SUPER YANG-MILLS, MATRIX MODELS AND GEOMETRIC TRANSITIONS

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PLAN OF THE TALK

1. SPECTRUM OF BPS DOMAIN WALLS IN $\mathcal{N}=1$ SYM

2. RELATION BETWEEN MATRIX MODELS AND CALABI-YAU SPACES

INCLUDING SOME INTERESTING OPEN QUESTIONS.

0. GENERAL SET-UP

$U(N)$
$\mathcal{N}=1$ SYM with adjoint chiral multiplets X_i and superpotential

$W = \mathrm{tr}\, V(X_i)$

$\Longleftrightarrow$

N IIB D5-branes wrapped on $\mathbb{P}^1 \subset CY_3$

	0	1	2	3	4	5	6	7	8	9
D5	×	×	×	×	×	×				
$\mathbb{P}^1$					×	×				
CY_3					×	×	×	×	×	×

QUANTUM CORRECTIONS TO F-TERMS

Matrix model with potential W

$\int dX_i \, e^{-\frac{n}{S} W}$
planar

$S=$ coupling of MM
$=$ glueball superfield of SYM

$\longleftrightarrow$

3-form flux through $S^3 \subset \widetilde{CY_3}$
No D-branes
(only the complex structure is needed to compute F-terms)

1. SPECTRUM OF BPS DOMAIN WALLS

Pure U(N) SYM

$$U(1)_R \xrightarrow[\text{instantons}]{} \mathbb{Z}_{2N} \xrightarrow[\text{dynamical chiral symmetry breaking}]{} \mathbb{Z}_2$$

There are N vacua $|k\rangle$, $0 \leq k \leq N-1$, and domain walls interpolating between them. The domain walls $(|k\rangle, |k+1\rangle)$ are like D-branes on which confining strings can end.

BPS :

$$T_{(|k\rangle, |k+1\rangle)} = 2N^2 \Lambda^3 \sin\frac{\pi}{N}$$

dynamically generated scale

$$\underset{N\to\infty}{\simeq} 2\pi \Lambda^3 . N \propto \frac{1}{g_s}$$

$U(N)$ SYM with adjoint X

$$dW = g \,\mathrm{tr}\,(X - x_+)(X - x_-)$$

We now have $\frac{1}{6}N(N^2+11)$ vacua corresponding to various patterns of gauge and chiral symmetry breaking.

Let us focus on the $2N$ vacua for which the low energy theory is pure SYM with scale Λ.

- $|k,+\rangle$ corresponding to $\langle X\rangle_{cl} = x_+ \mathbb{1}$.
- $|k,-\rangle$ corresponding to $\langle X\rangle_{cl} = x_- \mathbb{1}$.

DOMAIN WALLS :

1) $(|k,+\rangle, |k',+\rangle)$ or $(|k,-\rangle, |k',-\rangle)$

2) $(|k,+\rangle, |k',-\rangle)$: "classical" domain walls

$$T_{cl} \underset{N\to\infty}{\simeq} N^2 \propto \frac{1}{g_s^2}$$: "CLOSED STRING SOLITONS"

Domain walls :

$D_{\pm}$ are D branes $(|k,\pm\rangle, |k+1,\pm\rangle)$

S are solitons $(|+\rangle, |-\rangle)$

QUANTUM SPACE OF PARAMETERS :

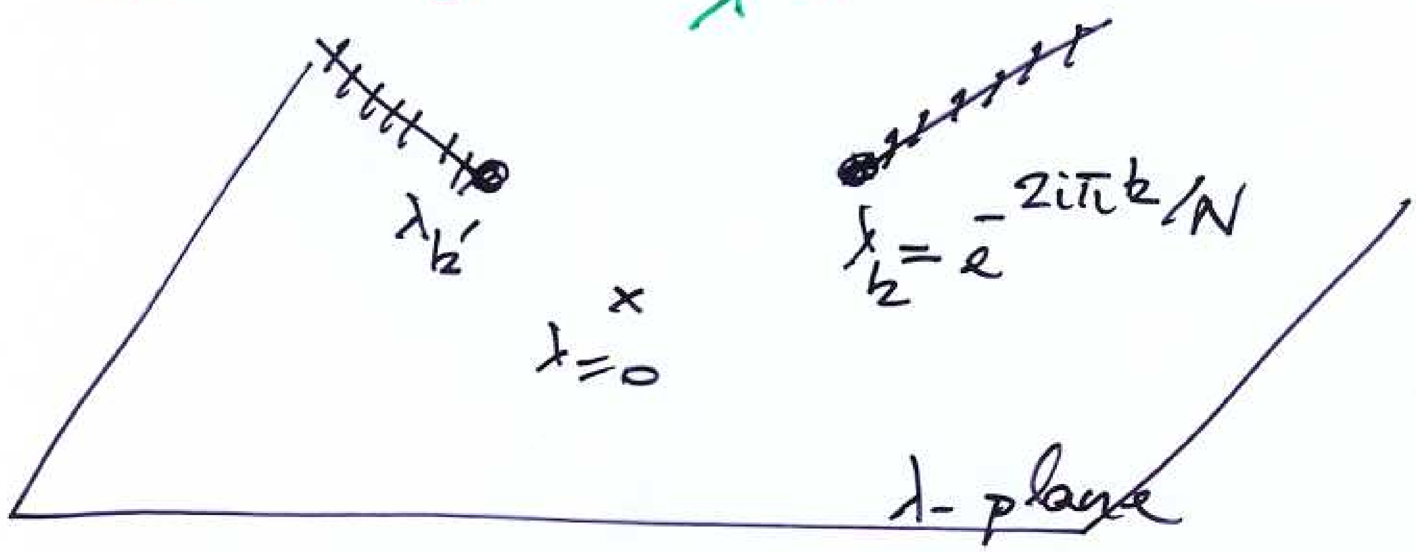
We can interpolate smoothly between $|k,+\rangle$ and $|k,-\rangle$, WITHOUT CHANGING $|k',+\rangle$ OR $|k',-\rangle$ FOR $k \neq k'$.

This is a strong coupling phenomenon.

CONCRETELY : $W = \frac{1}{2} m X^2 + \frac{1}{3} g X^3$

$$\lambda = \frac{8 g^2 \Lambda^3}{m^3}$$

$$W_{low}^{|k,\pm\rangle} = \frac{2N}{3}\Lambda^3 \, \frac{1}{\lambda}\left[1 \mp \left(1 - \lambda e^{\frac{2i\pi k}{N}}\right)^{3/2}\right]$$



$$\left(|k,+\rangle \,,\, |k+1,+\rangle\right) \quad : \text{D-brane}$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\left(|k,-\rangle \,,\, |k+1,+\rangle\right) \quad : \text{closed string soliton.}$$

We see that we can have a smooth interpolation between a D-brane and a closed string soliton, which can of course be seen directly from the tension

$$T_{(|a\rangle,|b\rangle)} = N \left| W_{low}^{|a\rangle} - W_{low}^{|b\rangle} \right| .$$

At the branching point we have

$$T_D \sim T_S \propto \sqrt{N} \propto \frac{1}{\sqrt{g_s}} \;!$$

7b 14

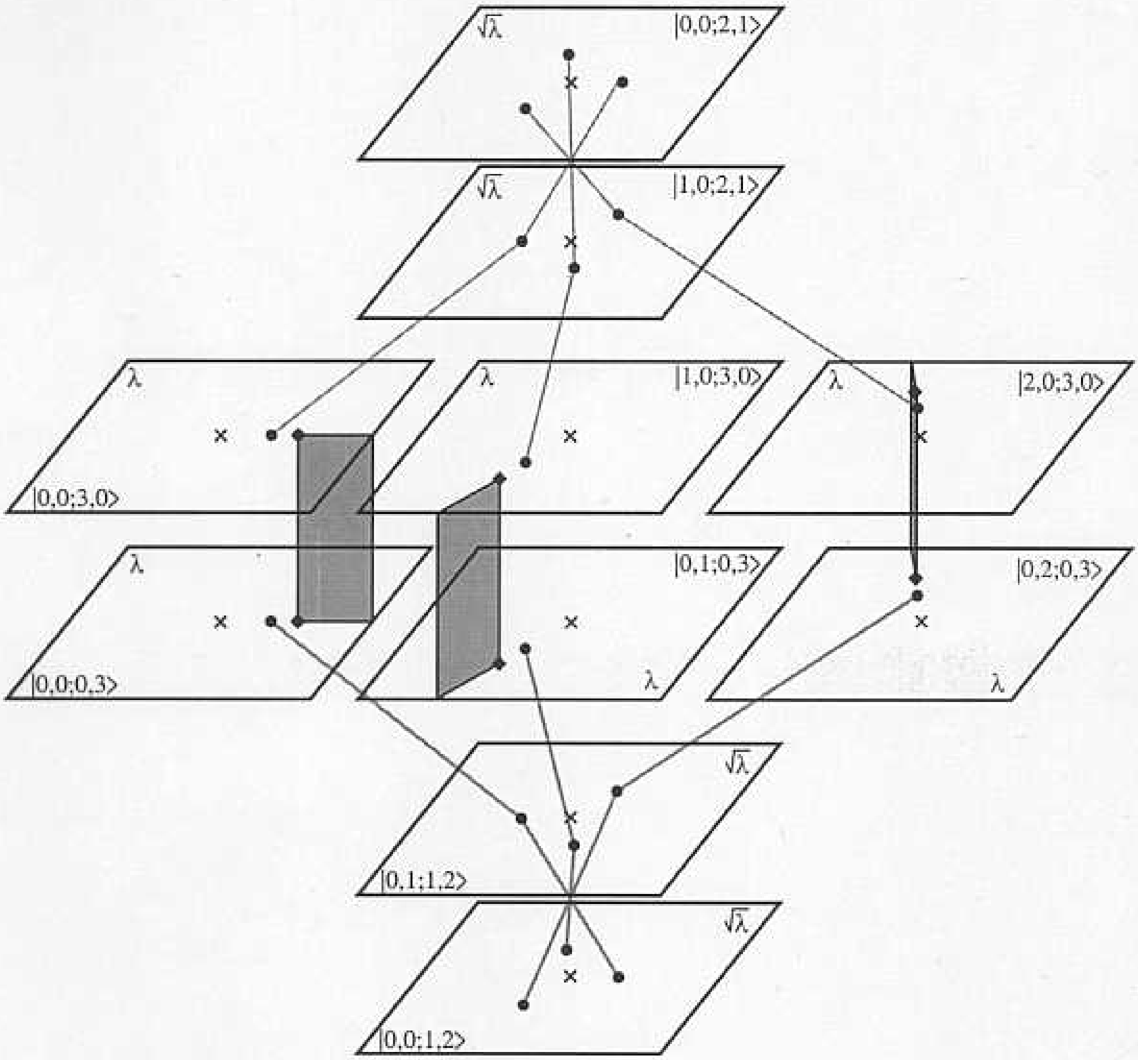


Figure 1: Sketch of the quantum parameter space $\mathcal{M}_q$ for gauge group U(3). The sheets are labeled by a state $|k_1, k_2; N_1, N_2\rangle$ and parametrized by λ (if $N_1 N_2 = 0$) or by $\sqrt{\lambda}$ (if $N_1 N_2 \neq 0$). The cross denotes the classical $\lambda = 0$ points. The black dots represent singularities with a massless monopole that are found for $\sqrt{\lambda} = 2\sqrt{2}e^{-i\pi q/3}/3$, q integer. Each singularity of that type link three sheets, and the gray lines represent the corresponding three-fold indentification. The black squares at $\lambda = e^{-2i\pi k/3}$ represent singularities with a massless glueball. They are the three branching points of the three cuts joining the sheets $|k, 0; 3, 0\rangle$ and $|0, k; 0, 3\rangle$.

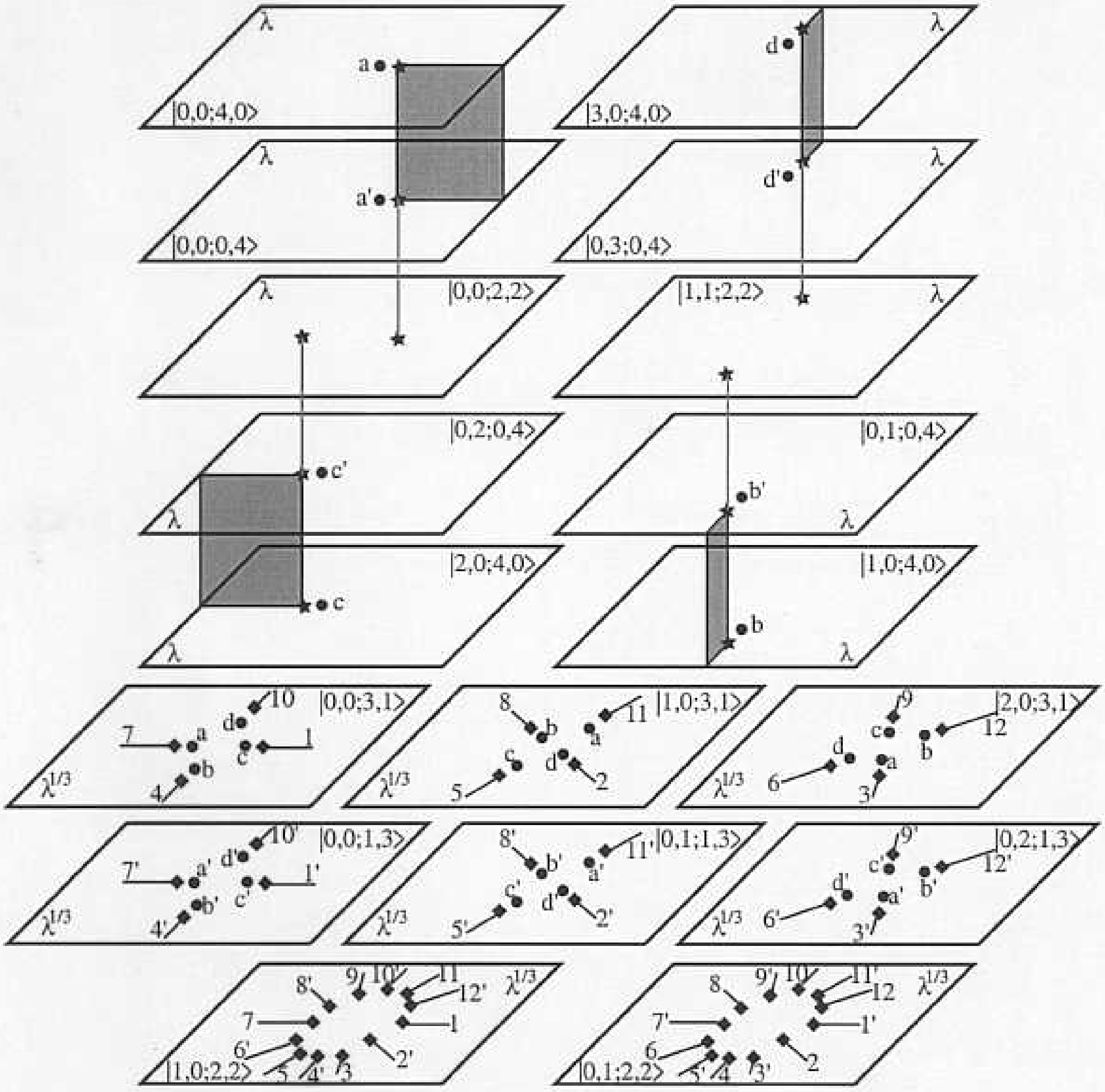


Figure 2: Sketch of the quantum parameter space $\mathcal{M}_q$ for gauge group U(4). The sheets are parametrized by λ or $\lambda^{1/3}$. The dots, squares and stars represent singularities with a massless monopole (at $\lambda = (4/5)e^{-ik\pi/2}$), a massless glueball (at $\lambda^{1/3} = (3^{1/4}/2^{1/6})e^{-ik\pi/6}$), or both (at $\lambda = e^{-ik\pi/2}$), respectively. Due to the complexity of the diagram, we have not been able to represent explicitly all the identifications between sheets. It is understood that singularities and branch cuts with the same label are identified.

CONCLUSION :

1. D-branes and solitonic branes can be smoothly connected.
2. New exotic extended objects, with e.g. $T \propto \frac{1}{\sqrt{g_s}}$ (or other fractional powers) do exist.
3. We have a simple framework (semi-classical SYM with one adjoint) to study solitonic branes very explicitly.

2. MATRIX MODELS AND CY SPACES

CY_3 two coordinate patches z, ω_1, ω_2
z', ω'_1, ω'_2

$$z' = \frac{1}{z} \qquad \mathbb{P}^1 \subset CY_3$$

$$\omega'_1 = z^{1-M}\, \omega_1$$

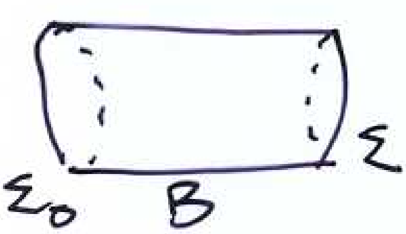
$$\omega'_2 = \underbrace{z^{1+M}\, \omega_2}_{\mathcal{O}(-1+M) \oplus \mathcal{O}(-1-M) \to \mathbb{P}^1} + \partial_\omega E(z, \omega_1)$$

: M adjoints

$$E(z,\omega) = \sum_{i=-\infty}^{+\infty} \underbrace{E_i(\omega)}_{\text{entire functions of } \omega.}\, z^i$$

Superpotential

1. One D5 brane : $W(\underset{\text{2-cycle}}{\Sigma}) = \int_B \underset{\text{holomorphic 3-form}}{\Omega} + \text{cste}$



$$\partial B = \Sigma - \Sigma_0$$

2. Preserve SUSY for N D-branes

Result:

$$W(X_1, \dots, X_M) = \frac{1}{2i\pi} \oint_{C_0} z^{-1-M} E\left(z, \sum_{j=1}^{M} X_j z^{j-1}\right) dz$$

$N \times N$ hermitian matrices (non-commuting)

small contour encircling 0

One matrix : $W(X) = E_1(X)$.

Two matrices : pick any $W(x, y)$ for x, y c-numbers, and use Weyl ordering

$$x^i y^j \longrightarrow \frac{i!\, j!}{(i+j)!} \oint_{C_0} \frac{dz}{2i\pi} z^{-1-j} (X + Yz)^{i+j}$$

Clearly we have a huge class of matrix models.

To solve the models, we have to BLOW-DOWN the geometry (birational mapping $\mathbb{P}^1 \to$ points) and then DEFORM with certain normalizability conditions.

BLOWING DOWN

$$CY_3 \underset{\pi^{-1}}{\overset{\pi}{\rightleftarrows}} CY_3^0$$

CY_3^0 should encode the classical MM, i.e. the algebra of eqs. of motion $\operatorname{tr} dW = 0$.

Example. $W(X,Y) = XY^2 + V(X) + U(Y)$

$$V'(X) = -F_1(X^2) - XF_2(X^2)$$
$$U'(Y) = -G(Y^2)$$

Algebra

$$\{X,Y\} = G(Y^2)$$
$$Y^2 = F_1(X^2) + XF_2(X^2)$$

there are $\max(\deg V'+2,\ 2(\deg V')(\deg U'))$ 1-dim irrep and $[\frac{1}{2}(\deg V'-1)]$ 2-dim irrep.

CY_3^0: $x_2[(x_2 - F_1(x_1))^2 - x_1F_2^2(x_1)] =$

$$x_4^2 - x_1x_3^2 - G(x_2)[(x_2 - F_1(x_1))x_3 + x_4F_2(x_1)]$$

DEFORMING

$$\widetilde{CY_3} : \quad CY_3^0 + S \sum c_{abcd}\, x_1^a x_2^b x_3^c x_4^d = 0$$

The allowed monomials are such that the divergences in $\oint \Omega$ are either S-independent or linear in S and logarithmic.

\# independent c_{abcd}
= # of adjustable parameters
= # of filling fractions in the MM
= # of irrep of $\operatorname{tr} dW = 0$.

✓ checked for $W(X,Y) = XY^2 + V(X) + U(Y)$

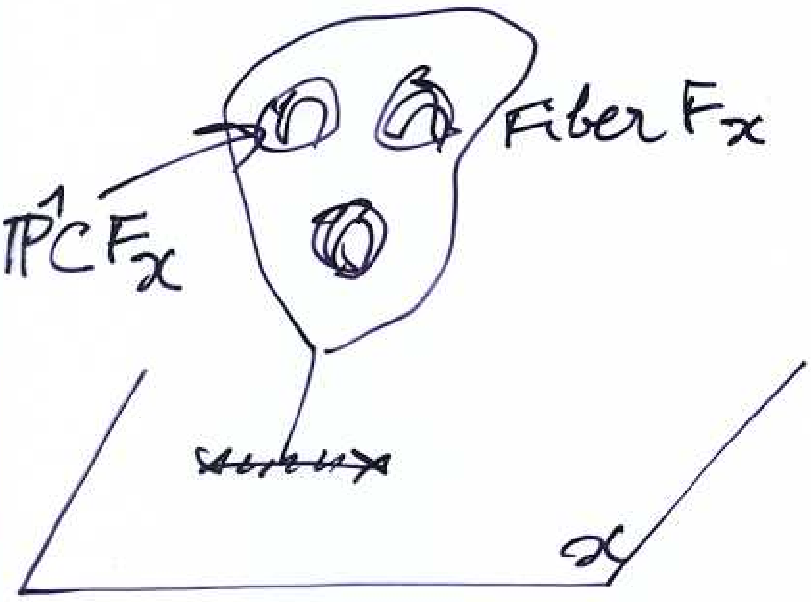
$$\# = \max\left(\deg V' + 2,\ 2(\deg V')(\deg U')\right) + \left[\tfrac{1}{2}(\deg V' - 1)\right]$$

We can also get the resolvents

$$g^X(x) = \langle \mathrm{tr} \frac{1}{x - X} \rangle$$

$$g^Y(y) = \langle \mathrm{tr} \frac{1}{y - Y} \rangle$$

from the geometry.



$$\int_{S^2 \subset F_x} i_X \Omega = g^X(x) - \hat{g}^X(x)$$

= discontinuity accross the branch cuts

$\propto$ density of eigenvalues.

CONCLUSIONS.

There is a rich interplay between algebraic geometry and matrix models, with many non-trivial results.

A most important open question is to prove that the blow down of CY_3 exists or does not exist.

If it does, then we have solved a huge class of multi-matrix models.

$$W(x_1, \dots, x_M) = \frac{1}{2\pi i} \oint_{C_0} z^{-M-1} E\left(z, \sum_{i=1}^{M} x_i z^{i-1}\right) dz$$

If it does not, then the picture of geometric transitions must be modified, with fundamental consequences for our understanding of the structure of $\mathcal{N}=1$ vacua in String theory.

Homework

$$z' = 1/z$$

$$w'_1 = z^{1-M} w_1$$

$$w'_2 = z^{1+M} w_2 + \partial_w E(z, w_1)$$

1. Find the blow-down geometry. Compare with the algebra
$$\mathrm{tr}\, dW = 0$$
where $$W = \frac{1}{2i\pi} \oint_{C_0} z^{-M-1} E\left(z, \sum_{i=1}^{M} X_i z^{i-1}\right) dz$$

2. Find the deformed geometry. Compare with the number of irrep. of
$$d\,\mathrm{tr}\, W = 0.$$

3. Compute the resolvents from the geometry. Compare with the loop equations.