MULTILOOP SUPERSTRING AMPLITUDES USING THE PURE SPINOR FORMALISM

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I. Introduction

I.A. Problems with RNS and GS approaches

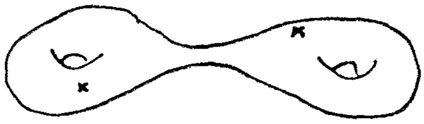
RNS: Spacetime susy only after summing over spin structures

$\Rightarrow$ divergences near boundary of moduli space for fixed spin structure

$\Rightarrow$ surface terms $A_{\substack{\text{spin}\\ \text{structure}}} = \int d\tau \frac{\partial}{\partial \tau}(\quad)$ cannot be ignored

$\Rightarrow$ amplitude depends on locations of picture-changing op's

("correct" locations can be determined from unitarity)



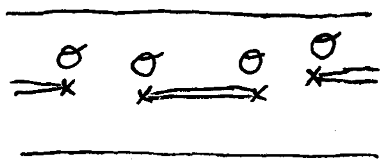
(Verlinde + Verlinde
Atick, Moore, Rabin, Sen)

Also, amplitudes involving external Ramond states are more difficult to compute.

Up to now, have explicit expressions up to 2-loop 4-point scattering of NS states. (D'Hoker + Phong, Iengo + Zhu)

GS: Scattering amplitudes have been computed only in light-cone gauge. (Green + Schwarz, Mandelstam)

Need to insert non-covariant operator $\mathcal{O} = P_L + P_i \theta \sigma^i \theta + P_R \theta\theta\theta\theta$ at light-cone interaction points



Complications from contact terms between $\mathcal{O}(y)\,\mathcal{O}(z)$ has prevented computations except for 4-point tree and one-loop amp's.

(Greensite + Klinkhammer
Mandelstam
Green + Seiberg)

I. B. References to pure spinor approach

1985

covariant GS (Green + Schwarz)

RNS (Friedan, Martinec + Shenker; Verlinde + Verlinde)

Harmonics (Sokatchev; Nissimov + Pacheva; Kallosh + Morozov)

Siegel

Twistors (Witten; Sorokin, Tkach, Volkov, Zheltukhin; NB; Tonin; Howe - pure spinors in D=10)

1990

Hybrid formalism (NB; NB + Vafa; NB + Siegel)

1995

AdS R-R backgrounds (NB + Vafa + Witten; NB, Bershadsky, Hauer, Zhukov, Zwiebach; NB, Gukov, Vallilo; NB + Chandia; Vallilo)

D=10 Pure Spinor formalism (NB; NB + Vallilo; NB + Chandia; NB + Howe; NB + Pershin; Oda + Tonin)

2000

Twistors (Marsher + A. Schwarz; Grassi + van Nieuwenhuizen; NB + Cherkov (in preparation); Matone, Mazzucato, Oda, Sorokin, Tonin)

"Extended" pure spinor formalism (Grassi, Policastro, Porrati + van Nieuwenhuizen; Aisaka + Kazama; Chesterman)

D=11 "pure" spinors (Cederwall et al; NB; Howe + Tsimpis; Grassi + Vanhove (in progress))

In components, can gauge-fix

$$A_\alpha(x,\theta) = e^{ik\cdot x}\left((\gamma^m\theta)_\alpha a_m + (\gamma^m\theta)_\alpha(\gamma_m\theta)_\beta \chi^\beta + \dots\right)$$

$$A_m(x,\theta) = e^{ik\cdot x}\left(a_m + (\gamma_m\theta)_\alpha \chi^\alpha + \dots\right)$$

where $k^2 = k\cdot a = \not{k}\chi = 0$ and $\dots$ involves products of k^m and a_m or χ

Superfield strengths:

$$W^\alpha(x,\theta) = \gamma^{m\alpha\beta}(D_\alpha A_m - \partial_m A_\alpha) = e^{ik\cdot x}\left(\chi^\alpha + (\gamma_{mn}\theta)^\alpha F^{mn} + \dots\right)$$

$$F_{mn}(x,\theta) = D\gamma_{mn}W = \partial_{[m}A_{n]} = e^{ik\cdot x}\left(F_{mn} + \dots\right)$$

Integrated open superstring vertex op's $\int dz\, U(z)$ are defined by requiring that $\boxed{Q\,U(z) = \partial V(z)}$.

For massless states, $V = \lambda^\alpha A_\alpha(x,\theta)$

$$\Rightarrow \boxed{U(z) = \partial\theta^\alpha A_\alpha(x,\theta) + \Pi^m A_m(x,\theta) + d_\alpha W^\alpha(x,\theta) + N_{mn}F^{mn}(x,\theta)}$$

In components, $U = \partial x^m a_m + (\frac{1}{2}p\gamma^{mn}\theta + N^{mn})F_{mn} + \dots$

Lorentz current $M^{mn} = \underset{k=+4}{\frac{1}{2}p\gamma^{mn}\theta} + \underset{k=-3}{N^{mn}}$ has same level as $M^{mn} = \underset{k=+1}{\psi^m\psi^n}$

BRST cohomology has been proven to correctly reproduce superstring spectrum. (NB / NB+Chandia)

BRST inv. in curved background implies low-energy eqns. for background superfields (NB+Howe)

Can compute tree amplitudes using normalization

$$\langle (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\rangle = 1.$$

Amplitudes agree with RNS prescription (NB+Vallilo)

II. Review of Pure Spinor Formalism

II.A. Worldsheet action and OPE's

Type IIB: $S = \int d^2z \left[-\frac{1}{2}\partial x^m \bar\partial x_m - p_\alpha \bar\partial \theta^\alpha - \bar p_\alpha \partial \bar\theta^\alpha + w_\alpha \bar\partial \lambda^\alpha + \bar w_\alpha \partial \bar\lambda^\alpha \right]$

where $\boxed{\lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta = 0 \text{ and } \bar\lambda^\alpha \gamma^m_{\alpha\beta} \bar\lambda^\beta = 0}$ "pure spinor"

$\alpha,\beta = 1$ to 16, $(\Gamma^m)_A{}^B = \begin{pmatrix} 0 & \gamma^m_{\alpha\beta} \\ \gamma^{m\alpha\beta} & 0 \end{pmatrix}$, $\gamma^{(m}_{\alpha\beta}\gamma^{n)\beta\gamma} = 2\eta^{mn}\delta_\alpha^\gamma$, $\eta_{mn}\gamma^m_{(\alpha\beta}\gamma^n_{\gamma)\delta} = 0$

$f^{(\alpha\beta)} = f^m \gamma_m^{\alpha\beta} + f^{mnpqr}\gamma^{\alpha\beta}_{mnpqr}$, $f^{[\alpha\beta]} = f^{mnp}\gamma^{\alpha\beta}_{mnp}$

$\lambda\gamma^m\lambda = 0 \Rightarrow \lambda^\alpha$ has 11 indep. comp's $\boxed{\text{e.g. } \lambda^\alpha = \left(\lambda^+, \lambda_{[ab]}, \frac{1}{\lambda^+}\epsilon^{abcde}\lambda_{[bc]}\lambda_{[de]}\right) \text{ for } a,b = 1 \text{ to } 5.}$

$\Rightarrow w_\alpha$ has gauge invariance

$\delta w_\alpha = (\gamma^m\lambda)_\alpha \Omega_m \Rightarrow w_\alpha$ only appears in the gauge-invariant combinations $\boxed{N_{mn} = \frac{1}{2} w\gamma_{mn}\lambda}$ and $\boxed{J = w_\alpha \lambda^\alpha}$

Can use unconstrained (non-covariant) definition of λ^α to compute the manifestly Lorentz-covariant OPE's:

$$N_{mn}(y)\,\lambda^\alpha(z) \to \frac{(\gamma_{mn}\lambda)^\alpha}{2(y-z)}, \quad J(y)\,\lambda^\alpha(z) \to \frac{\lambda^\alpha}{y-z}$$

$$N_{mn}(y)N_{pq}(z) \to \frac{-3}{(y-z)^2}\eta_{q[m}\eta_{n]p} + \frac{1}{y-z}\left(\eta^{p[n}N^{m]q} - \eta^{q[n}N^{m]p}\right)$$

$$J(y)\,N_{mn}(z) \to \text{regular}, \quad J(y)\,J(z) \to \frac{-4}{(y-z)^2}$$

$$J(y)\,T(z) \to \frac{-8}{(y-z)^3} + \frac{J(z)}{(y-z)^2}, \quad N_{mn}(y)\,T(z) \to \frac{N_{mn}(z)}{(y-z)^2}$$

$T = -\frac{1}{2}\partial x^m \partial x_m - p_\alpha \partial\theta^\alpha + w_\alpha \partial\lambda^\alpha$ has no central charge
$c = 10 + 16(-2) + 11(+2) = 0$

J has -8 ghost-number anomaly

N_{mn} is an $SO(9,1)$ current algebra of level -3

II.B. BRST operator and vertex operators

Open superstring spectrum described by ghost-number +1 states in cohomology of $\boxed{Q = \int dz\, \lambda^\alpha d_\alpha}$

$$\boxed{d_\alpha \equiv p_\alpha - \tfrac{1}{2}(\gamma^m\theta)_\alpha \partial x_m - \tfrac{1}{8}(\gamma^m\theta)_\alpha(\theta\gamma_m\partial\theta) \text{ is Dirac constraint of cov. GS superstring}}$$

Free-field OPE's of x^m and $(p_\alpha, \theta^\alpha)$ $\Rightarrow$

$$d_\alpha(y)\, d_\beta(z) \to -\frac{\gamma^m_{\alpha\beta}}{y-z}\Pi_m$$

$$d_\alpha(y)\, \Pi^m(z) \to \frac{\gamma^m_{\alpha\beta}}{y-z}\partial\theta^\beta$$

$$d_\alpha(y)\, \partial\theta^\beta(z) \to \frac{1}{(y-z)^2}\delta_\alpha^\beta$$

$$d_\alpha(y)\, A(x(z),\theta(z)) \to \frac{1}{y-z} D_\alpha A(x,\theta)$$

$\boxed{\Pi^m = \partial x^m + \tfrac{1}{2}\theta\gamma^m\partial\theta \text{ is supersymmetric momentum}}$

$\boxed{D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \tfrac{1}{2}(\gamma^m\theta)_\alpha \partial_m}$

$$Q^2 = \int dz\, \lambda^\alpha\lambda^\beta(-\gamma^m_{\alpha\beta}\Pi_m) = 0 \text{ by pure spinor constraint}$$

Massless states have zero conf. wt. at zero momentum

$$\Rightarrow \quad V = \lambda^\alpha A_\alpha(x,\theta)$$

$$QV = 0 \Rightarrow \lambda^\alpha\lambda^\beta D_\beta A_\alpha = 0 \Rightarrow \gamma^{\alpha\beta}_{mnpqr} D_\beta A_\alpha = 0$$

$$\Rightarrow \boxed{D_{(\alpha}A_{\beta)} = \gamma^m_{\alpha\beta} A_m \text{ for some } A_m(x,\theta)}$$

$$\delta V = Q\Omega = \lambda^\alpha D_\alpha \Omega \Rightarrow \boxed{\delta A_\alpha = D_\alpha\Omega \text{ and } \delta A_m = \partial_m \Omega}$$

$\Rightarrow$ $A_\alpha(x,\theta)$ and $A_m(x,\theta)$ describe on-shell gauge superfields of super-Maxwell theory

$$\boxed{\nabla_\alpha = D_\alpha + A_\alpha, \quad \nabla_m = \partial_m + A_m, \quad \{\nabla_\alpha, \nabla_\beta\} = \gamma^m_{\alpha\beta}\nabla_m}$$

III. Functional Integration

III.A. Measure factor for pure spinors

After using OPE's to integrate over non-zero worldsheet modes, how does one integrate over zero modes of pure spinors?

Although $[d^{11}\lambda]^{\alpha_1 \dots \alpha_{11}} = d\lambda^{\alpha_1} \wedge d\lambda^{\alpha_2} \wedge \dots \wedge d\lambda^{\alpha_{11}}$ is not Lorentz invariant, it is related to Lorentz. inv. measure $[\mathcal{D}\lambda]$ of ghost-number +8 by the formula

$$\boxed{[d^{11}\lambda]^{\alpha_1 \dots \alpha_{11}} = [\mathcal{D}\lambda]\, \mathcal{P}^{[\alpha_1 \dots \alpha_{11}]}_{((\beta\gamma\delta))} \lambda^\beta \lambda^\gamma \lambda^\delta}$$

$$\mathcal{P}^{[\alpha_1 \dots \alpha_{11}]}_{((\beta\gamma\delta))} \lambda^\beta \lambda^\gamma \lambda^\delta = \epsilon^{\alpha_1 \dots \alpha_{16}} (\lambda\gamma^m)_{\alpha_{12}} (\lambda\gamma^n)_{\alpha_{13}} (\lambda\gamma^p)_{\alpha_{14}} (\gamma_{mnp})_{\alpha_{15}\alpha_{16}}$$

$((\beta\gamma\delta))$ denotes symmetric and γ-matrix traceless

Can prove above formula using that $(\lambda\gamma^m)_{\alpha_1} [d^{11}\lambda]^{\alpha_1 \dots \alpha_{11}} = 0$.

Similarly, can use $N^{mn} = \frac{1}{2} w\gamma^{mn}\lambda$ and $J = w_\alpha \lambda^\alpha$ to prove

$[d^{11}N]^{[m_1 n_1] \dots [m_{10} n_{10}]} = dN^{m_1 n_1} \wedge dN^{m_2 n_2} \wedge \dots \wedge dN^{m_{10} n_{10}} \wedge dJ$

is related to Lorentz. inv. measure $[\mathcal{D}N]$ of ghost-number -8 by

$$\boxed{\begin{aligned} &[d^{11}N]^{[m_1 n_1] \dots [m_{10} n_{10}]} = \\ &[\mathcal{D}N]\Big[(\lambda\gamma^{m_1 n_1 m_2 m_3 m_4}\lambda)(\lambda\gamma^{m_5 n_5 n_2 m_6 m_7}\lambda)(\lambda\gamma^{m_8 n_8 n_3 n_6 m_9}\lambda)(\lambda\gamma^{m_{10} n_{10} n_4 n_7 n_9}\lambda) \\ &\qquad + \text{permutations}\Big] \end{aligned}}$$

Can prove above formula using that $(\lambda\gamma_{m_1})_\alpha [d^{11}N]^{[m_1 n_1] \dots [m_{10} n_{10}]} = 0$.

So $[\mathcal{D}\lambda]$ and $[\mathcal{D}N]$ are natural measure factors for integrating over 11 λ's and 11 gauge-inv. comb's of w.

III. B. Picture-changing operators

As in RNS formalism, integration over bosonic zero modes diverges unless one inserts delta-functions. In RNS formalism, $\delta(\beta)$ and $\delta(\gamma)$ come from picture-changing operators $Z = Q(\beta)\,\delta(\beta) = e^{\varphi}\partial x^m \psi_m + \ldots$ and $Y = c\partial\delta(\gamma) = c\partial\xi e^{-2\varphi}$.

In pure spinor formalism, picture-changing operators are

$$\boxed{\begin{aligned} Z_B &= Q(B_{mn}N^{mn})\,\delta(B_{pq}N^{pq}) = \tfrac{1}{2}B_{mn}(\lambda\gamma^{mn}d)\,\delta(B_{pq}N^{pq}) \\ Z_J &= Q(J)\,\delta(J) = \lambda^\alpha d_\alpha\,\delta(J) \\ Y_C &= C_\alpha\theta^\alpha\,\delta(C_\beta\lambda^\beta) \end{aligned}}$$

B_{mn} is a constant 2-form and C_α is a constant spinor

Can check that $QZ_B = QZ_J = QY = 0$ and that ∂Z_B, ∂Z_J and ∂Y_C are BRST-trivial.

Can also prove that Z_B and Y_C are indep. of choice of B_{mn} and C_α up to a BRST-trivial quantity

$$C_\alpha \to C_\alpha + \Lambda_\alpha \Rightarrow Y_C \to Y_{C+\Lambda} = Y_C + Q\left[(\Lambda_\alpha\theta^\alpha)(C_\beta\theta^\beta)\partial\delta(C_\gamma\lambda^\gamma)\right]$$

Also, susy variation of Y_C is BRST-trivial.

So up to surface terms, amplitudes are super-Poincaré covariant and independent of locations of (Z_B, Z_J, Y_C) and choices of (B_{mn}, C_α)

But unlike in RNS, surface terms can be ignored since amplitudes have no divergences near boundary of moduli space.

III. c. Construction of b ghost

To compute g-loop amplitudes, need b ghost of -1 ghost-number satisfying $\{Q, b(u)\} = T(u)$ so that

$$\{Q, \int d^2u\, b(u)\, \mu_s(u)\} = \int d^2u\, T(u)\, \mu_s(u) = \frac{\partial}{\partial \tau_s},$$

$\mu_s(u)$ = Beltrami differential for Teichmüller parameter τ_s

Since w_α only appears in combinations with zero ghost-number, cannot construct b ghost satisfying $\{Q, b\} = T$.

But using $Z_B = \frac{1}{2}(\lambda \gamma^{mn} d) B_{mn}\, \delta(BN)$ of $+1$ ghost number, can construct b ghost in non-zero picture satisfying

$$\boxed{\{Q, b_B\} = T Z_B} \quad \Rightarrow b_B \text{ has } \underline{\text{zero}} \text{ ghost-number}$$

$$\boxed{\begin{aligned} b_B = {} & B\,(d\,d\,\Pi + d\,N\,\partial\theta + N\,N + N\,\Pi\,\Pi)\,\delta(BN) \\ & + B\,B\,(d\,d\,d\,d + d\,d\,N\,\Pi + N\,N\,\Pi\,\Pi + N\,N\,d\,\partial\theta)\,\partial\delta(BN) \\ & + B\,B\,B\,(d\,d\,d\,d\,N + d\,d\,N\,N\,\Pi)\,\partial^2\delta(BN) \\ & + B\,B\,B\,B\,(d\,d\,d\,d\,N\,N)\,\partial^3\delta(BN) \end{aligned}}$$

All terms in b_B carry $+2$ conf. wt. and $+4$ "engineering dimension" where $[\lambda^\alpha, \theta^\alpha, x^m, d_\alpha, N_{mn}]$ carries $[0, \frac{1}{2}, 1, \frac{3}{2}, 2]$ "engineering dimension"

$\partial^L \delta(BN) \equiv \frac{\partial^L}{\partial (BN)^L}\, \delta(BN)$ defined to carry $-2L$ engineering dimension

b_B is manifestly supersymmetric and is Lorentz-invariant up to a BRST-trivial quantity.

IV. Super-Poincaré Covariant Loop Amplitudes

IV.A. g-loop prescription

At genus g, need 11 Y's and 11g Z's to absorb zero modes of λ^α and w_α.

$$\mathcal{A}_g = \prod_{P=1}^{3g-3} \int d^2\tau_P \left\langle \left| \prod_{P=1}^{3g-3} \int d^2 u_P\, b_{B_P}(u_P)\, \mu_P(u_P) \prod_{P=3g-2}^{10g} Z_{B_P}(z_P) \prod_{R=1}^{g} Z_J(w_R) \prod_{I=1}^{11} Y_{C_I}(y_I) \right|^2 \right.$$

$$\left. \prod_{r=1}^{N} \int d^2 t_r\, U(t_r, \bar{t}_r) \right\rangle$$

$$b_{B_P} = B_P d d \Pi\, \delta(B_P N) + B_P B_P dddd \frac{\partial}{\partial(B_P N)} \delta(B_P N) + \ldots$$

$$Z_{B_P} = \tfrac{1}{2}(\lambda B_P d)\,\delta(B_P N)\ , \quad Z_J = (\lambda d)\,\delta(J)\ , \quad Y_{C_I} = (C_I \theta)\,\delta(C_I \lambda)$$

$$U(t,\bar{t}) = e^{ik\cdot x} \left| \partial\theta^\alpha A_\alpha(\theta) + \Pi^m A_m(\theta) + d_\alpha W^\alpha(\theta) + N_{mn} F^{mn}(\theta) \right|^2$$

Since all worldsheet fields have conf. wt. 0 or 1, partition function cancels since

$$\int D^{10}x \left| \int D^{16}\theta\, D^{16}p\, D^{11}\lambda\, D^{11}w \right|^2 e^{-S} = \left| \det{}^{-5}(\bar{\partial}_0)\, \det{}^{16}(\bar{\partial}_0)\, \det{}^{-11}(\bar{\partial}_0) \right|^2 = 1$$

To compute correlation functions, separate off zero modes as

$$d_\alpha(z) = \sum_{R=1}^{g} d_\alpha^R\, \omega_R(z) + \hat{d}_\alpha(z)\ , \quad N_{mn}(z) = \sum_{R=1}^{g} N_{mn}^R\, \omega_R(z) + \hat{N}_{mn}(z)$$

$\omega_R(z)$ are g holomorphic one-forms

Use OPE's for $\hat{d}_\alpha$ and $\hat{N}_{mn}$ to integrate out non-zero modes

Ex: $$\langle d_\alpha(z)\, \Pi_m(y) \ldots \rangle = \sum_{R=1}^{g} d_\alpha^R\, \omega_R(z) \langle \Pi_m(y) \ldots \rangle$$
$$+ \partial_z \log E(z,y) \langle \gamma_{m\alpha\beta}\, \partial\theta^\beta(y) \ldots \rangle + \ldots$$

$E(z,y)$ is holomorphic prime form

Although $\partial_z \log E(z,y)$ is not single-valued, correlation functions are single-valued after integration over the zero modes of (d_α, θ^β) and $(\lambda^\alpha, \omega_\beta)$.

To integrate over these zero modes, use measure factor

$$\int d^{16}\theta \int [D\lambda] \prod_{R=1}^{g} \int d^{16} d^R \int [DN^R]$$

$$\Rightarrow \mathcal{A}_g = \int d^{16}\theta \int [D\lambda] \prod_{R=1}^{g} \int d^{16} d^R \int [DN^R] \; f(\lambda, \theta, d^R, N^R, J^R, B_p, C_I)$$

where $f = \lambda^{\alpha_1} \ldots \lambda^{\alpha_{8g+3}} f_{\alpha_1 \ldots \alpha_{8g+3}}(\theta, C_I, B_p) \prod_{p=1}^{10g} \delta(B_p N) \prod_{R=1}^{g} \delta(J) \prod_{I=1}^{11} \delta(C_I \lambda)$

Since $\mathcal{A}_g$ is independent of B_p^{mn} and $C_{I\alpha}$, can integrate over all choices of B_p^{mn} and $C_{I\alpha}$ to obtain Lorentz inv. formula

$$\mathcal{A}_g = \int d^{16}\theta \prod_{R=1}^{g} \int d^{16} d^R \, (\mathcal{P}^{-1})^{((\alpha_1 \alpha_2 \alpha_3}_{[\rho_1 \ldots \rho_{11}]} \left(\left(\frac{\partial}{\partial B_p}\right)^{10g} \right)^{\alpha_4 \ldots \alpha_{8g+3}))}$$

$$\left(\prod_{I=1}^{11} \frac{\partial}{\partial C_{I\rho_I}} \right) f_{\alpha_1 \ldots \alpha_{8g+3}}$$

Can verify that this reproduces correct tree amp's and 4-point one-loop massless amplitude.

$$\mathcal{A}_{g=1} = \int d^2\tau \, (\mathrm{Im}\,\tau)^{-5} \prod_{r=2}^{4} \int d^2 t_r \prod_{r<s} G(t_r - t_s)^{2k_r \cdot k_s}$$

$$\left| (\mathcal{P}^{-1})^{((\alpha\beta\gamma))}_{[\rho_1 \ldots \rho_{11}]} \, (\gamma_{mnpqr})_{\alpha\beta} \int (d^5\theta)^{\rho_1 \ldots \rho_{11}} A_\gamma(\theta) \, (W(\theta) \gamma^{mnp} W(\theta)) \, F^{qr}(\theta) \right|^2$$

Expanding superfields in components gives expected R^4 term.

V. Vanishing Theorems

V.A. Non-renorm. theorem and perturbative finiteness

Thm: N-point g-loop massless amp's vanish for $N \leq 3$ and $g > 0$

$N=0 \Rightarrow$ no cosmological constant
$N=1 \Rightarrow$ no tadpoles
$N=2 \Rightarrow$ massless states stay massless

} Implies finiteness near boundary of moduli space

$N=3 \Rightarrow$ no coupling constant renormalization (Martinec '86)

Assuming factorization and absence of unphysical divergences in interior of moduli space, non-renormalization theorem implies perturbative finiteness of superstring amplitudes.

In RNS, no proof because of unphysical poles in susy currents

In GS, no proof because of possible contact terms from light-cone operators in interior of moduli space.

Mandelstam has proven finiteness by combining features of RNS and GS formalisms.
(Mandelstam '92)

In pure spinor formalism, can easily prove non-renorm. thm. by counting fermionic zero modes of d_α.

At g-loops (assume $g>1$), need to get $16g$ d_α zero modes from

$$(Z_B)^{7g+3}(Z_J)^g \to (d)^{8g+3}$$

$$(b_B)^{3g-3} \to (d)^{8g-8+\frac{4M}{3}} \frac{\partial^M}{\partial(\theta N)^M}$$

$\Rightarrow$ N vertex op's must provide $5 - \frac{4M}{3}$ d_α's and M N_{mn}'s

$U = |\partial\theta^\alpha A_\alpha + \Pi^m A_m + d_\alpha W^\alpha + N_{mn} F^{mn}|^2 \Rightarrow$ Amplitude vanishes for $N < 4$

V.B. R^4 term and Type IIB S-duality

Green-Gutperle and Green-Vanhove have conjectured that R^4 term in Type IIB low-energy effective action appears in an $SL(2,\mathbb{Z})$-invariant combination as

$$\mathcal{S}_{eff} = \int d^{10}x \sqrt{g}\ (e^{-2\phi} + \zeta(3) + \text{instanton contributions})\ R^4$$

$\Rightarrow$ No perturbative R^4 contributions above one-loop.

Using RNS formalism, conjecture was recently proven to two-loops (D'Hoker+Phong '02, Iengo+Zhu '02)

Using pure spinor formalism, can easily prove absence of multiloop R^4 contributions to all loops.

To provide $16g$ d_α zero modes, vertex op's must provide $5-\frac{4M}{3}$ d_α's and M N_{mn}'s

$\Rightarrow$ Four vertex op's contribute

$$\int d^2z_1 \left| d_\alpha W_1^\alpha(\theta) + N_{mn} F_1^{mn}(\theta)\right|^2 \prod_{T=2}^{4} \int d^2 z_T \left| N_{pq} F_T^{pq}(\theta)\right|^2$$

To provide 16 θ^α and 16 $\bar\theta^\alpha$ zero modes, vertex op's must contribute 5 θ^α and 5 $\bar\theta^\alpha$ zero modes since $\left|\prod_{I=1}^{11} Y_{C_I}\right|^2$ contributes 11 θ^α and 11 $\bar\theta^\alpha$ zero modes.

Since $W^\alpha(\theta) = F_{mn}\theta + (\partial_p F_{mn})\theta^3 + \ldots$ and $\mathcal{F}_{mn}(\theta) \to F_{mn} + (\partial_p F_{mn})\theta^2 + \ldots$ amplitude has terms $\left|(\partial F)^2 F^2\right|^2 \sim \partial^4 R^4$

$\Rightarrow$ no multiloop contributions to R^4 or $\partial^2 R^4$ terms.

VI. Conclusions

Multiloop amplitude computations using pure spinor formalism are simpler than in RNS formalism

- No sum over spin structures
- Surface terms can be ignored
- No unphysical poles from φ chiral boson
- Partition functions cancel
- Amplitudes with Ramond states not more complicated
- Fermionic zero modes make it easy to prove vanishing theorems related to perturbative finiteness and S-duality conjectures

$$\langle \varphi Q Y \varphi + Y \varphi^3 \rangle$$

$$Q\varphi = \varphi \star \varphi$$