# a-maximization and RG flows

or

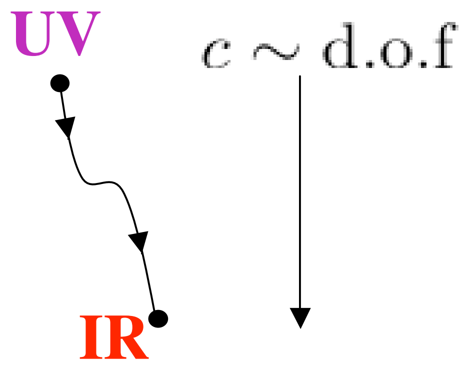
# Towards a c-theorem in Four Dimensions

Based on works with Brian Wecht,

and, to appear, with Barnes, Wecht, and Wright.

# RG Flows and the c-theorem

General intuition, made precise in **2d** by Zamolodchikov:




• Quantity, c, counts # massless d.o.f. of CFT.
Endpoints of all RG flows should satisfy:

$$c_{UV} > c_{IR}$$

• Stronger: can extend to monotonically decreasing c-function,
$\dot{c}(g(t)) < 0$ along entire RG flow to IR. $(t = -\log \mu)$

• Strongest (Zamolodchikov proved in **2d**): RG flows are **gradients**:

$$\dot{g}^I(t) \equiv -\beta^I(g) = -G^{IJ}\frac{\partial c(g)}{\partial g^J} \longrightarrow \dot{c}(g) = -G^{IJ}\frac{\partial c}{\partial g^I}\frac{\partial c}{\partial g^J} < 0$$

positive definite!

# What about in four dimensions?

Cardy conjecture for quantity that counts # massless d.o.f. of 4d **CFT**s: coefficient ***a:***

$$\langle T^{\mu}_{\mu} \rangle = c(\text{Weyl})^2 + a(\text{Euler}).$$

no! yes?

4d "a-theorem" **conjecture**: **Endpoints** of all 4d RG flows satisfy

$a_{UV} > a_{IR}$ and also $a_{CFT} > 0$ **True in every known example!!!** Powerful result **if** really generally true. Proof?

Some promising attempts but, unlike 2d, no general and generally accepted, complete proof as of yet.

Perhaps stronger claims also true? 4d RG = gradient flow with positive definite metric? Investigated by Osborn and collaborators.

# N=1 susy RG flows: exact results via R-symmetry

$$R(Q_i) = \frac{2}{3}\Delta(Q_i) = \frac{2}{3} + \frac{1}{3}\gamma_i(g(t))$$

**running R charges**

**Exact** beta functions proportional to R-charge violation:

$$\beta_{NSVZ}(g^{-2}) \sim T_2(G) + \sum_i T_2(r_i)(R_i - 1)$$

$U(1)_R$ anomaly

$$\beta(h) = \frac{3}{2}h(R(W) - 2)$$

h=superpotential coupling, beta fn related to R-charge violation of W.

RG flow to SCFT fixed point: $U(1)_R \to U(1)_{R_*}$

**conserved in IR SCFT**

Knowing $U(1)_{R_*}$ exactly determines $\Delta_*(Q_i)|_{SCFT}$!

## Finding the superconformal $U(1)_{R_*}$ general SCFTs:

**Problem**: R can mix with flavor symmetries, $R = R_0 + \sum_I s_I F_I$

**which** is the superconformal one $U(1)_{R_*} \subset SU(2,2|1)$?

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**Solution** (KI, B. Wecht '03) **: to uniquely determine** $U(1)_{R_*}$

**locally maximize:** $a_{trial}(R) = 3\text{Tr } \mathrm{U}(1)_{\mathrm{R}}^3 - \text{Tr } \mathrm{U}(1)_{\mathrm{R}}$

**over all conserved, possible R-symmetries.**

't Hooft anomalies,
**exactly computable!**

(We proved this, using susy + CFT unitarity)

Value of $a_{trial}(R)$ at local maximum is the conformal anomaly

$a_{SCFT}$ appearing in Cardy's conjecture!

Anselmi, Freedman, Erlich, Grisaru, Johansen

**"a-maximization"**

## Aside on connecting to AdS/CFT, won't be otherwise used here:

$$AdS_5 \times H_5: \qquad a = c \sim \frac{N^2}{vol(H_5)}$$

Henningson, Skenderis
Gubser

very restricted subset of N=1 SCFTs!

"a-maximization ~ volume minimization".

Superconformal R-symmetry = U(1) isometry of $H_5$

Non-R flavor symmetries = $C_4$ 3-cycles of $H_5$

Can clearly distinguish, superconformal R-symm not ambiguous.

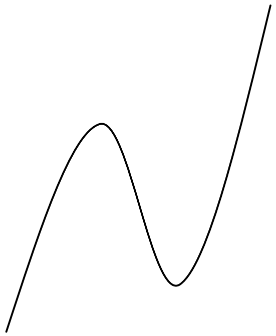
Mathematics of connection to $a_{trial} = 3Tr\ R^3 - TrR$ looks nontrivial. K.I., B Wecht ; Chris Herzog and co. '03

Monotonically decreasing a-function iff weak-energy condition satisfied in bulk. Freedman, Gubser, Pilch, Warner; Anselmi Giradello Porrati, Zaffaroni

**Quick example:** Consider a free chiral superfield $\Phi$

$$a = 3(r-1)^3 - (r-1)$$

This has extrema at $r=2/3$ and $r=4/3$.



**max!**

**min!**

**correct value for free field.**

**And it's basically just as easy for interacting theories!**

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**General observation:** Since we're maximizing a cubic function, R-charges, chiral primary operator dimensions, and central charges must **always** be quadratic irrationals:

$$\frac{n+\sqrt{m}}{p}$$

**Quantized, so cannot depend on any continuous moduli!**

# $a$-maximization almost proves the $a$-theorem!

Since relevant deformations generally break the flavor symmetries,

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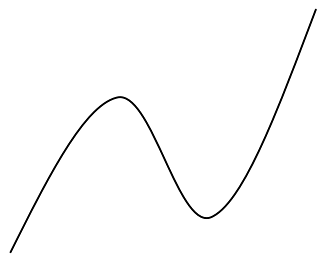

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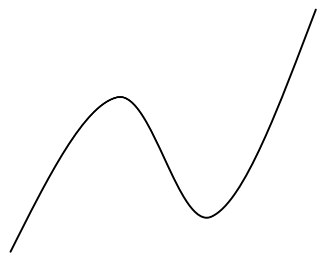

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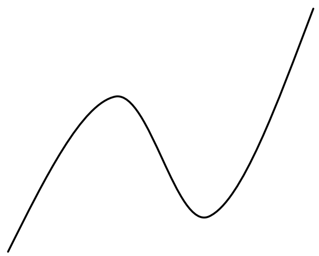


Trying to close these loopholes. a-thm checks also in examples where accidental symmetries are crucial (Kutasov, Parnachev, Sahakyan).

## A big zoo of examples: SCFTs obtainable from SQCD with fundamentals + adjoints

(KI, B. Wecht)

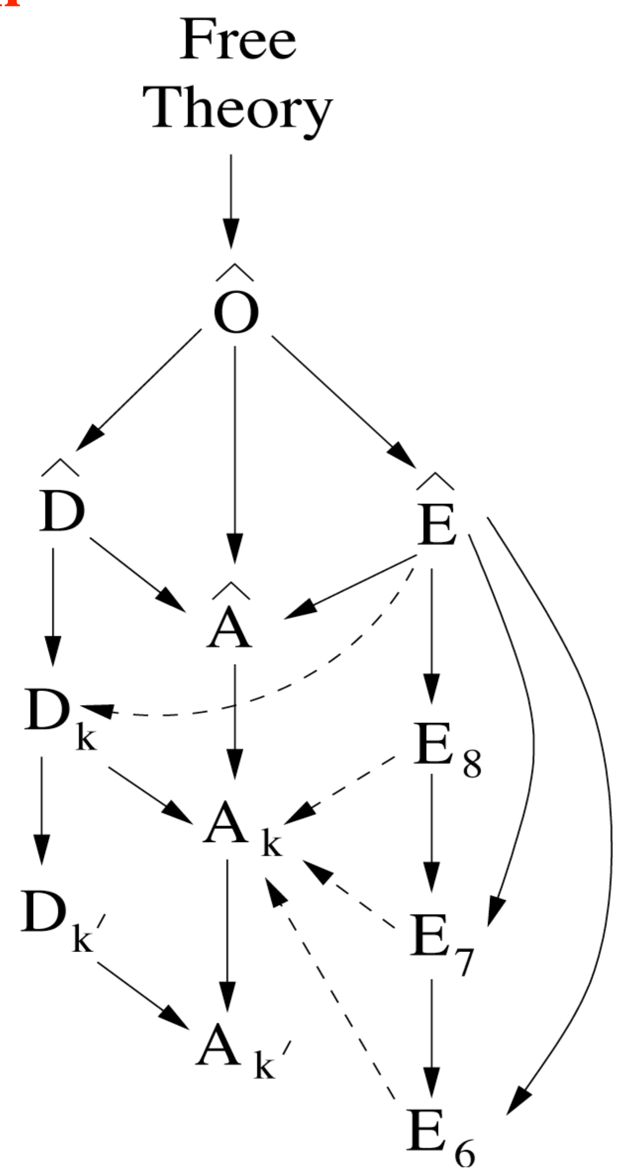
Determine operator dimensions, and classify which superpotentials are relevant, and when. Find it coincides with **Arnold's ADE** classification!

$$W_{\widehat{O}} = 0 \qquad W_{\widehat{A}} = TrY^2$$

$$W_{\widehat{D}} = TrXY^2 \quad W_{\widehat{E}} = TrY^3$$

$$W_{E_8} = Tr(X^3 + Y^5)$$ etc.

**Lots of new SCFTs!**




**All flows indeed compatible with the *a*-conjecture,**

$$\boxed{a_{UV} > a_{IR}}$$

# Extend $a_{UV} > a_{IR}$ to a monotonically decreasing a-function along entire RG flow?

**Kutasov:** Recall our argument that $a_{IR} < a_{UV}$ because $\mathcal{F}_{IR} \subset \mathcal{F}_{UV}$.

Implement $\mathcal{F}_{IR} \subset \mathcal{F}_{UV}$ IR constraint on R-charges via **Lagrange multipliers**, interpreted as **running couplings** along RG flow to IR.

a-maximization, holding Lagrange multipliers fixed gives candidate

**a-function along RG flow:** $a(\lambda) = a(R,\lambda)|_{R(\lambda)}$.

**extremizing solution.**

**E.g.**

$$a = 2|G| + \sum_i |r_i|[3(R_i-1)^3 - (R_i-1)] - \lambda(T(G) + \sum_i T(r_i)(R_i-1))$$

**This Lagrange multiplier enforces anomaly freedom in the IR.**

Extremizing w.r.t. the R-charges, yields $R_i(\lambda) = 1 - \frac{1}{3}\sqrt{1 + \frac{\lambda T(r_i)}{|r_i|}}$

Interpret as R-charges along flow from UV to IR: $\lambda = 0 \rightarrow \lambda_*$

Since $\quad R_i = \frac{2}{3} + \frac{1}{3}\gamma_i \quad$ this gives $\quad \gamma_i(\lambda) = 1 - \sqrt{1 + \frac{\lambda T(r_i)}{|r_i|}}$

Expanding for small $\lambda$ **agrees with perturbation thy**. provided that

$\lambda = \frac{g^2}{2\pi^2}|G| + \dots \quad$ Interpret $\lambda$ as coupling (in some scheme).

Computing $\quad a(\lambda) = a(R, \lambda)|_{R(\lambda)}$ :

$\lambda$

$$a(\lambda) = 2|G| - \lambda T(G) + \frac{2}{9}\sum_i |r_i| \left(1 + \frac{\lambda T(r_i)}{|r_i|}\right)^{3/2}.$$

$a(\lambda)$

Interpolating a-function. **Monotonically decreasing** along RG flow:

$$\frac{da}{d\lambda} = -[T(G) + \sum_i T(r_i)(R_i - 1)] \sim \beta_{NSVZ}(g) < 0.$$

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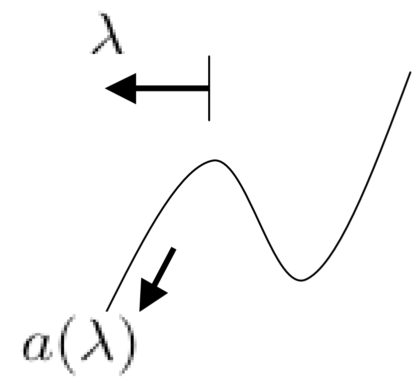


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Let's extend this approach, writing generally for G or W interactions

Barnes, KI
Wecht, Wright

$$a(R_i, \lambda_I) = 3Tr\ R^3 - Tr\ R + \sum_J \lambda_J \widehat{\beta}_J(R)$$

$$\widehat{\beta}_G(R) = -(T(G) + \sum_i T(r_I)(R_i - 1)) \sim \beta_{NSVZ}(g^{-2})$$

$$\widehat{\beta}_W = R(W) - 2 = \frac{2}{3h}\beta(h)$$

IR constraints on R-charges = proportional to exact beta functions.

Extremizing over the R-charges, holding Lagrange multipliers fixed, gives **running R-charges** $R_i(\lambda)$ and thus anomalous dimensions. (Always, non-trivially, recover the leading perturbative expressions.) Plugging $R_i(\lambda)$ into above expression gives **a-function** such that

$$\frac{\partial a(\lambda)}{\partial \lambda_I} = \widehat{\beta}_I$$

**Suggests gradient flow, as in 2d!**

Compute 4d analog of Zamolodchikov metric for our **a-function**:

$$\frac{\partial a}{\partial g} = G_{gg}\beta_{NSVZ}(g) \qquad \text{and} \qquad \frac{\partial a}{\partial h} = G_{hh}\beta_{W}(h)$$

(Work to leading order, so ignore cross terms.)

Use leading order dictionary between the $\lambda$ and the couplings, foung by comparing $R_i(\lambda)$ with perturbative anomalous dims.

**We find precise numerical agreement, with leading, perturbative metrics found by Freedman and Osborn!**

(Computed from $\beta_{\mu\nu} = G_{IJ}\partial_\mu g^I \partial_\nu g^J$ king couplings spatially dependent.)

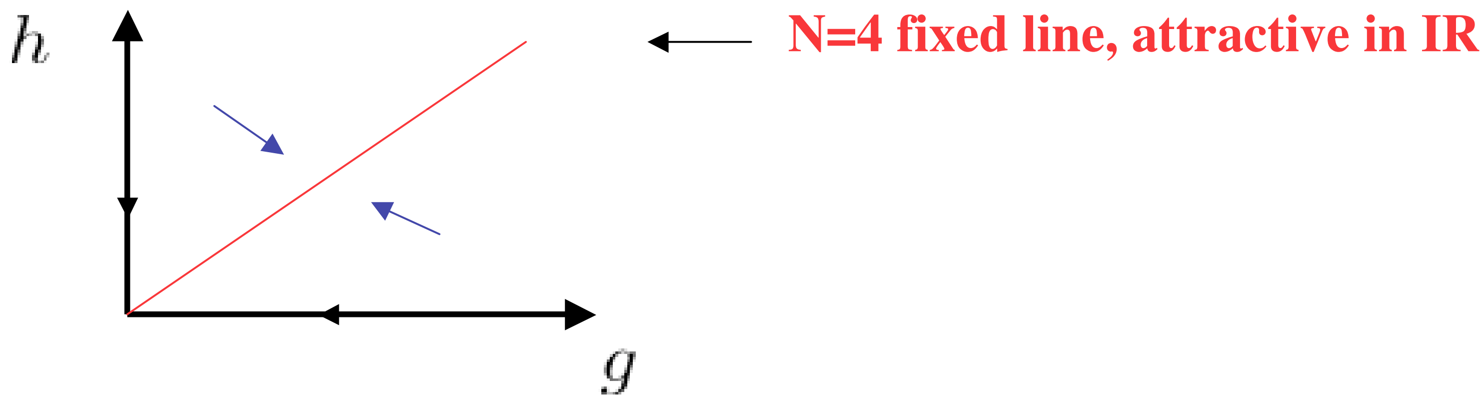
**gauge**

$$G_{gg} = \frac{4|G|}{g^2} + O(g^0), \qquad G_{hh} = \frac{1}{24\pi^2} + O(h^2)$$

**Yukawa**

**Looks promising for this a-function, and stronger a-theorem claim!**

Quick example: N=1 with 3 adjoints and $W = h\mathrm{Tr}\left(\Phi_1[\Phi_2, \Phi_3]\right)$




$$a = 2|G| + 3|G|(3(R-1)^3 - (R-1)) - \lambda_G(T(G) + 3T(G)(R-1)) + \lambda_W(3R-2)$$

extremize

$$R = 1 - \frac{1}{3}\sqrt{1 + \frac{\lambda_G T(G)}{|G|} - \frac{\lambda_W}{|G|}}$$

Running R charges.

Flows attracted to N=4 fixed line: $\lambda_G T(G) = \lambda_W$

$a(\lambda_G, \lambda_W)$ decreases along flows to fixed line. At weak coupling, can obtain

$\lambda_G(g,h)$ and $\lambda_W(g,h)$ dictionary explicitly. Map flows beyond weak coup.?

Would like to better understand $\lambda_G(g,h)$ nd $\lambda_W(g,h)$ tions.

E.g. for magnetic dual of SQCD find $$R_M = 1 \pm \frac{1}{3}\sqrt{1 - \frac{\lambda_W}{N_f^2}}$$

→ $\lambda_W = 0$ for both free theory, R=2/3, and interacting case R=4/3.

$\boxed{a(R_M)}$

$\lambda_W,\ h^2$

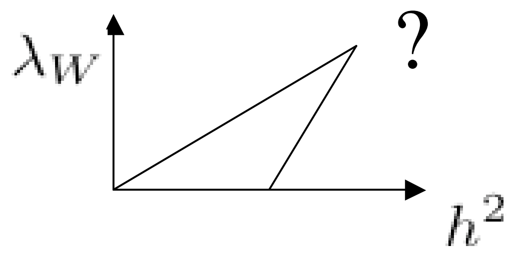
**free** $\lambda_W,\ h^2 = 0$ ⟶

$N_f = \frac{3}{2}N_c$

⟵ $\lambda_W = 0,\ h^2 \neq 0$

**interacting**

$N_f = 3N_c$




$$G_{hh} = \frac{d\lambda_W(h^2)}{d(h^2)}$$ ← positive definite?

**(over simplifying a bit, G a matrix)**

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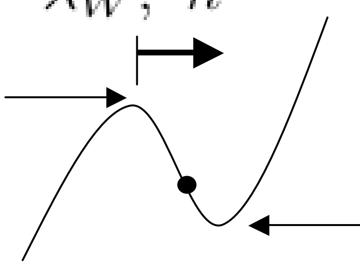

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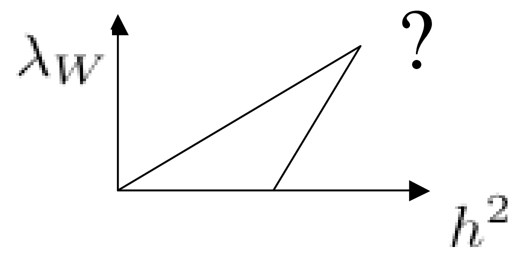


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## Conclude/Summary:

Use **a-maximization** to fix the superconformal R-symmetries.
Get new, exact results for previously mysterious SCFTs.

Find new SCFTs and explore RG flows among them.

The a-theorem is almost proved for SCFTs. Loopholes closing.

Can obtain new, general results about 4d SCFTs, e.g.
R and central charges always of general form: $\frac{n+\sqrt{m}}{p}$

Evidence for stronger 4d analogs of 2d c-theorem: monotonically decreasing a-function, and gradient RG flow. Interesting agreement with earlier perturbative computations of metric. But no proof yet that metric is always postive definite.