Symmetries and conservation laws in two dimensional string theory

1. Motivation

2. Symmetries and conserved charges in open string theory

3. Conserved charges in two dimensional string theory

5. Application: Holes and black holes

6. Lessons for critical string theory

Although we shall carry out our analysis in bosonic string theory, similar analysis can be done in type 0B superstring theory.

Two dimensional string theories have:

1. Continuum description based on world-sheet theory

2. Matrix model description

$\rightarrow$ free fermions moving under an inverted harmonic oscillator potential

$$h(q,p) = \frac{p^2}{2} - \frac{q^2}{2} + \frac{\mu}{g_s}$$

This provides a good laboratory for testing out various general ideas in string theory.

(*e.g.* open-closed string duality conjecture)

However even in two dimensional string theory several questions do not have fully satisfactory answers.

1. Free fermion description has both fermion and hole like states

Fermions $\rightarrow$ D0-branes in the continuum theory

What are the hole states?

2. The continuum description has black hole solutions.

What is the description of the black hole in the matrix model?

The free fermion description of the theory has infinite number of conserved charges.

$$e^{(k-l)x^0}\, h(q,p)\,(q-p)^k(q+p)^l \equiv e^{(k-l)x^0} W_{k,l}$$

A given state in the fermionic theory carries specific values of these charges.

Q. How do we compute these conserved charges for a configuration in the continuum description?

Comparison of the conserved charges in the matrix model and the continuum description

$$\downarrow$$

will help us identify a matrix model configuration in the continuum theory and vice versa

Conserved charges of open string theory from rigid gauge transformations in closed string theory

Example: Consider rigid space-time translation

(a special closed string gauge transformation that leaves flat space-time background invariant)

This generates global symmetries of open string theory

Associated conserved charges are total momentum and energy of the D-brane.

We shall now describe a general method for constructing a conserved charge carried by a D-brane for an arbitrary rigid gauge transformation of the closed string theory.

Closed string field
$\leftrightarrow$ a ghost number 2 state $|\Phi\rangle$ of the first quantized closed string satisfying

$$(b_0 - \bar{b}_0)|\Phi\rangle = 0, \qquad (L_0 - \bar{L}_0)|\Phi\rangle = 0$$

Closed string gauge transformation parameters
$\leftrightarrow$ a ghost number 1 state $|\Lambda\rangle$ of the first quantized closed string satisfying

$$(b_0 - \bar{b}_0)|\Lambda\rangle = 0, \qquad (L_0 - \bar{L}_0)|\Lambda\rangle = 0$$

Gauge transformation law:

$$\delta|\Phi\rangle = (Q_B + \bar{Q}_B)|\Lambda\rangle + \mathcal{O}(\Phi)$$

Thus $\delta|\Phi\rangle = 0$ at $|\Phi\rangle = 0$ if

$$(Q_B + \bar{Q}_B)|\Lambda\rangle = 0$$

$\rightarrow$ rigid gauge transformations leaving the $|\Phi\rangle = 0$ background invariant.

Can we construct a conserved charge associated with this symmetry?

x^0: time coordinate

$x \equiv ix^0$: euclidean time

p: momentum conjugate to x

Suppose we have a BRST invariant $|\Lambda\rangle$ at some fixed value $p = c$.

Define $|\Lambda(p)\rangle$ as a family of states carrying momentum p such that

$$|\Lambda(p = c)\rangle = |\Lambda\rangle$$

Then

$$(Q_B + \bar{Q}_B)|\Lambda(p)\rangle = (p - c)|\phi(p)\rangle$$

for some $|\phi(p)\rangle$.

Now consider a D-brane configuration described by boundary state $|\mathcal{B}\rangle$

$|\mathcal{B}\rangle$: a ghost number 3 closed string state satisfying:

$$(Q_B + \bar{Q}_B)|\mathcal{B}\rangle = 0$$

$$(b_0 - \bar{b}_0)|\mathcal{B}\rangle = 0, \qquad (L_0 - \bar{L}_0)|\mathcal{B}\rangle = 0$$

Then

$$\begin{aligned} & \langle\mathcal{B}|(c_0 - \bar{c}_0)(Q_B + \bar{Q}_B)|\Lambda(p)\rangle \\ = \; & \langle\mathcal{B}|\{(c_0 - \bar{c}_0), (Q_B + \bar{Q}_B)\}|\Lambda(p)\rangle \\ = \; & 0 \end{aligned}$$

since $\{(c_0 - \bar{c}_0), (Q_B + \bar{Q}_B)\}$ does not contain any zero mode of $(c_0 - \bar{c}_0)$ and the corresponding matrix element vanishes.

$$\langle\mathcal{B}|(c_0 - \bar{c}_0)(Q_B + \bar{Q}_B)|\Lambda(p)\rangle = 0$$

$$(Q_B + \bar{Q}_B)|\Lambda(p)\rangle = (p - c)|\phi(p)\rangle$$

Thus

$$(p - c)\langle\mathcal{B}|(c_0 - \bar{c}_0)|\phi(p)\rangle = 0$$

Define

$$F(x) = \int dp e^{-ipx}\langle\mathcal{B}|(c_0 - \bar{c}_0)|\phi(p)\rangle$$

$$\rightarrow \partial_x \left(e^{icx}F(x)\right) = 0$$

$e^{icx}F(x)$: conserved charge

World-sheet description of two dimensional string theory

$$s_{ws} = s_X + s_{ghost} + s_L$$

$$s_X = \frac{1}{2\pi} \int d^2z \, \partial_z X \bar{\partial}_z X, \quad X = iX^0$$

$$s_{ghost} = \frac{1}{2\pi} \int d^2z (b\bar{\partial}_z c + \bar{b}\partial_z \bar{c})$$

$$s_L = \frac{1}{2\pi} \int d^2z \left(\partial_z \varphi \bar{\partial}_z \varphi + \mu e^{2\varphi} \right)$$

$+$ a linear dilaton field $\Phi = Q\varphi$ with $Q = 2$

$$\rightarrow c_L = 1 + 6Q^2 = 25$$

For $\varphi << 0$ the Liouville field behaves like a free scalar field with background charge.

This string theory has infinite number of elements of BRST cohomology at ghost number 1 labelled by SU(2) quantum numbers (j, m)

$$(Q_B + \bar{Q}_B)|\Lambda_{j,m}\rangle = 0$$

$$2j \in Z, \quad (j-m) \in Z, \quad -(j-1) \leq m \leq (j-1)$$

$|\Lambda_{j,m}\rangle$ carries X-momentum $p = 2m$.

$\rightarrow$ for every such $|\Lambda_{j,m}\rangle$ we can define a conserved charge $F_{j,m}$ such that

$$\partial_x \left(e^{2imx} F_{j,m}(x) \right) = 0$$

Define

$$Q_{j,m}(x^0) = F_{j,m}(ix^0)$$

Then

$$\partial_{x^0} \left(e^{-2mx^0} Q_{j,m}(x^0) \right) = 0$$

Algorithm for constructing $Q_{j,m}(x^0)$:

1. Take the cohomology element $|\Lambda_{j,m}\rangle$. This has $p = 2m$.

2. Define $|\Lambda_{j,m}(p)\rangle$ such that it has X-momentum p and $|\Lambda_{j,m}(p = 2m)\rangle = |\Lambda_{j,m}\rangle$.

3. Define $|\phi_{j,m}(p)\rangle$ through:

$$(Q_B + \bar{Q}_B)|\Lambda_{j,m}(p)\rangle = (p - 2m)|\phi_{j,m}(p)\rangle$$

4. Then

$$F_{j,m}(x) == \int dp e^{-ipx} \langle \mathcal{B}|(c_0 - \bar{c}_0)|\phi_{j,m}(p)\rangle$$

5. $Q_{j,m}(x^0) = F_{j,m}(ix^0)$

The two dimensional string theory has a D0-brane

1. Neumann boundary condition on X^0.

2. Usual boundary condition on ghosts $b, c, \bar{b}, \bar{c}$.

3. 'Dirichlet' boundary condition on φ.

Since the Liouville field is interacting it is more appropriate to describe the boundary condition on φ as an abstract boundary CFT.

Open string spectrum in Liouville sector has only the Virasoro descendants of the vacuum.

The boundary state of this D0-brane has the form:

$$|\mathcal{B}\rangle = |\mathcal{B}\rangle_X \otimes |\mathcal{B}\rangle_L \otimes |\mathcal{B}\rangle_{ghost}$$

$|\mathcal{B}\rangle_X$, $|\mathcal{B}\rangle_{ghost}$: standard $|\mathcal{B}\rangle_L$: known

We can now deform the CFT of X by boundary operator

$$\lambda \int dt \cos(X(t)) = \lambda \int dt \cosh(X^0(t))$$

This generates the rolling tachyon solution with energy $(\mu/g_s)\cos^2(\pi\lambda)$.

New boundary state

$$|\mathcal{B}(\lambda)\rangle = |\mathcal{B}(\lambda)\rangle_X \otimes |\mathcal{B}\rangle_L \otimes |\mathcal{B}\rangle_{ghost}$$

Following the procedure given earlier we can now explicitly compute the charges $Q_{j,m}(x^0)$ for this boundary state.

$$\begin{aligned} Q_{j,m}(x^0) &= \frac{\mu^j}{g_s}(-1)^{2m}\, e^{2mx^0}\, \cos^2(\pi\lambda) \\ &\quad \sum_{l=|m|}^{j-1} b_j^{l+m,l-m} \sin^{2l}(\pi\lambda)\,, \end{aligned}$$

where

$$b_j^{l+m,l-m} = -\frac{1}{(2j)!(2j-1)!}\left[\alpha_{j,m}^{l-|m|} - \alpha_{j-1,m}^{l-|m|}\right]$$

$$\begin{aligned} \alpha_{j,m}^s &= -1 + \\ &\sum_{\mu=0}^{s} \frac{(j+|m|)!(j-|m|)!(j-|m|-1-\mu)!}{((j-|m|-\mu)!)^2\mu!(2|m|+\mu)!(s-\mu)!} \\ &\times \frac{(-1)^{s-j+|m|}}{(j-|m|-1-s)!} \qquad \text{for } s \leq j-|m|-1\,, \\ \alpha_{j,m}^s &= 0 \quad \text{for } s > j-|m|-1\,. \end{aligned}$$

A single D0-brane corresponds to a single fermion excitation in the matrix model.

$\longrightarrow$ described by the hamiltonian

$$h(q,p) = \frac{p^2}{2} - \frac{q^2}{2} + \frac{\mu}{g_s}$$

Compare energies of the D0-brane and the matrix model:

$$E = \frac{\mu}{g_s}\cos^2(\pi\lambda) \quad \longrightarrow \quad q = -\sqrt{\frac{2\mu}{g_s}}\sin(\pi\lambda)\cosh(x^0)$$

$$p = -\sqrt{\frac{2\mu}{g_s}}\sin(\pi\lambda)\sinh(x^0)$$

$\longrightarrow$ the charges $W_{k,l} = h(q,p)(q-p)^k(q+p)^l$ take values:

$$W_{k,l} = (-1)^{k+l} e^{-(k-l)x^0} \left(\frac{\mu}{g_s}\right)^{1+\frac{k+l}{2}} 2^{\frac{k+l}{2}} \cos^2(\pi\lambda)\sin^{k+l}(\pi\lambda)$$

Comparing the x^0 and λ dependence of $W_{k,l}$ with that of $Q_{j,m}$ we can determine $Q_{j,m}$ in terms of $W_{k,l}$ and vice versa.

$$Q_{j,m} = \sum_{l=|m|}^{j-1} \mu^{j-l-1}\, 2^{-l}\, {g_s}^l\, b_j^{l+m,l-m} W_{l-m,l+m}$$

These relations can be inverted to give $W_{k,l}$ in terms of $Q_{j,m}$.

Although these relations have been derived using single fermion excitations, once the relationship between the charges in the two descriptions have been established, they must hold for all states.

Example: A hole state of energy $E = \frac{\mu}{g_s} \sinh^2(\pi\alpha)$ corresponds to removing a fermion from the trajectory with $E = -\mu \sinh^2(\pi\alpha)/g_s$:

$$q = -\sqrt{\frac{2\mu}{g_s}} \cosh(\pi\alpha) \cosh(x^0)$$
$$p = -\sqrt{\frac{2\mu}{g_s}} \cosh(\pi\alpha) \sinh(x^0)$$

Corresponding charges in the matrix model:

$$W^h_{k,l} = (-1)^{k+l} e^{-(k-l)x^0} \left(\frac{\mu}{g_s}\right)^{1+\frac{k+l}{2}} 2^{\frac{k+l}{2}} \sinh^2(\pi\alpha) \cosh^{k+l}(\pi\lambda)$$

Using the relation between $W_{k,l}$ and $Q_{j,m}$ gives:

$$Q^h_{j,m}(x^0) = \frac{\mu^j}{g_s} (-1)^{2m} e^{2mx^0} \sinh^2(\pi\alpha) \sum_{l=|m|}^{j-1} b_j^{l+m,l-m} \cosh^{2l}(\pi\alpha) \,,$$

On the other hand if the hole state is described by a D-brane then $Q^h_{j,m}$ can be computed from the corresponding boundary state by using the earlier formula derived for a general D-brane.

Knowledge of $Q^h_{j,m}$ gives constraints on the boundary state describing the hole.

Any proposal for the hole state in the continuum description must satisfy these constraints.

Unfortunately these constraints do not determine the form of the boundary state uniquely.

The proposal by Douglas et. al. and Gaiotto-Itzhaki-Rastelli satisfies these constraints.

$\mu \to 0$ **limit:**

$$Q_{j,m} = \sum_{l=|m|}^{j-1} \mu^{j-l-1}\, 2^{-l}\, {g_s}^l\, b_j^{l+m,l-m} W_{l-m,l+m}$$

In $\mu \to 0$ limit only the $l = j-1$ term contributes.

$$\begin{aligned} & Q_{j,m} \\ = & \ 2^{-(j-1)}\, (g_s)^{j-1}\, b_j^{j+m-1,j-m-1} W_{j-1-m,j-1+m} \\ = & \ \frac{2^{-j}\, (g_s)^{j-1}}{(2j-1)!(j+m)!(j-m)!} (q+p)^{j+m}(q-p)^{j-m} \end{aligned}$$

$\to$ agrees with the previous results on the relationship between the symmetry generators of the matrix model and the continuum theory.

Charges and Asymptotic Fields

Not all configurations have descriptions as D-branes.

Example: The black hole in two dimensional string theory

How can we compute the charges $Q_{j,m}$ carried by the black hole?

Sources $\to$ an asymptotic closed string field configuration for large negative φ.

This field configuration should contain information about the conserved charges.

(Analog of Gauss law / ADM formula)

ADM formula for $Q_{j,m}(x^0)$

$|\phi_{j,m}(p)\rangle$ carries some fixed Liouville momentum K ($K = 2i(j-1)$)

$|\phi_{j,m}(p,q)\rangle$: A family of states with Liouville momentum q such that

$$|\phi_{j,m}(p, q = K)\rangle = |\phi_{j,m}(p)\rangle$$

Then

$$(Q_B + \bar{Q}_B)|\phi_{j,m}(p,q)\rangle = (q - K)|\Omega_{j,m}(p,q)\rangle$$

$\rightarrow$ defines $|\Omega_{j,m}(p,q)\rangle$

Then

$$\begin{aligned} F_{j,m}(x) &= -i \lim_{\varphi\to\infty} \Big(e^{-2(1+j)\varphi} \int \frac{dp}{2\pi} e^{-ipx} \\ &\quad \int \frac{dq}{2\pi} e^{(2-iq)\varphi} \langle \Phi | (c_0 - \bar{c}_0) | \Omega_{j,m}(p,q)\rangle \Big) \end{aligned}$$

$|\Phi\rangle$: Closed string field

$Q_{j,m}(x^0) = F_{j,m}(ix^0)$

Apply this formula to two dimensional black holes.

Result: (for $\mu = 0$)

$Q_{j,m} = 0$ for $m \neq 0$

$Q_{1,0} = M$ (mass)

$Q_{2,0} = 0$

$Q_{3,0} = 0$??

$Q_{j,0} = 0$ for $j \geq 4$

We shall proceed by assuming that $Q_{j,0} = 0$ $\forall$ $j \geq 2$

Q. Is there a configuration of free fermions carrying such charges?

Consider a system of fermions of energy E_1, E_2, ... and holes of energy ϵ_1, ϵ_2, ...

For this system:

$$Q_{j,0} = \frac{1}{(j!)^2(2j-1)!}\left(-\sum_k(-E_k)^j + \sum_r(\epsilon_r)^j\right)$$

Mass $= Q_{1,0} = \sum_k E_k + \sum_r \epsilon_r$

For odd j the contributions from holes and fermions add.

Small $Q_{j,0}$ for $j > 1$

$\rightarrow$ large number of fermions and holes each with very small energy.

$\rightarrow$ suggests that the black hole consists of a large number of fermions and holes, each with energy close to zero (the fermi level).

If this is the case then a D0-brane, represented by a fermion of finite energy, should not feel the black hole at the classical level.

Is this true?

In the DBI approximation, the D0-brane world-volume action is:

$$\int d\tau e^{-\Phi} \sqrt{G_{\mu\nu} \partial_\tau X^\mu \partial_\tau X^\nu}$$

X^0: time, $\qquad X^1 \equiv \varphi$

$G_{\mu\nu}$: string metric $\qquad \Phi$= dilaton

Thus the D0-brane sees an effective metric

$$e^{-2\Phi} G_{\mu\nu}$$

Now compare $e^{-2\Phi}G_{\mu\nu}$ for the black hole background and the usual linear dilaton background.

Linear dilaton:

$$e^{-2\Phi}ds^2 = e^{-4\varphi}\left(-(dx^0)^2 + d\varphi^2\right)$$

Black Hole:

$$\begin{aligned} e^{-2\Phi}ds^2_{BH} &= e^{-4\varphi}\Big(-(1-ae^{4\varphi})(dx^0)^2 \\ &\quad +(1-ae^{4\varphi})^{-1}d\varphi^2\Big) \end{aligned}$$

Define $w = -\frac{1}{4}\ln(e^{-4\varphi}-1)$

$$e^{-2\Phi}ds^2_{BH} = e^{-4w}(-(dx^0)^2 + dw^2)$$

Thus $e^{-2\Phi}ds^2$ and $e^{-2\Phi}ds^2_{BH}$ describe the same metric.

In fact both metric are known to be flat.

Lessons for Critical String Theory

The open string field theory on a D0-brane in 2D string theory sits inside open string field theory on any D0-brane as a universal subsector.

This involves arbitrary ghost and X excitations and only excitations by Virasoro algebra in the $c = 25$ CFT describing the rest of the scalars.

Many interesting classical solutions sit inside this subspace.

Example: Tachyon vacuum, Tachyon lump, Rolling tachyon

Thus understanding the open string field theory (and its conservation laws) on the D0-brane of 2D string theory automatically gives us information about an important subsector of open string field theory for critical strings.