CHASING

m/f - THEORY

NIKITA NEKRASOV

IHES

related work:

- Gerasimov - Shatashvili
- Dijkgraaf, Gukov, Neitzke, Vafa

(M)athematical M-theory

Challenges:

* Topological string/gauge theory duality

* S-duality in topological strings

* Holomorphic anomaly

* Sum up string perturbation theory & Quantum foam

Type II superstring compactified on CY manifold (also het dual)

Dixon, Kaplunovsky, Louis

Antoniadis, Gava, Narain

Bershadsky, Cecotti, Ooguri, Vafa + Taylor

Harvey, Moore

Becker, Becker, Strominger

Strominger

Antoniadis, Pioline, Taylor,

Antoniadis, Minasian, Theisen, Vanhove

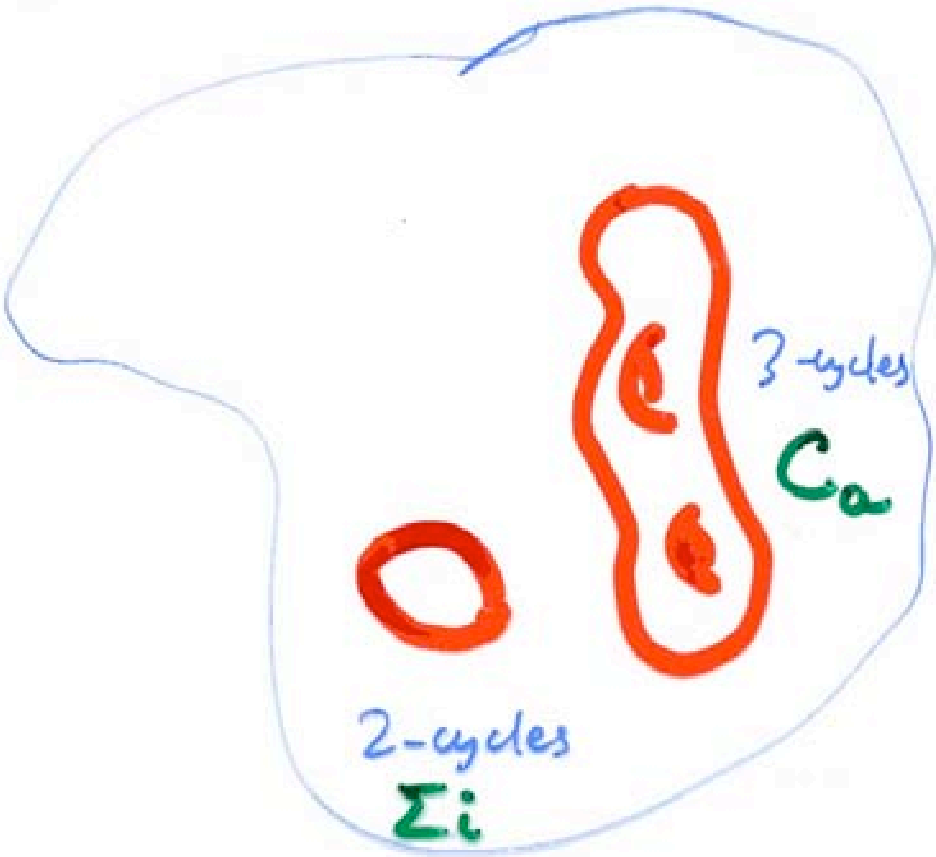
...

Two kinds of moduli:

$$\mathcal{M}_V \times \mathcal{M}_H$$

vector multiplets

hypermultiplets



Topological STRING (IN A MODERN SENSE)

should be thought of a packaging procedure which encodes the exact geometry of $\mathcal{M}_V \times \mathcal{M}_H$

(and its quantization)

IIA perspective

Vector multiplets

$h^{1,1}(X)$

Kähler form

$$\phi^i \longleftrightarrow \int_{\Sigma_i} (i J + B^{NS})$$

$$A^i \longleftrightarrow \int_{\Sigma_i} C^{(3)}_{RR}$$

in addition, $C^{(1)}_{RR}$ goes to the gravity multiplet (graviphoton)

Hypermultiplets (IIA)

$$h^{2,1}(X) + 1 \equiv \dim_{\mathbb{C}} H^3(X, \mathbb{R})$$

universal multiplet
$(\phi,\ a$ – dual of B^{NS}, $\int_X C^{(3)}_{RR} \wedge \Omega)$

(3,0) holomorphic form

$\left(\begin{array}{l}\text{complex}\\ \text{structure}\\ \text{moduli}\end{array},\ \int_{C_a} C^{(3)}_{RR} \bmod \ldots\right) \sim \int_X C^{(3)}_{RR} \wedge \frac{\partial \Omega}{\partial t^a}$

$\Omega^{(2,1)}$ – form

$t^a \sim \int_{A_a} \Omega$ mod rescaling

$C_a = (A_a, B^a)$

symplectic basis of 3-cycles

$\mathcal{M}_V$ is special Kähler

$\mathcal{M}_H$ is quaternionic-Kähler

(bad name
as it needs
not be
Kähler
$Sp(n)Sp(1)$
$\not\subset U(2n)$)

Geometry of vector multiplets

is ENCODED in the

prepotential

$$\mathcal{F}_0(t) = \frac{1}{6}\int \tilde{k}\wedge\tilde{k}\wedge\tilde{k} + \text{w.s. instantons}$$

$$\tilde{k} = k + iB$$

$$t^a = \int_{\Sigma_a} iK + B$$

(More invariantly

t^a

$$t^D_a = \int_{\Sigma^\vee_a} \frac{(iK+B)^2}{2} + \dots)$$

W.S. INSTANTONS = genus zero Gromov-Witten invariants

Higher genus Gromov-Witten invariants :

$$\mathcal{F}_g(t) = \sum_{\beta \in H_2(X,\mathbb{Z})} e^{-\int_\beta \tilde{k}} \times$$

$$\times\, n_{\beta,g}$$

$\uparrow$

# of holomorphic maps of genus g curves to X which land in β

Give rise to terms in

4d effective action:

$$\int \mathcal{F}_g(t)\, W^{2g}\, d^4\theta$$

$i,j = 1,2$

$$W^{ij}_{\mu\nu} = T^{ij}_{\mu\nu} - R_{\mu\nu\lambda\rho}\,\theta^i \sigma^{\lambda\rho} \theta^j + \ldots$$

graviphoton field strength

Weyl multiplet

$$t^a = (\text{scalar})^a + \tfrac{1}{2} \hat{F}^a_{\lambda\rho}\, \theta^i \sigma^{\lambda\rho} \theta^j \varepsilon_{ij} + \ldots$$

vector field field strength

Mirror picture (IIB)

$\mathcal{F}_0(t)$

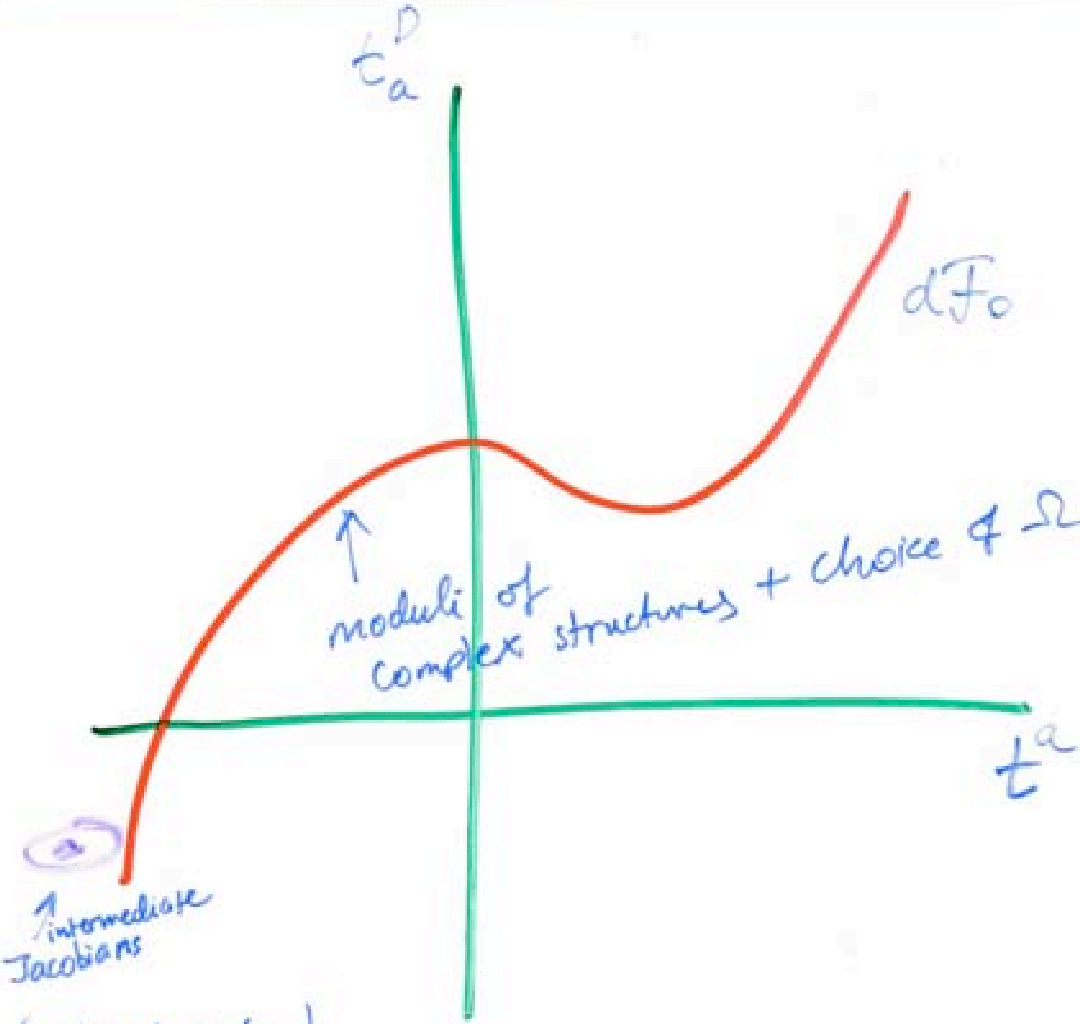
classical computation, no
w.s. corrections

complex moduli

$$t^a = \int_{A_a} \Omega$$

up to
rescaling

$$\frac{\partial \mathcal{F}_0}{\partial t^a} = \int_{B_a} \Omega$$



(Hitchin's System) - Donagi

Lagrangian submanifold

$$Z_B^{pert}(t, g_B) = e^{\sum_g g_B^{2g-2} F_g(t)}$$

$g_B^2 \sim \hbar$ ~ quantization

holomorphic anomaly & BCOV + Witten's quantization

How to calculate F_g's ?

BCOV
Kodaira-Spencer Theory of GRAVITY

(B model)

$$\frac{1}{2}\int A \frac{1}{\partial}\bar{\partial} A + \frac{1}{6}\int_X (A \wedge A)' \wedge A$$

$A = x + A_0(x)$ → orthogonal

↳ harmonic representative

D-branes in topological strings

↓

D-INSTANTON EFFECTS

A-model ⟶ Lagrangian branes (3-cycles)

invented by E.W.

B-model → sheaves, supported on complex submanifolds (pts, curves, divisors, X)

invented by M.K

These non-perturbative effects change hypermultiplet moduli space metric (there are also loop effects, but...)

Yet we claim that there is some intimate relation between these and closed topological string amplitudes, which are related to $\mathcal{M}_V$

TO COMPARE

$$\tilde{\mu}_V \text{ vs. } \mu_H$$

NEED TO TAKE

CARE OF JACOBIAN(X)

ETC.

This relation becomes clear if we embed topological strings into TypeIIB superstrings and employ

S-duality

Then A model worldsheet instanton sum would turn into the D1-instanton sum, which is B model object

By relating physical string couplings to those of topological string we arrive at the dictionary:

$$g_A = \frac{1}{g_B}$$

$$k_A + iB_{NS} = \frac{k_B}{g_B} + i B_{RR}$$

Consider now

A D1 INSTANTON WRAPPING

HOLOMORPHIC CURVE

$$\Sigma \subset X$$

AND N D(-1) INSTANTONS

LOCATED AT SOME POINTS ON X

INSTANTON FACTOR

$$e^{-\int_{\Sigma}\left(\frac{K_B}{g_B}+iB_{RR}\right) - \frac{N}{g_B}} = \exp\left(-\int_{\Sigma}(k_A+iB_{NS}) - Ng_A\right)$$

IT TURNS OUT THAT THE SUM OVER

D(-1)'S WITH THE

WEIGHT $e^{-N g_A}$

REPRODUCES MULTI-COVERING + MAPS TO POINTS ON THE **A** SIDE

THE SIMPLEST EXAMPLE:

GW THEORY OF

$\mathbb{C}^3 = \mathbb{R}^6$

also equiv. version

$$\exp\left(2\cdot\sum_{g=0}^{\infty} g_A^{2g-2} \int_{\overline{\mathcal{M}}_g} c_{g-1}^3(\mathbb{H})\right) \quad \times \frac{(\varepsilon_1+\varepsilon_2)(\varepsilon_1+\varepsilon_3)(\varepsilon_2+\varepsilon_3)}{\varepsilon_1\varepsilon_2\varepsilon_3}$$

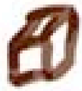
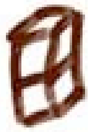
$$= M(q) = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n}$$

$q = -e^{i g_A}$

QUESTION TO E.W.

WHAT IS

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n} \quad ?$$

$\sum_{\text{3d partitions}} q^{\#\text{boxes}} = 1 +$  $+$  $+ \dots$

$q \qquad 3q^2$

$=$

PARTITION FUNCTION OF DONALDSON-THOMAS THEORY

$=$

$$\sum q^{ch_3} \int_{\text{moduli space of ideal sheaves}} 1$$

Ideal sheaves $\rightsquigarrow$ singular U(1) INSTANTONS

solutions to

F-hint

$$\begin{cases} F_A^{0,2} = \bar{\partial}_A^{+} \bar{\varphi} \\ F_A^{2,0} = \partial_A \varphi \\ k^2 \wedge F_A^{1,1} = [\varphi, \bar{\varphi}] \end{cases}$$

THESE EQUATIONS WANT 2 EXTRA DIMENSIONS

BPS equations of D5 brane theory

" D1 bound states "
D(-1)

Losev, Nekrasov, Marshakov
Iqbal, Okounkov, Vafa, NN;
NN, Ooguri, Vafa; Maulik, Okounkov, NN, Pandharipande
Kapustin

From the 6d gauge theory one can derive instanton measure and localize it using toric symmetries

$$\sum_{\text{3d partitions}} q^{\#\text{boxes}} \mu_\pi(\varepsilon_1, \varepsilon_2, \varepsilon_3) =$$

$$= M(q)^{-\frac{(\varepsilon_1+\varepsilon_2)(\varepsilon_1+\varepsilon_3)(\varepsilon_2+\varepsilon_3)}{\varepsilon_1\varepsilon_2\varepsilon_3}}$$

equivariant vertex measure (rational function of ε)

proven by Maulik, Okounkov, Pandaripande NN

M/F-hint

THIS INSTANTON MEASURE HAS NATURAL 7d (8d) GENERALIZATION

$$q_\alpha = e^{\beta \varepsilon_\alpha} \quad \alpha = 1, 2, 3$$

$$q_4 = (q_1 q_2 q_3)^{-1} \qquad q_\pm = q q_4^{\mp \frac{1}{2}}$$

$$\sum_{\pi - \text{3d Young diagrams}} q^{\#\text{boxes}} \mu_\pi(q_1, q_2, q_3) =$$

$$= \prod_{\alpha=1}^{4} \prod_{a,b=1}^{\infty} \left(\frac{1 - q_-^{a} q_\alpha^{b}}{1 - q_+^{a} q_\alpha^{b}} \right)$$

EXP. FACT

CONJECTURALLY THIS COUNTS K-THEORETIC (OR TRULY MEMBRANE) GROMOV-WITTEN INVARIANTS OF $\mathbb{C}^3$

Givental
Coates

Cumrun talked about extensions of this duality (in the toric case) to the configurations involving

D(1) - INSTANTONS

AND ABOUT QUANTUM SPACE-TIME FOAM INTERPRETATION

$$Z_A \sim \sum_{\text{"geometries"}} e^{-\frac{1}{g_A^2}\text{volume}}$$

NOW SUPPOSE WE

COMBINE PERTURBATIVE

AND NON-PERTURBATIVE

EFFECTS OF A or B

TYPE TOPOLOGICAL

STRINGS

TAKE **B** MODEL, FOR EXAMPLE:

$$Z^{B}_{full} = Z\big(\text{complex moduli}, \; g_B \, ; \, k_8\big)$$

$$\underbrace{\text{perturbative STRING AMPLITUDES}}_{\substack{\wr\wr \\ H^3(X)}}$$

D(-1) $\to g_B$

D1-INST. $\to k_8$

$\wr\wr$

$H^2(X)$

A MODEL:

$$Z^{A}_{full} = Z\big(\text{kähler moduli}, \; g_A \, ; \, C^{(3)}\big)$$

$H^2(X)$

D2-inst $\to C^{(3)}$

$H^3(X)$

Strominger's theorem:

if 2 functions are defined
on the same space
and both come from
string theory $\Rightarrow$ they are equal

We claim something similar:

$$g_A = 1/g_B \text{ , etc.}$$

$$\text{but } Z^A_{full} = Z^B_{full}$$

(Also, should be more precise about D-brane charges $\rightarrow$ only half is needed)

WHAT WE HAD IN 6D:

PERT. B MODEL: Kodaira-Spencer

NON-PERT. COMPLETION: Kähler (Q. foam)

$\Downarrow$

FULL CALABI-YAU METRIC

TAKE

$$CY \times S^1 = X^7$$

$$\downarrow$$

"G_2" – manifold

$$k \wedge d\varphi + \mathrm{Re}\,\Omega = \Phi$$

$$\swarrow$$

closed 3-form,

INVARIANT UNDER

G_2 rotations

of a frame

"INVERSE CONSTRUCTION":

$$S = \int_{X^7} h^{ij}\, \Phi_i \wedge \Phi_j \wedge \Phi \; + \; \int_{X^7} \sqrt{h}$$

$$\Phi_i = \Phi_{ijk}\, dx^j \wedge dx^k$$

$$d\Phi = 0 \qquad \Phi = x + dB$$

minimize w.r.t. h_{ij}

$$\Rightarrow \int_{X^7} \left(\det_{ij} (\phi_i \phi_j \phi) \right)^{1/9}$$

Hitchin's functional,
whose critical points
give metrics of G_2-holonomy
(if non-degenerate)

Upon reduction on a circle

$$\Phi = K \wedge d\varphi + \Theta$$

$$\int_{Y^6} K^3 + \int_{Y^6} \mu^A \wedge \Theta_A \wedge \Theta + \int_{Y^6} \varepsilon((\mu,\mu) + 6)$$

Kähler gravity

$\mu^A = \mu^A{}_B dx^B$

$\mu^A{}_A = 0$

ε – volume form

(upon elimination of μ, ε)

Hitchin's functional for 3-forms in 6d, whose critical points are Calabi-Yau manifolds (perhaps not Kähler)

We also need an analogue of M-theory 3-form, to allow holomorphicity

$$S \to S + \frac{1}{2}\int C\,dC + \int \Phi\, dC$$

"Explains" holomorphic anomaly equation (Shatashvili-Gerasimov, unpublished)

Gukov's superpolen'!

THERE IS ALSO
AN 8 DIMENSIONAL
HITCHIN'S
FUNCTIONAL .....

$$\int \tfrac{1}{2} C \wedge dC + \phi\, dC + \ldots$$

should also explain

terms like

$$\int \frac{\hat{\Omega}_B \wedge dk_B}{g_B^2}$$

$$\int \frac{\hat{\Omega}_A \wedge dk_A}{g_A^2}$$

which are predicted

from geom. transitions etc.

Amusing predictions of S-duality:

$$\sum_{N=0}^{\infty} e^{-\mu N} \, CS^{T^*S^3}_{U(N),\ \frac{2\pi i}{k+N} = g_A - \text{fixed}}$$

$$\sim \exp\Big(- \frac{1}{g_B^2} \mu^2 \log \mu + \frac{1}{12} \log \mu + \sum_{h=2}^{\infty} \frac{B_{2h}}{2h(2h-2)} \mu^{2-2h} g_B^{2h-2} \Big)$$

$g_B \sim k + N \to 0$ critical level (Langlands)

Another challenge for

(M)athematical M-theory :

Explain equality :

GW on $\begin{array}{c} \widetilde{\mathbb{C}^2/\Gamma} \otimes \mathcal{O}(-1) \\ \downarrow \\ \mathbb{P}^1 \end{array}$ =

$$Z^{5d} = \Lambda^{c_2} \int_{\mathcal{M}_G} 1$$

K-theor. equiv. integral

moduli of G-instantons on $\mathbb{R}^4$

LATE BIRTHDAY PRESENT FOR SIR MICHAEL

$G \longleftrightarrow \Gamma$

Mackay

CONCLUSION —

— m/f - THEORY IS NOT FAR...

— need to understand tons of things

(membranes, "M5"

$e^{+i\int C + i\Phi}$ $\quad e^{\int *\Phi} \ldots$)