Supersymmetric Backgrounds
From
Generalized Calabi-Yau Manifolds

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Type II sugra on $M_{10} = M_4 \times M_6$

Minimal supersymmetry

M$_6$ is CY
SU(3) holonomy
$\nabla_m \eta = 0$

Turn on fluxes

M$_6$ CY
SU(3) structure
$\nabla_m^{(T)} \eta = 0$
Torsion $\sim$ flux

CY

Generalized CY

Hitchin '02
Outline

• Basic notions about SU(3) structure
• Torsion versus fluxes
• Generalized complex geometry / Generalized CY
• Generalized CY from supersymmetry equations
• Conclusions / open questions
Basic notions about SU(3) structure

- No flux \(\text{M}_6\) is CY \(\Rightarrow\) SU(3) holonomy

- 9 SU(3) invt. spinor \(\eta\)

\[
\eta^\dagger \gamma_{mn} \gamma \eta = i \, w_{mn}
\]

\[
\eta^\dagger \gamma_{mnp} (1 + \gamma) \eta = i \, \Omega_{mnp}
\]

- \(\eta\) is cov. constant: \(r_m \eta = 0\)

\[
dw = 0
\]

\[
d\Omega = 0
\]

\[
\epsilon^{1,2}_{10} = \theta^{1,2} \otimes \eta \rightarrow \mathcal{N} = 2
\]

- Turn on flux \(\Rightarrow\) back-reaction

\[M_{10} = \text{M}_4 \times \text{M}_6 \]

\[
[ds_{10}^2 = e^{2A(y)} \eta_{\mu\nu} dx^{\mu} dx^{\nu} + ds_6^2(y)]
\]

- \(N = 2 \rightarrow N = 1\) (relation between \(\pi^1\) \(\theta^2\))

\(\text{M}_6\) acquires torsion \(\Rightarrow\) SU(3) structure

- 9 SU(3) invt. spinor \(\eta\)

\[
\eta^\dagger \gamma_{mn} \gamma \eta = i \, w_{mn}
\]

\[
\eta^\dagger \gamma_{mnp} (1 + \gamma) \eta = i \, \Omega_{mnp}
\]

- \(r_m^{(T)} \eta = 0\)

\[
r_m^{(T)} w = 0 \quad dw \neq 0 / T
\]

\[
r_m^{(T)} \Omega = 0 \quad d\Omega \neq 0 / T
\]

Gauntlett, Martelli, Pakis, Waldram ’02
Torsion versus fluxes

Torsion: \( dw = \text{Im} (W_1 \Omega) + W_4 \mathcal{A} w + W_3 \)

SU(3) reps

\[
\begin{align*}
1 & \oplus 1 & 3 & \oplus 3 & 6 & \oplus 6 \\
1 & \oplus 1 & 3 & \oplus 3 & 8 & \oplus 8
\end{align*}
\]

\( d \Omega = W_1 w^2 + W_5 \mathcal{A} \Omega + W_2 \mathcal{A} w \)

<table>
<thead>
<tr>
<th></th>
<th>1 ( \oplus 1 )</th>
<th>3 ( \oplus 3 )</th>
<th>6 ( \oplus 6 )</th>
<th>8 ( \oplus 8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Torsion</td>
<td>1 ( (W_1) )</td>
<td>2 ( (W_4, W_5) )</td>
<td>1 ( (W_3) )</td>
<td>1 ( (W_2) )</td>
</tr>
<tr>
<td>( H_3 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>IIA: ( F_{2n} )</td>
<td>2 ( (F_0, F_2, F_4) )</td>
<td>2 ( (F_2, F_4) )</td>
<td>0</td>
<td>1 ( (F_2, F_4) )</td>
</tr>
<tr>
<td>IIB: ( F_{2n+1} )</td>
<td>1 ( (F_3) )</td>
<td>3 ( (F_1, F_3, F_5) )</td>
<td>1 ( (F_3) )</td>
<td>0</td>
</tr>
</tbody>
</table>

If also \( W_1 = 0 \) \( \rightarrow \) IIB: \( d \Omega = W_5 \mathcal{A} \Omega \) (true in all susy vacua)

M\(_6\) is complex

M\(_6\) is “twisted symplectic”

In IIB \( W_2 = 0 \) (integrability of complex structure)

In IIA \( W_3 \rightarrow H^6 \) (symplectic geometry)

Is there a mathematical construction that extends complex and symplectic geometry?
**Generalized complex geometry**

- Usual differential geometry \( \rightarrow \) tangent bundle sections vector fields \( X \)
- Want differential geometry on \( T \otimes T^* \) sections are \( X^+ \), \( \zeta \)

Natural metric \( I \) on \( T \otimes T^* : (X^+ \zeta, X^+ \zeta) = i_X \zeta \)

\[
I = \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix}
\]

On \( T \otimes T^* \) define **Generalized Almost Complex Structure (GACS)** \( J : T \otimes T^* \rightarrow T \otimes T^* \)

\[ J^2 = -1_{2d} \]
\[ J^\dagger \mid J = I \]

### GACS

<table>
<thead>
<tr>
<th>Projectors</th>
<th>GACS</th>
<th>T</th>
<th>T ( \otimes ) ( T^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi_$ = (1_d $ i J) )</td>
<td>J s.t. ( J^2 = -1_d )</td>
<td>( J ) s.t. ( J^2 = -1_{2d} )</td>
<td>( \Pi_$ = (1_{2d} $ i J) )</td>
</tr>
<tr>
<td>( \pi_$ [X, \pi_$ Y] = 0 )</td>
<td>( \Pi_$ [\pi_$ (X+\zeta), \pi_$ (Y+\zeta)] = 0 )</td>
<td>( \Pi_$ [\pi_$ (X+\zeta), \pi_$ (Y+\zeta)] = 0 )</td>
<td></td>
</tr>
</tbody>
</table>

Aside: GCG naturally incorporates B-field:

\[
e^B (X^+ \zeta) = X^+ \zeta + i_X B \quad e^B (X^+ \zeta), e^B (Y+\eta)]_C = e^B [X^+\zeta, Y+\eta]_C \quad \text{automorphisms of} \ [\ , \]_C
\]

How is complex geometry embedded in generalized complex geometry?
A GACS has the form

\[
\begin{pmatrix}
J & P \\
L & K
\end{pmatrix}
\]

Demanding
\[
J^2 = -1_{2d}
\]
\[
J^* J = I
\]

Get conditions
\[
K = -J^t
\]
\[
P^t = -P
\]
\[
L^t = -L
\]

It is easy to guess how ACS _ GACS :

\[
J_1 = \begin{pmatrix} J & 0 \\ 0 & -J^t \end{pmatrix}
\]

Integrability of GACS \( J_1 \) is integrable (it is a CS in \( T \)) \( M \) is complex

But GACS have more...

Consider
\[
J_2 = \begin{pmatrix} 0 & -w^{-1} \\ w & 0 \end{pmatrix}
\]

Integrability of \( J_2 \) \( dw = 0 \). If \( w \) is non-degenerate \( M \) is symplectic

Complex manifolds
Symplectic manifolds
\[\Rightarrow\] admit GCS

Complex: locally equivalent to \( \mathbb{C}^{d/2} \)

Symplectic: locally equivalent to \( (\mathbb{R}^d, w) \); \( w = dx \wedge dx + \cdots + dx \wedge dx \)

Generalized complex: locally equivalent to \( \mathbb{C}^k \_ (\mathbb{R}^{d-2k}, w) \) \( k \): rank. \( k=0 \) for symplectic \( k=d/2 \) for complex
**GACS ↔ Pure spinors of Clifford (d,d)**

<table>
<thead>
<tr>
<th></th>
<th>T</th>
<th>T ⊗ T*</th>
</tr>
</thead>
<tbody>
<tr>
<td>Algebra</td>
<td>Clifford 6</td>
<td>Clifford (6,6)</td>
</tr>
<tr>
<td>Representation</td>
<td>({\gamma^m, \gamma^n} = g^{mn})</td>
<td>({\gamma^m, \gamma^n} = 0, {\gamma^m, \gamma_n} = \delta^m_n)</td>
</tr>
<tr>
<td>in terms of forms</td>
<td>(\Omega)</td>
<td>(\Omega)</td>
</tr>
<tr>
<td>Clifford vacuum</td>
<td>(\gamma^i = dz^i \land \gamma = g_{-} i \tau_{ij})</td>
<td>(\gamma^m = dx^m \land \gamma_m = \tau_{am})</td>
</tr>
<tr>
<td>Pure spinor</td>
<td>(\gamma^i \Omega = 0)</td>
<td>(\gamma^i \Omega = \gamma_\Omega = 0)</td>
</tr>
<tr>
<td>Basis</td>
<td>(\Omega: (3,0))</td>
<td>(\Omega: (3,0))</td>
</tr>
<tr>
<td></td>
<td>(\gamma^- \Omega: (2,0))</td>
<td>(\gamma^- \Omega: (3,1))</td>
</tr>
<tr>
<td></td>
<td>(\gamma^+ \Omega: (1,0))</td>
<td>(\gamma^+ \Omega: (3,2))</td>
</tr>
<tr>
<td></td>
<td>(\gamma^{jk} \Omega: (0,0))</td>
<td>(\gamma^{jk} \Omega: (3,3))</td>
</tr>
<tr>
<td>Spinors $p,0$ forms</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Spinors $p,q$ forms</td>
<td></td>
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</tbody>
</table>

- On a manifold with SU(3) structure in T, 2 pure spinors of Clifford(6,6)
- 1-1 correspondence between pure spinors and GACS J

\[
e^{iw} = 1 + iw - \frac{1}{2}w^2 - i\frac{1}{3!}w^3
\]

\((\gamma_m + i W_{mn} \gamma^n)e^{iw}=0\)

\[
\Omega; \ e^{iw}
\]

Integrability of J, 9 v and \(\xi\) s.t

\[
d\varphi = (\nu + \xi \land) \varphi
\]

Generalized Calabi-Yau → A manifold that has one closed pure spinor \((d\varphi = 0)\)
Supersymmetry equations written in terms of pure spinors

\[ \delta \psi_m : \begin{align*}
IIA & : D_m^{I\!\!A} \eta_+ + e^\phi (\bar{\Omega} F^{I\!\!A})_m \eta_+ + e^\phi \left[ (F^{I\!\!A} e^{i\omega})_n + (F^{I\!\!A} e^{i\omega})_0 g_{mn} + (F^{I\!\!A} e^{i\omega})_{mn} \right] \gamma^p \eta_- = 0 \\
IIB & : D_m^{I\!\!B} \eta_+ + e^\phi (e^{-i\omega} F^{I\!\!B})_m \eta_+ + e^\phi \left[ (F^{I\!\!B} \Omega)_n + (F^{I\!\!B} \Omega)_0 g_{mn} + (F^{I\!\!B} \Omega)_{mn} \right] \gamma^p \eta_- = 0
\end{align*} \]

NSNS sector: \( D_m^H \eta \equiv \left( \nabla_m + \frac{1}{8} H_{mnp} \gamma_{np} \right) \eta \quad \text{We are not in CY} \quad 0 \neq \nabla_m \eta \sim W \eta \)

\[ D_m^H \eta_+ = (W_4 + W_5 + iH^{(3)})_m \eta_+ + \left[ (W_1 + iH^{(1)}) g_{mn} + (W_3 + iH^{(6)} + W_2)_{mn} \right] \gamma^p \eta_- \]

RR fluxes:

\[ F^{I\!\!A} = F_0 + F_2 + F_4 \quad F^{I\!\!B} = F_1 + F_3 + F_5 \]

\[ (F \Omega)_{mn} \equiv (F \not\omega)_{mn} = \left( F_{i_1 \ldots i_k} \Omega_{abc} \gamma^{i_1 \ldots i_k} \gamma^{abc} \right)_{mn} \]

IIA $ \Leftrightarrow $ IIB

\[
F^A \underbrace{\Leftrightarrow}_{\text{Take coefficient of term with 2 $\gamma$'s}} F^B \\
\text{Ex: } (F_1 \Omega)_{mn} = \frac{1}{2} F^a \Omega_{amn}
\]

exchange of two pure spinors -- action of mirror symmetry
Susy vacua are all twisted generalized Calabi-Yau's!

**IIA**
\[ c^{-\beta} d(c^{\beta} e^{i\omega}) = H \cdot c^{i\omega} \]
\[ e^{-\gamma} d(e^{\gamma} \Omega) = H \cdot \Omega + F_A e^{i\omega} \]

**IIB**
\[ c^{-\beta} d(c^{\beta} e^{i\omega}) = H \cdot c^{i\omega} + F_B e^{i\omega} \]
\[ e^{-\gamma} d(e^{\gamma} \Omega) = H \cdot \Omega \]

\[ e^{\phi} e^{i\omega} \] is "twisted" closed  \[ \rightarrow \]  \[ e^{\gamma} \Omega \] is "twisted" closed

\[ M \] is twisted symplectic  \[ \rightarrow \]  \[ \Omega \] is complex

**Susy vacua are all twisted generalized Calabi-Yau's!**

**Caveat:** Hitchin considered twisting by \[ H \]

Hitchin's twisting  \[ [d + H \wedge] \] \[ \rightarrow \]  \[ \text{Associated GACS integrable:} \]

\[ [X + \zeta, Y + \eta] \]

\[ \Psi \] \[ \rightarrow \]

Twisting by \( H \) in supergravity equations does not work à la Hitchin

Strominger '86
(Maldacena - Nuñez)
Conclusions / Open questions

• Type II supersymmetric vacua are twisted generalized Calabi-Yau’s

• What is the meaning of supergravity twisting by H?

• How to compactify on these manifolds? (evade no-go theorems)

• Moduli spaces?

Generalized complex geometry is a nice tool for a systematic description of flux backgrounds. But maybe strings see SO(d,d) structure of $T \otimes T^*$ and we will learn more…