)}_{RR} + \tau H^{(3)}_{NS}) \wedge \Omega(z) \equiv \int_M G \wedge \Omega(z); \\ K(z, \bar{z}) &= -\log \int_M \Omega(z) \wedge \bar{\Omega}(\bar{z}); \\ V(z) &= e^K \left(|DW|^2 - 3|W|^2 \right). \end{aligned}$$

Here z parameterizes complex structure moduli of M, $\tau = C^{(0)} + ie^{-D}$ is the axion-dilaton, and $\Omega(z)$ is the holomorphic three-form on M. We will ignore Kähler moduli, whose potential is determined non-perturbatively.

By varying the choice of flux, we get a simple ensemble of effective theories, which can be studied systematically. Since one can understand the dualities between vacua within this sector (they arise from CY geometry), counting these vacua would give us a lower bound for the total number of vacua.

Furthermore, by known dualities (Maldacena, Gopakumar, Vafa, Klebanov, Strassler, ...), these effective potentials describe a good deal of nonperturbative physics, such as gauge theory instanton effects. It is not completely crazy to claim that most of the choices not having to do with explicit low energy gauge symmetry can be dualized into this choice. If so, this multiplicity would be the dominant factor in the total multiplicity of vacua.

Ideally, we would count nonsupersymmetric vacua with hierarchically small supersymmetry breaking, and cosmological constant $\Lambda \sim 10^{-120} M_{pl}^4$. This is close to being a a well posed mathematical problem (the only issue is what cuts we have to make), and using the methods we will discuss shortly, we believe one can get significant results for it.

Clearly we should start more simply, however, and count the $\mathcal{N} = 1$ supersymmetric vacua. There have been good guesses for their number, but no controlled calculation, up to now.

Besides a choice of CY and orientifolding, a flux compactification sector is characterized by a quantized flux

$$N^\alpha \equiv N^\alpha_{RR} + \tau N^\alpha_{NS} = \int_{\Sigma_\alpha} F^{(3)}_{RR} + \tau H^{(3)}_{NS}.$$

A vacuum is then a particular solution of

$$DW_N(z) = 0.$$

These are related by dualities, which act both on flux and moduli,

$$(N, z) \sim (N', z').$$

We can take one representative from each duality class, by only considering vacua for which the moduli z to live within a fundamental region of the duality group. On doing this, N^α_{RR} and N^α_{NS} can be arbitrary integers, so the first point is to understand why the total number of allowed choices is finite.

In principle, we might have to make cuts to get finiteness. Two physically reasonable ones are to remove decompactification limits (the large complex structure limit), or to put a cut on the cosmological constant, $\Lambda = -3e^K|W|^2 \geq -|\Lambda_{min}|$. For simplicity, we won't do this unless forced to.

Other considerations can also cut down the possible choices of flux. An intriguing aspect of the IIb problem is that the IIb Chern-Simons coupling

$$\int C^{(4)} \wedge F^{(3)} \wedge H^{(3)}$$

leads to the following constraint from tadpole cancellation for $C^{(4)}$ (Gukov *et.al.*, Giddings *et.al.*):

$$\int F \wedge H + N(D3 \text{ branes}) = N(O3 \text{ planes}), \tag{1}$$

The numbers $N(O3)$, $N(D3)$ are positive for supersymmetric vacua. Furthermore, one can show that

$$0 < \int G \wedge *G \propto \int F \wedge H$$

for supersymmetric vacua. This might suggest that the number of fluxes satisfying (1) should be finite.

However, this is not true, because $\int F \wedge H$ involves the intersection form,

$$\int F \wedge H = \sum_i N_{RR}^{Ai} N_{NS}^{Bi} - N_{RR}^{Bi} N_{NS}^{Ai},$$

which is an **indefinite** form. Thus,

$$0 < \int F \wedge H \leq L_{max}$$

has an infinite number of solutions.

In fact, (Trivedi & Tripathy 0301139) have found infinite series of supersymmetric vacua on $K3 \times T^2$ (ignoring dependence on Kähler moduli).

So, finiteness of the number of vacua was not established. On the other hand, the series found by T & T decompactifies. So, putting a "cut" which removes decompactification limits saves it, in this example.

One can show (Ashok and Douglas 0307049) that all infinite series of IIb supersymmetric flux vacua, run off to limits of moduli space, in which the conditions $D_i W_N = 0$ change rank. The only known examples are decompactification (large complex structure) limits. So, this cut should suffice.

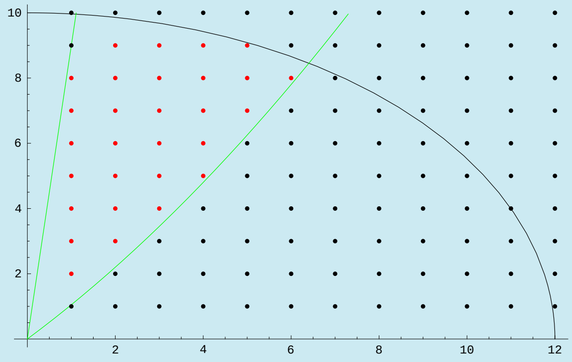
Home Page
Title Page
Page 17 of 28
Go Back
Full Screen
Close
Quit

3. New results

In (Ashok & Douglas 0307049) we have counted the flux vacua, under one major simplifying assumption: we ignore flux quantization, and compute the "volume" in "flux space"

$$\mathrm{vol} = \int d^{b_3} N_{RR}\ d^{b_3} N_{NS}$$

for which supersymmetric vacua exist. This gives the leading estimate for the number of vacua as the total flux becomes large.



Since the overall scale of the flux is set by the tadpole constraint

$$L \equiv N(O3 \text{ planes}) - N(D3 \text{ branes}) = \int F \wedge H \equiv \eta_{\alpha\beta} N^{\alpha}_{RR} N^{\beta}_{NS},$$

our computation gives the leading estimate,

$$N_{vac}(L \leq L_{max}) \sim c L^{b_3} + \mathcal{O}(L^{b_3 - 1}),$$

for large L. Here $b_3 = \dim H^3(M)$ is a Betti number of M. In examples, $N(O3) \sim 32$, so L can be large.

The number of vacua can be defined as the expected number of supersymmetric vacua (solutions of $DW = 0$), in the ensemble of flux vacua of total flux L,

$$\begin{aligned} I_{vac} &= \sum_{vacua} 1 \\ &= \sum_{\eta N N = L} \int_{\mathcal{F}} d^n z \; \delta(DW(z)) \; |\det D^2 W(z)| \\ &\equiv \int_{\mathcal{F}} d^n z \; \left\langle \delta(DW(z)) \; |\det D^2 W(z)| \right\rangle_L . \end{aligned}$$

Duality can be treated most simply by restricting the integral over moduli space to a fundamental region $\mathcal{F}$, and doing an unrestricted sum over all fluxes.

Our basic technique is to treat the flux ensemble, as a limit of a Gaussian ensemble of superpotentials, defined as a distribution

$$d\mu[W] = \int d^{2K}N \;\, e^{-Q_{\alpha\beta}N^\alpha \bar{N}^\beta} \delta(W(z) - \sum_\alpha N^\alpha \Pi_\alpha(z))$$

with

$$\Pi_\alpha(z) = \int_{\Sigma_\alpha} \Omega(z).$$

In words, we sum over all possible flux superpotentials, taking the fluxes out of a Gaussian distribution.

Any expectation value in this ensemble can be computed in terms of a two-point function,

$$\left\langle W(z_1) W^*(\bar{z}_2) \right\rangle,$$

which gives the joint expectation value for the product of superpotentials at two points in moduli space.

The natural two-point function in the IIb flux problem is

$$\begin{aligned}
\left\langle W(z_1)W^*(\bar{z}_2)\right\rangle &= \sum_G e^{-\int \alpha G\wedge *G}\, W(z_1)\bar{W}(\bar{z}_2) \\
&\sim \int dG\ e^{-\int \alpha G\wedge *G}(G\wedge\Omega(z_1))\ (\bar{G}\wedge\bar{\Omega}(\bar{z}_2)) \\
&= -\frac{1}{\alpha}\int \Omega(z_1)\wedge\bar{\Omega}(\bar{z}_2), \\
&= \frac{1}{\alpha}e^{-K(z_1,\bar{z}_2)},
\end{aligned}$$

(using a standard formula from special geometry), where $K(z_1,\bar{z}_2)$ is the Kähler potential, regarded as an independent function of the holomorphic and antiholomorphic variables.

This respects the $Sp(b_3,\mathbb{Z})$ group of possible duality symmetries. It also allows us to fix the tadpole condition $L=\eta NN$, by doing a Laplace transform,

$$\left\langle \ldots \right\rangle_{\text{fixed } L} = \int d\alpha\ e^{\alpha L}\int d^K N\ e^{-\alpha G\wedge *G}\ldots . \int d\alpha\ e^{\alpha L}\left\langle \ldots \right\rangle_{\text{fixed } \alpha}.$$

(A subtle point is that this "action" is not positive definite. One can however justify its use, by an analytic continuation. This works because $\int F\wedge H > 0$ for susy vacua.)

Using

$$\left\langle W(z_1)W^*(\bar{z}_2)\right\rangle = e^{-K(z_1,\bar{z}_2)},$$

one finds that the number of any given type of vacuum, can be computed as the integral of a density constructed from the Kähler form ω and curvature R of moduli space. These can be determined using techniques developed in the study of mirror symmetry.

The simplest computations are of analytic expectation values, which can be computed using Wick's theorem. In particular, the continuous flux approximation to the "supergravity index," which counts vacua with signs (the sign of the determinant of the fermion mass matrix),

$$\begin{aligned} I_{vac} &= \sum_{vacua} (-1)^F \\ &= \int_{\mathcal{F}} d^n z \,\left\langle \delta(DW(z))\;\det D^2W(z)\right\rangle, \end{aligned}$$

as an integral of an "index density"

$$d\mu_I(z) = \left\langle \delta(DW(z))\;\det D^2W(z)\right\rangle$$

counting the contribution of supersymmetric vacua which stabilize the moduli at the point z.

Computing this, simply requires computing quantities such as

$$\left\langle D_{1a} D_{1b} W(z_1) \bar{D}_{2\bar{c}} \bar{D}_{2\bar{d}} W^*(\bar{z}_2) \right\rangle|_{z_1=z_2=z} = R_{a\bar{c}b\bar{d}} + g_{b\bar{c}} g_{a\bar{d}} + g_{a\bar{c}} g_{b\bar{d}},$$

and taking a determinant.

We obtain for the index counting all vacua with flux up to L,

$$\begin{aligned} I_{vac}(L \le L_{max}) &= \frac{(2\pi L)^{b_3}}{12 \cdot b_3!} [c_n(\Omega M \otimes \mathcal{L})] \\ &= \frac{(2\pi L)^{b_3}}{12\pi^n n! b_3!} \int_{\mathcal{F}} \det(-R - \omega), \end{aligned}$$

where $\mathcal{F}$ is a fundamental region in the complex structure moduli space,

For example, for T^6 (with symmetrized period matrix), $K = b_3 = 20$, and

$$I = \frac{1}{181440 \cdot 12 \cdot 20!} (2\pi L)^{20} \sim 2 \times 10^{21} \text{ for } L = 32.$$

The index provides a lower bound for N_{vac}. N_{vac} is also computable, by doing more complicated Gaussian integrals. This allows one to study whether I_{vac} is a good approximation to N_{vac}; it seems to be.

4. Conclusions

We gave a general method for computing expectation values in the ensemble of effective supergravity theories arising from IIb compactification with flux on a Calabi-Yau, and computed numbers of supersymmetric vacua.

Expectation values involving a single point in configuration space (the index density, or density for a specified type of vacuum, including non-supersymmetric vacua), can be expressed entirely in terms of the Kähler form, curvature and its derivatives. "Multi-point" expectation values (such as expected distances between vacua, or in principle the tunneling action) can also be computed, if we can do the appropriate Gaussian integral.

The main limitation of the technique, is that in some cases it is not a good approximation to ignore flux quantization. For example, if the number of cycles K is greater than the number of fluxes L, one finds $I \sim L^K/K! \to 0$, but this is probably an artifact of our approximation.

It may be possible to do sums over quantized fluxes this way. One can also estimate the number of vacua for $K > L$ by summing over subsets of fluxes. This leads to the estimate

$$N_{vac} \sim \sum_{n=1}^{K} \binom{K}{n} \frac{(2\pi c L)^n}{n!} \sim e^{\sqrt{2\pi c K L}} \sim \mathcal{O}(10^{100}) \text{ for } L = 32, K \sim 250.$$

So is 10^{100} a large number of vacua or not ? We would suggest that it is **not**, that the known constraints on a realistic model (not including matching all the couplings) are probably satisfied by a fraction of vacua which is comparable to 10^{-100}.

What are these constraints?

- Supersymmetry breaking. Flux ensembles will include **large numbers** of **metastable nonsupersymmetric vacua** (even without adding antibranes), roughly comparable to the number of supersymmetric vacua we found. *
- Vacua must be stable on cosmological time scales: $O(?)$ *
- The hierarchy. Supersymmetric gauge theories (and flux superpotentials) can generate exponentially small hierarchies without fine tuning. So, this fraction might also be $O(1)$. *
- Given low energy supersymmetry breaking, one needs $\Lambda \sim 10^{-60} M_{susy}^4$. This could be realized statistically, along the lines of (Bousso and Polchinski, 0004134). It is reasonable to expect the cosmological constant in non-supersymmetric vacua to have a uniform distribution around 0, since there is nothing in a generic effective field theory which picks out 0. Thus, this cuts down the number by $O(10^{-60})$. *

- Low energy supersymmetry breaking. *
- Stability. *
- Cosmological constant. *
- As we discussed, the standard model gauge group and matter content are probably not hard, $O(10^{-10})$.
- Sufficient inflation: $O(?)$. *
- Supersymmetry breaking compatible with known phenomenological constraints: $O(?)$.

* indicates tests which can be made if we know the effective potential.

Of course, the flux potential is just an approximation to the exact string/M theory potential. In general, estimating the number of true string/M theory vacua meeting any of these tests, is not yet feasible.

However, the flux ensemble contains enough physics, to think it gives a good picture of the true ensemble of effective potentials arising from string/M theory compactification. As our understanding progresses, we can formulate better ensembles of effective potentials.

Given an **ensemble of effective field theories**, such as the ensemble of quiver gauge theories we discussed, one can in principle apply **all** the tests, and estimate the number of models from the ensemble, which realize the Standard Model. Again, one can start with simple ensembles, and systematically improve them to take into account progress in the study of string/M theory compactification.

Features/mechanisms which apply to $N >> 1$ vacua in an ensemble, can be considered **natural** in that ensemble. Thus, we can develop an idea of "stringy naturalness."

Using these ideas, we can try to find out whether candidate scenarios for string phenomenology are natural, whether or not string theory is predictive, and in what ways. Perhaps this will lead to the prediction that $O(1)$ vacua will work; perhaps not.