# Semiclassical Relativistic Strings in $AdS_5 \times S^5$ and Long Scalar Operators in $\mathcal{N}=4$ SYM

based on: S. Frolov, A.T. (2003)

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hep-th/0403120

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hep-th/0404133

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$\mathcal{N}=4$ $SU(N)$ SYM

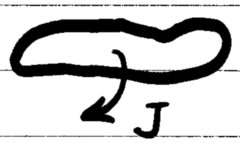
$$N=\infty, \quad \lambda = g^2_{YM} N = \text{fixed}$$

emiclassical "fast-moving" closed strings in $AdS_5 \times S^5$ $\longleftrightarrow$ $\mathrm{Tr}\,(D\ldots D\Phi \ldots D\ldots D\Phi\ldots)$

alks by Polyakov, Bianchi, Dixon

# Motivation:

- solve $\mathcal{N}=4$ SYM ($N=\infty$):
  spectrum of dim's $\Delta = \underline{\Delta(\lambda)} = ?$
  duality to $AdS_5 \times S^5$ string $\mapsto$
  hidden <u>2-d</u> (integrable) structure

- <u>quantitative</u> check of duality
  in sector of <u>non</u>-BPS string states:
  $$E(\lambda, J, \ldots) = \Delta(\lambda, J, \ldots)$$
   $\longleftrightarrow$ $Tr(\Phi^J \ldots)$

- Precise <u>map</u> between
  string and SYM states/operators ?
  How strings "emerge" from gauge theory?

- Use duality as a <u>guide</u>: to "guess" non-p
  (exact) SYM dilatation operator $\to \Delta($

- Lessons for less susy cases? ...
  New methods to compute anom. dim. in QCD?

# Plan:

- Basics of AdS/CFT
- Review of duality checks for spinning strings
- $SU(2)$ sector: ($J_1, J_2$ spins)
  - effective $\sigma$-model from spin chain/SYM
  - effective $\sigma$-model from string action
- $O(6)$ sector: general rotating and pulsating strings in $S^5$ → phase spac[e] $G_{2,6}$ $\sigma$-model.
- Conclusions
- Beyond two leading orders? reconstructing dilatation operator in $SU(2)$ sect[or]

Do not write in this margin

# $\mathcal{N}=4$ SYM : $SU(N)$

$N \to \infty$, $\lambda = g_{YM}^2 N$ = fixed

$$\mathcal{L} = \mathrm{Tr}\left[F_{\mu\nu}^2 + (\partial\phi_I)^2 + [\phi,\phi]^2 + \text{ferm.}\right]$$

$\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6$ : $\Phi_1 = \phi_1 + i\phi_2$

adjoint rep.

$\Phi_2 = \phi_3 + i\phi_4$

$\Phi_3 = \phi_5 + i\phi_6$

$SO(2,4) \times SO(6)$

Conformal $\times$ R-symmetry

Local ops : $\mathcal{O} = \mathrm{Tr}\left(F_{\mu\nu}^2 \ldots DF \ldots \Phi \ldots \Psi \ldots\right)$

Dimensions : $\Delta = n + \underline{\gamma(\lambda)} = ?$

- "Protected" sector : $\gamma(\lambda) = 0$

**BPS** or CPO $\qquad \mathrm{Tr}\left(\phi_{\{I_1} \ldots \phi_{I_n\}}\right)$

$\mathrm{Tr}\, \Phi_1^J$ : $\Delta = J$

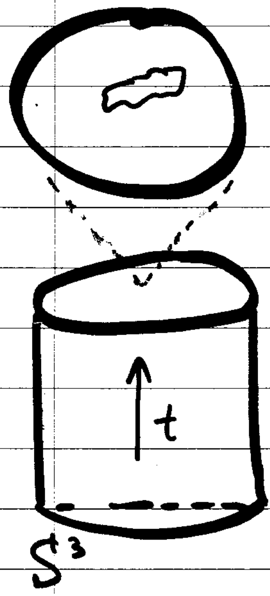
- "**Near-BPS**" : $\mathrm{Tr}\left(\Phi_1^J \Phi_2^k\right) + \ldots$

$J \gg k$

- **Non-BPS** : $\mathrm{Tr}\left(\Phi_1^{J_1} \Phi_2^{J_2}\right) + \ldots$

$J_1 \sim J_2$

# $AdS_5 \times S^5$ string theory:




$S^5$

$$X_I X_I = 1$$
$$I = 1, \dots, 6$$

$AdS_5$

$$\eta^{MN} Y_M Y_N = -1$$
$$\eta^{MN} = (-++++-)$$

$$SO(6) \times SO(2,4) \quad + \text{susy}$$

$$I = T \int_0^{2\pi} d\sigma \int d\tau \left( \partial Y^M \partial Y^N \eta_{MN} + \partial X^I \partial X^I + \text{fermions} \right)$$

$$\boxed{T = \frac{R^2}{2\pi\alpha'} \equiv \frac{\sqrt{\lambda}}{2\pi}}$$

2-d CFT
integrable

Spectrum of string states?

<u>**$AdS_5$ energy**</u> $\quad E = E(Q_i, n_k, \underline{T})$

$Q_i$ : $\quad 2 + 3$ Cartan charges of $SO(2,4) \times SO(6)$

$$(\underbrace{S_1, S_2}_{AdS_5 \text{ spins}} ; \underbrace{J_1, J_2, J_3}_{S^5 \text{ spins}}) + \text{higher charges } n_k$$

# Duality:

quantum string states $\longleftrightarrow$ quantum SYM states in $R \times S^3$

$\parallel$

Tr(...) operators in $R^4$

$$\boxed{E_{AdS}(\lambda, J, \ldots) = \Delta(\lambda, J, \ldots)}$$

$$E = \sum_k \frac{c_n}{(\sqrt{\lambda})^n}$$ "$\alpha'$ - expansion"

$$\Delta = \sum_n a_n \lambda^n$$ perturbation theory

GK

Generic states : $E \sim \frac{1}{\sqrt{\alpha'}} \sim \sqrt[4]{\lambda}$

$\lambda \to \infty$ $\longrightarrow$ $\lambda \to 0$ <u>interpolation</u> $\Delta \sim \lambda + \lambda^2 + \ldots$

<u>How to check</u>?

- BPS sector : protected states :
  sugra modes (point-like strings) $\longleftrightarrow$ CPO's $\mathrm{Tr}\, \Phi_{\{i_1} \ldots \Phi_{i_n\}}$

- Near-BPS sector :
  nearly point-like strings $\longleftrightarrow$ $\mathrm{Tr}(\Phi^J \ldots)$

- Non-BPS : "large" closed strings $\longleftrightarrow$ ?

# Key idea:

Berenstein
Maldacena
Nastase
(2002)

look at subsectors
of states with large (semiclassical)
quantum number: $J \sim T \sim \sqrt{\lambda}$

Gubser
Klebanov
Polyakov
(2002)

New limit:

$$\boxed{J \to \infty, \qquad \tilde{\lambda} \equiv \frac{\lambda}{J^2} = \text{fixed}} \qquad (\lambda' = \tilde{\lambda})$$

Semiclassical string states $\oplus$ fluctuations

$\leftrightarrow$ "Long" SYM operators $\mathrm{Tr}\,(\underbrace{\Phi \ldots \Phi}_{\sim J})$

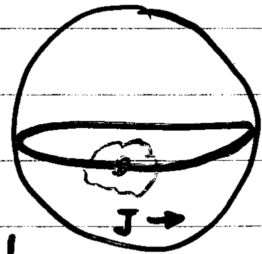
$$\boxed{\Delta \overset{?}{=} E = J + f(\lambda, J)}$$

duality map becomes more explicit:

- Near BPS $\equiv$ BMN sector $\mathrm{Tr}(\Phi_1^J \Phi_2 \ldots)$

semiclassical state: point-like

$E = J$




Fluctuation's spectrum: Quantitative match

BMN
Gross, Mikhailov, Roiban
Santambrogio - Zanon

$$E_n = J + \sqrt{1 + \tilde{\lambda} n^2}\, N_n + O\left(\frac{1}{J}\right)$$

Analytic in $\tilde{\lambda}$ !

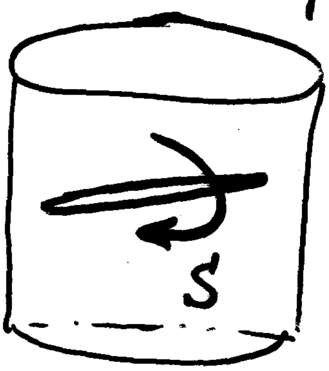
- **Non-BPS** **single-spin** states:
  **interpolating** **functions** of $\lambda$
  (quantum string corrections **not** suppressed at $J \to \infty$)

$J \equiv S$

**Qualitative** agreement: $J$-dependence matches!

Example:

GK




**Folded string in $AdS_5$**

$$\boxed{E = S + f(\lambda)\ln S + \dots}$$

$S \to \infty$

$$\mathcal{O} = \mathrm{Tr}\left(\Phi^* D_+^S \Phi\right)$$

**String side**:

$$f(\lambda) = a_0\sqrt{\lambda} + a_1 + \frac{a_2}{\sqrt{\lambda}} + \dots$$

classical energy non-analytic in $\lambda$

GKP

quantum corrections not suppressed

Frolov
A.T.

**SYM**:

$$\boxed{\Delta = S + f(\lambda)\ln S + \dots}$$

$$f(\lambda) = b_1\lambda + b_2\lambda^2 + b_3\lambda^3 + \dots$$

**Same** $S$-dependence (!)

but $f(\lambda)$ hard to match

(But can try to Pade interpolate)

Kotikov
Lipatov
Velizhanin
Onishchenko

# Non-BPS states:

Quantitative agreement (as in BMN case)?

Key idea: consider states with several large angular momenta

Frolov, A.T. 0304255

Multi-spin solutions with large $J$ in $S^5$:

$$E = \sqrt{\lambda}\,\mathcal{E}(\omega_i) \qquad \sqrt{\lambda} \sim \text{tension}$$

$$J_i = \sqrt{\lambda}\,\omega_i \,, \qquad \boxed{J = \sum_i J_i}$$

$$E = E(J_i, \lambda)$$

$$= J + c_1 \frac{\lambda}{J} + c_2 \frac{\lambda^2}{J^3} + \dots$$

$$\boxed{E = J\left(1 + c_1 \tilde{\lambda} + c_2 \tilde{\lambda}^2 + \dots\right)}$$

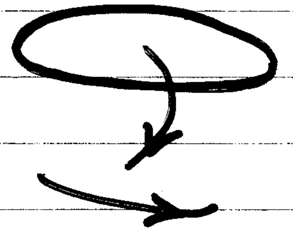
analytic in $\boxed{\tilde{\lambda} \equiv \frac{\lambda}{J^2}}$ (no $\sqrt{\lambda}$'s)

$c_n = c_n\left(\frac{J_i}{J}\right)$ finite in the limit

$$\boxed{J_i \to \infty \,, \qquad \tilde{\lambda} = \text{fixed}}$$

# Examples:

- circular string in $S^3 \subset S^5$
  with $J_1 = J_2$



$$\mathcal{X}_1 = X_1 + i X_2 = \cos n\sigma \, e^{i\omega\tau}$$

$$\mathcal{X}_2 = X_3 + i X_4 = \sin n\sigma \, e^{i\omega\tau}$$

$$t = \varkappa \tau \qquad \varkappa^2 = \omega^2 + n^2, \qquad E = \sqrt{\lambda}\,\varkappa, \quad J = \sqrt{\lambda}\,\omega$$

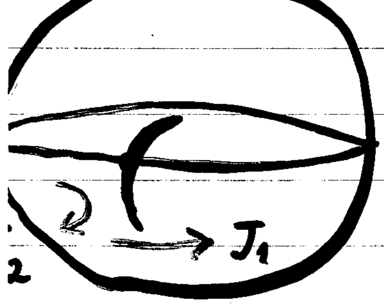
$$\boxed{\tilde{\lambda} = \frac{\lambda}{J^2}}$$

$$\begin{aligned} E_{class} &= \sqrt{J^2 + n^2 \lambda} \\ &= J\left(1 + \frac{n^2}{2}\tilde{\lambda} - \frac{n^4}{8}\tilde{\lambda}^2 + \dots\right) \end{aligned}$$

"relativistic" or small tension expansio

cf. $E = \sqrt{\frac{2}{\alpha'} J}$ (in flat space)

- folded rotating string orbiting in $S^5$:
  (elliptic 1-d sin-gordon problem)




$$E = J\left(1 + c_1 \tilde{\lambda} + c_2 \tilde{\lambda}^2 + \dots\right)$$

$J \to \infty$

$$c_n = c_n\left(\frac{J_1}{J_2}\right)$$ expressed in elliptic integrals

- generic 3-spin solutions: integrable Neumann system

Arutyunov, Frolov, Russo, A.T. 0307191

Compare to SYM dimensions ? "

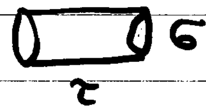
String theory limit: $J \to \infty$, $\tilde{\lambda} = \text{fixed} \sim \frac{1}{\omega^2}$

semiclassical expansion

string $\underline{\alpha'} \sim \frac{1}{\sqrt{\lambda}} \sim \frac{1}{J\sqrt{\tilde{\lambda}}}$ corrections

formally suppressed at $J \to \infty$

Moreover, analytic in $\underline{\tilde{\lambda}}$ Frolov, A.T. 0306130

corrections to soliton energy on 

in 2-d finite theory (fermions are crucial !)

with massive fluctuations $m \sim \omega \sim \frac{1}{\sqrt{\tilde{\lambda}}}$ :

regular $\frac{1}{m^2}$ expansion]

$$E_{\text{Quant.}} = J \left[ 1 + \tilde{\lambda} \left( \underline{c_1} + \frac{d_1}{J} + \dots \right) + \tilde{\lambda}^2 \left( \underline{c_2} + \frac{d_2}{J} + \dots \right) + \dots \right]$$

$\boxed{J \to \infty, \quad \tilde{\lambda} = \text{fixed}}$ limit well-defined on string side :

$\boxed{E_{\underline{\text{class}}}(\lambda, J) = \underline{\text{exact}} \text{ at } J \to \infty}$

Assume same limit well-defined in SYM

and compare to dimensions of

$\mathcal{O} = \text{Tr}\,(\Phi_1 \dots \Phi_2 \dots \Phi_1 \dots)$ with same R-charges $J_1, J_2$

**SYM**: in practice $\Delta(\lambda, J)$ computed perturbatively in $\lambda$

**assume** can interchange the limits:

**expand** in $\underline{\lambda}$, **then** $J \to \infty$ :

$$\Delta = J + \underline{\lambda}\left(\frac{a_1}{J} + \frac{b_1}{J^2} + \dots\right) + \underline{\lambda}^2\left(\frac{a_2}{J^3} + \frac{b_2}{J^4} + \dots\right) + \dots$$

1-loop        2-loop

Compare $a_1, a_2, \dots$ to $c_1, c_2, \dots$ in E

How to compute anom. dim's of very **long operators**?

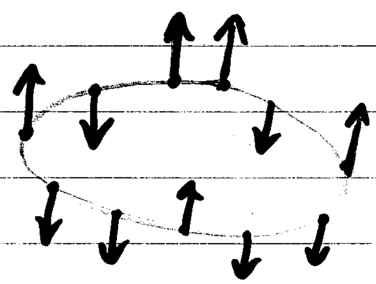
Crucial observation: Minahan, Zarembo 0212208

dilatation operator (1-loop, $N=\infty$)

= Hamiltonian of integrable spin chain

**SU(2) sector** : $\mathrm{Tr}\left(\Phi_1^{J_1}\Phi_2^{J_2}\right) + \dots$

$\Phi_1 : |\uparrow\rangle$    $\Phi_2 : |\downarrow\rangle$



Periodic 1-d chain $XXX_{1/2}$

($J = J_1 + J_2$ sites)

$$\boxed{D_1 = H_{\text{Heisenberg}} = \frac{\lambda}{(4\pi)^2}\sum_{\ell=1}^{J}\left(1 - \vec{\sigma}_\ell \cdot \vec{\sigma}_{\ell+1}\right)}$$

Use <u>Bethe ansatz</u> in <u>thermodynamic</u>
<u>$J \to \infty$</u> limit to find eigenvalues of $D_1$

$$\Delta_1 = \frac{\lambda}{J} \underline{a_1} + O\left(\frac{\lambda}{J^2}\right) \qquad a_1 \overset{?}{=} c_1$$

Remarkable agreement with String theory!

Beisert, Minahan, Staudacher, Zarembo 0306139

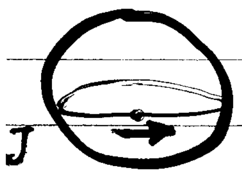
Circular $J_1 = J_2$ string + fluctuations

Folded and circular $(J_1, J_2)$ string:

matching of <u>functions</u> $\boxed{a_1 = c_1} = f\left(\frac{J_1}{J_2}\right)$

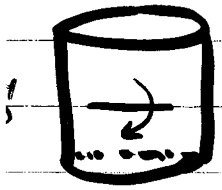
Beisert, Frolov, Staudacher, A.T. 0308117

Also: matching in <u>SL(2) sector</u>:



$$(S, J) \text{ string} \longleftrightarrow \mathrm{Tr}\left(D^S \Phi^J\right)$$

$$E = J + S + \frac{\lambda}{J} c_1\left(\frac{S}{J}\right) + \ldots$$



<u>SU(3) sector</u>: $(J_1, J_2, J_3)$ states: spins in $S^5$

Engquist, Minahan, Zarembo 0310188

$$\mathrm{Tr}\left(\Phi_1^{J_1} \Phi_2^{J_2} \Phi_3^{J_3}\right) + \ldots$$

Kristjansen 0402033

Matching of leading-order coeffs for particular example

**Why** it works? Understand matching in a **universal** way: beyond particular examp

"Derive" string $\sigma$-model from SYM?

**Precise** relation between string profiles and structure of dual operators?

String side interpretation of spin chain (operator ordering) direction?

ey idea: match effective 2-d actions on string and spin chain sides in the limit $\underbrace{J \to \infty, \quad \tilde{\lambda} \to 0}$

Kruczenski 0311203

ompare semiclassical (coherent) states of string o semiclassical (coherent) states of spin chain

erromagnetic spin chain: in $J \to \infty$ limit with $J_1/J_2$ = fixed $\mapsto$ large clusters of spins

$\uparrow\uparrow\uparrow\uparrow\uparrow \; \downarrow\downarrow\downarrow\downarrow\downarrow \; \uparrow\uparrow\uparrow\uparrow$

effective low-energy description in continuum limit $\to$ $\sigma$-model

lternative approach: match general solutions integrable structures of string $\sigma$-model and Bethe ansatz

Kazakov, Marshakov, Minahan, Zarembo (0402207)

# Coherent States

Harmonic oscillator: $\qquad [a, a^+] = 1$

$$a\,|u\rangle = u\,|u\rangle\,, \qquad u \in \mathbb{C}$$

**or**

$$|u\rangle = R(u)\,|0\rangle\,, \qquad a|0\rangle = 0$$

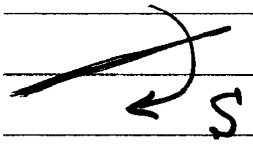
$$R(u) = \exp\left(u a^+ - u^* a\right) \quad \text{or} \quad e^{u a^+}$$

**or** states with minimal uncertainty or "classical"

$$(\Delta p)^2 = (\Delta q)^2 = \tfrac{1}{2} \qquad \hat{q} = \tfrac{1}{\sqrt{2}}(a + a^+)$$

$$(\Delta p)^2 = \langle u|\left(\hat{p} - \langle u|\hat{p}|u\rangle\right)^2|u\rangle \qquad \hat{p} = -\tfrac{i}{\sqrt{2}}(a - a^+)$$

**Example**: string in flat space




$$e^{\sqrt{S}\,a_1^+}|0\rangle \quad \text{vs} \quad (a_1^+)^S|0\rangle$$

$$E = \sqrt{\tfrac{2}{\alpha'}S} \quad ]$$

$$\langle u|\hat{q}|u\rangle = \tfrac{1}{\sqrt{2}}(u + u^*)$$

$$\langle u|\hat{p}|u\rangle = -\tfrac{i}{\sqrt{2}}(u - u^*)$$

$$\boxed{u(t) = \frac{q(t) + i\,p(t)}{\sqrt{2}}}$$

$$|u(t)\rangle = e^{-iH_0 t}|u\rangle$$

quantum states following classical harm. osc. eqs

$$|u\rangle \sim \sum_{n=0}^{\infty} \frac{u^n}{\sqrt{n!}}\,|n\rangle$$

# SU(2) : Spin coherent states

$$[S_3, S_\pm] = \pm S_\pm \quad , \quad [S_+, S_-] = 2 S_3$$

$$\boxed{S_i = \tfrac{1}{2}\sigma_i} \quad , \quad |0\rangle \equiv |\tfrac{1}{2}, +\tfrac{1}{2}\rangle = |\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix}$$

$$|u\rangle = e^{u S_+ - u^* S_-}|0\rangle \equiv R(u)|0\rangle$$

$$u = \frac{\theta}{2} e^{i\phi} \in \mathbb{C} \qquad \qquad \alpha|\uparrow\rangle + \beta|\downarrow\rangle$$

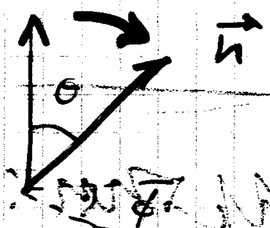
Equivalently: $\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

$$\vec{n}^2 = 1$$

$$\vec{n} = U^+ \vec{\sigma}\, U \qquad U = \begin{pmatrix}u_1\\u_2\end{pmatrix}$$

$$|\vec{n}\rangle = |u\rangle \qquad \vec{n} \in S^2 = SU(2)/U(1)$$

$$\boxed{|\vec{n}\rangle = R(\vec{n})|0\rangle}$$




$(0,0,1) \longleftrightarrow |\uparrow\rangle$

Key property:

$$\boxed{\langle\vec{n}|\,\vec{S}\,|\vec{n}\rangle = \tfrac{1}{2}\vec{n}}$$

<u>Coherent state path integral</u>: $[dp\,dq] \to [du]$ (cf. phase space)

$$\int [du]\, e^{iS(u)} \quad , \quad \boxed{S = \int dt \left( i\langle u|\tfrac{d}{dt}|u\rangle - \langle u|H|u\rangle \right)}$$

Berry phase or WZ term

Applied to $H = \lambda \sum_{\ell=1}^{L} (1 - \vec{\sigma}_\ell \vec{\sigma}_{\ell+1})$ :

$$S = \sum_{\ell=1}^{L} \int dt \left( \underbrace{\vec{C} \cdot \dot{\vec{n}}_\ell}_{\text{WZ}} - \underbrace{\frac{\lambda}{4} \left( \vec{n}_{\ell+1} - \vec{n}_\ell \right)^2}_{\langle H \rangle} \right)$$

$C \cdot \dot{n} = \cos\theta \, \dot{\phi}$ , $\quad dC = \varepsilon^{ijk} n_i \, dn_j \wedge dn_k$

monopole potential

Haldane

$\{ \vec{n}_\ell(t) \}$ : "Non-relativistic" action

$$\boxed{L \equiv J \to \infty , \quad \frac{\lambda}{J^2} \equiv \tilde{\lambda} = \text{fixed}} \; : \; \underline{\text{semiclassical limit}}$$

$$\vec{n}(t, \sigma) = \left\{ \vec{n}\left(t, \frac{2\pi}{J} \ell\right) \right\}$$

$$\boxed{S = J \int dt \int_0^{2\pi} \frac{d\sigma}{2\pi} \left( \vec{C} \cdot \dot{\vec{n}} - \frac{\tilde{\lambda}}{4} \vec{n}' \vec{n}' \right)} + (\ldots$$

"Landau - Lifshitz" $\sigma$- model : eqs. of motion

$$\dot{n}_i = \frac{\tilde{\lambda}}{2} \varepsilon_{ijk} \, n_j \, n_k''$$

classical ferromagnet

Same action on <u>string</u> side ?

<u>First</u> order in time derivative ?!

- isolate "<u>fast</u>" coordinate <u>$\alpha$</u> with $P_\alpha \sim J$
- gauge fix $t \sim \tau$ , $\tilde{\alpha} \sim \sigma$
- expand in velocities of "slow" "transverse" coordinates

# LL σ-model from string action:

## SU(2) sector $(J_1, J_2)$

String motion in $S^3 \subset S^5$ : $R_t \times S^3$

$$ds^2 = -dt^2 + dX_i dX_i^* , \quad |X_i|^2 = 1$$

$$X_1 = x_1 + i x_2 = u_1 e^{i\alpha}$$

$$X_2 = x_3 + i x_4 = u_2 e^{i\alpha} \qquad \boxed{|u_i|^2 = 1}$$

Large spins: $\alpha \sim J\tau$ — "fast" coordinate
$J_1 \sim J_2$ (orbital + internal spin)

$$dX dX^* = (d\alpha + C)^2 + \underbrace{du_i^* du_i + (u_i^* du_i)^2}_{CP^1 \simeq S^2}$$

$$C = -i\, u_i^* du_i$$

Hopf fibration $S^3 \sim S^1 \times S^2$

U(1) gauge freedom: $\alpha \to \alpha - \beta$

$$u_i \to \exp(i\beta)\, u_i$$

$$\boxed{\vec{n} = U^\dagger \vec{\sigma} U} , \quad U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\boxed{X_i dX_i^* = (d\alpha + C(n))^2 + \frac{1}{4} d\vec{n}\, d\vec{n}}$$

Gauge fixing: phase space action,
= physical ⊥, $\boxed{t = \tau, \quad P_\alpha = \text{const} = J}$

Kruczenski
Ryzhov
AT
0403120

# Equivalent approach:

first 2-d (or T-) duality $\alpha \to \tilde{\alpha}$

then static gauge $\tilde{\alpha} \sim J\sigma$ in Nambu action

(**NOT** conformal gauge)

Kruczenski, A.T. 0406189

$$\mathcal{L} = \sqrt{-g}\, g^{ab}\left(-\partial_a t \partial_b t + D_a \alpha D_b \alpha + D_a U_i^* D_b U_i\right)$$

$$D_a \alpha = \partial_a + C_a , \qquad D_a U_i = \partial_a U_i - i C_a U_i$$

$$C_a = -i U_i^* \partial_a U_i$$

$$|U_i|^2 = 1$$

2-d duality: $\alpha \to \tilde{\alpha}$

$$\sqrt{-g}\, g^{ab} D_b \alpha = -\varepsilon^{ab} \partial_b \tilde{\alpha} : \qquad P_\alpha D\alpha \to WZ$$

$$\mathcal{L} = \sqrt{-g}\left(-\partial t \partial t + \partial\tilde{\alpha}\partial\tilde{\alpha} + D U_i^* D U_i\right) + \underline{\varepsilon^{ab} C_a \partial_b \tilde{\alpha}}$$

origin of WZ term

$$\boxed{\mathcal{L}_{Nambu} = \varepsilon^{ab} C_a \partial_b \tilde{\alpha} - \sqrt{-h}}$$

$$h_{ab} = -\partial_a t \partial_b t + \partial_a \tilde{\alpha} \partial_b \tilde{\alpha} + D_a U_i^* D_b U_i$$

**Static gauge:** $\underline{t} = \underline{\tau}$, $\underline{\tilde{\alpha}} = \frac{J}{\sqrt{\lambda}}\sigma \equiv \frac{1}{\sqrt{\tilde{\lambda}}}\sigma$

After time rescaling:

$$= J \int d\tau \int_0^{2\pi} \frac{d\sigma}{2\pi}\left[\underline{C_0(U)} - \sqrt{(1+\tilde{\lambda}|D_1 U|^2)(1-\tilde{\lambda}^{-1}|D_0 U|^2) + \dots}\right]$$

$$\boxed{\mathcal{L} = -i U_i^* \partial_0 U_i - \frac{\tilde{\lambda}}{2} |D_1 U_i|^2 + O(\tilde{\lambda}^2)}.$$

or $\mathcal{L} = \vec{C}(\vec{n})\,\dot{\vec{n}} - \frac{\tilde{\lambda}}{4}\left(\partial_1 \vec{n}\right)^2$

Same $CP^1 \sim S^2$ action as on spin chain side

Explains matching of energies and states for 2-spin states (and fluctuations); also matching of integrable structures

Summary:

$(t, p_\alpha)$ or $(t, \tilde{\alpha})$ : "longitudinal" coordinates: gauge fixed

$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ or $\vec{n}$ : string "transverse" profile $\rightarrow$

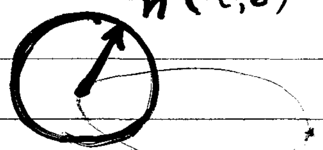
$\rightarrow |n\rangle = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}$ coherent state

$\tilde{\alpha} \sim \sigma$ : Spin chain direction

semiclassical string state $\longleftrightarrow$ coherent spin chain state or coherent operator $Tr(\Phi_1 \dots \Phi_2 \dots$

string side :

$S^2$ $\vec{n}(\tau,\sigma)$



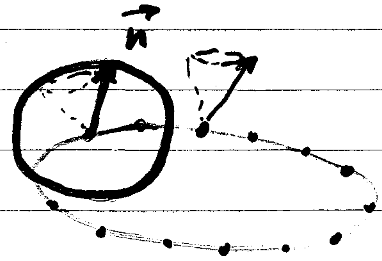
$S^1 \times S^2 \sim S$

$S^1 \sim \tilde{\alpha} \sim \sigma$

spin chain side:

$\vec{n} = u^+ \vec{\sigma}\, u$

$\vec{n}$



$\vec{n}_\ell(t)$

$\mathcal{O} = Tr \prod_{\ell=1}^{J} \left(u_{\ell i} \Phi_i\right) \quad (i=1,2)$

# Generalizations :

## SU(2) sector : higher orders in $\tilde{\lambda}$

**string side** : eliminate $\dot{n}^2, \ldots$ by field redef's ($\dot{n} \sim n''$)

$$\mathcal{L} = \vec{C}\,\dot{\vec{n}} - \frac{1}{4}\tilde{\lambda} n'^2 + \frac{1}{16}\tilde{\lambda}^2\left(n''^2 - \frac{3}{4}n'^4\right) - \frac{1}{32}\tilde{\lambda}^3\left(n'''^2 - \frac{7}{4}n'^2 n''^2 - \frac{25}{2}(n'n'')^2 + \frac{13}{16}n'^6\right) + \ldots$$

Kruczenski, Ryzhov, AT

**gauge theory** : 2-loop dilatation operator

Beisert, Kristjansen, Staudacher 030360

$$D_2 = \frac{\lambda^2}{(4\pi)^4}\sum_{\ell=1}^{L}\left(Q_{\ell,\ell+2} - 4Q_{\ell,\ell+1}\right)$$

$$Q_{\ell,k} = I - \vec{\sigma}_k \vec{\sigma}_\ell$$

Extract effective action for coherent state $|\vec{n}\rangle$ :

KRT

$$\Delta\mathcal{L} = \frac{1}{16}\tilde{\lambda}^2\left(n''^2 - \frac{3}{4}n'^4\right)$$

Precise matching of $\tilde{\lambda}^2$ terms !

Explains matching of energies at $\tilde{\lambda}^2$ for strings and Bethe ansatz eigenvalues first observed by Serban & Staudacher 0401057

Equivalent conclusion in integrability based approach of Kazakov, Marshakov, Minahan, Zarembo

Apparent disagreement at order $\tilde{\lambda}^3$ ?

Gauge theory knowledge incomplete:
YM takes different limit: $\{\lambda \to 0$, then $J \to \infty\}$

Proposal how to fix the problem: Beisert, Dippel, Staudacher 0405001

and very recent paper: Arutyunov, Frolov, Staudacher 0406256

Suggests also resolution of disagreement with $1/J$ corrections to BMN spectrum found by Callan, McLoughlin, Schwarz, Swanson, Wu 0307032

[Why problems started at 3 loops or $\tilde{\lambda}^3$ ?
Structure of dil. op. fixed uniquely by BMN limit at $\lambda$ and $\lambda^2$ orders only $\rightarrow$ unique effective action]

# Generalizations to other sectors: order $\tilde{\lambda}$

Stefanski, A... 0404133

- SU(3) sector $(J_1, J_2, J_3)$:

$$\mathcal{L} = -U_i^* \partial_0 U_i - \frac{\tilde{\lambda}}{2} |D_1 U_i|^2 \quad (i=1,2,3)$$

CP$^2$ model on both sides

Hernandez, Lopez 0403139

- SL(2) sector $(S, J)$

$$\vec{n} \to \vec{\ell}: \quad \ell_1^2 - \ell_2^2 - \ell_3^2 = 1$$

use dil. op. of Beisert, Staudacher

# General fast motion in $S^5$ $\leftrightarrow$ general scalar operators : SO(6) sector

Kruczenski, A.T.
0406189

Generic string states:
oscillations or pulsations

closely related approach
Mikhailov
0311019, 0402067
0404173

Large oscillation number $L$: large energy

$$E = L + \frac{\lambda}{L} c_1 + \frac{\lambda^2}{L^3} c_2 + \dots$$

Minahan
0209047

regular expansion : regular limit!

$$\boxed{L \to \infty, \quad \tilde{\lambda} = \frac{\lambda}{L^2} = \text{fixed}}$$

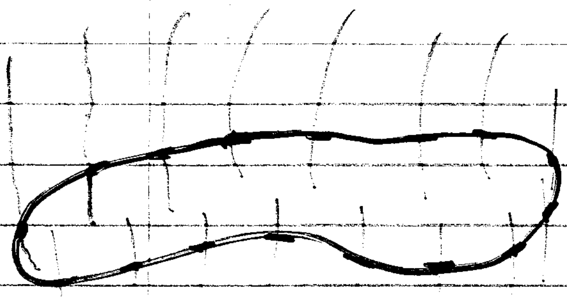
Engquist, Minahan
Zarembo 0310188
Minahan 0405243

Pulsating string : $c_1$ matched with SO(6)
Bethe ansatz eigenvalue

Beisert, Minahan, Staudacher, Zarembo

When regular limit in general?
Fast string (relativistic) motions : null worldsheet for $\tilde{\lambda} \to 0$

Mikhailov
motivated by
Mateos, Mateos
Townsend
0401058



each string bit follows (for $\tilde{\lambda} \to 0$) geodesic –
large circle in $S^5$
dual SYM operator : "locally BPS"

$$\boxed{\mathcal{O} = C_{m_1 \dots m_L} \, \mathrm{Tr}\left(\phi_{m_1} \dots \phi_{m_L}\right)}$$

$m = 1, \dots, 6$

Minahan
Zarembo

$D \sim H$ $\quad$ **SO(6)** spin chain

Natural coherent states: $\quad \prod_{\ell} |V_\ell\rangle \qquad (V \equiv u)$

$$|V\rangle = R(v)\, |0\rangle, \qquad R \in SO(6)$$

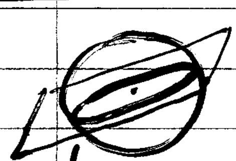
$$|0\rangle : \quad \mathrm{Tr}\left(\phi_1 + i\phi_2\right)^L \quad \text{or} \quad (1, i, 0, 0, 0, 0) \equiv V_0$$

Invariant under: $\quad H = SO(2) \times SO(4)$

Stefanski
A.T.

$$v \in G/H = \boxed{\frac{SO(6)}{SO(2) \times SO(4)}}$$

**Grassmanian** $\;$ **$G_{2,6}$** : $\quad$ 2-planes in $R^6$
or big circles in $S^5$
or moduli space of geodesics in $S^5$



**8-dimensional**: $\quad$ surface in **$CP^5$**

$$V_m : \quad \boxed{|V_m|^2 = 1}, \qquad V_m \in \mathbb{C}^6 \qquad (m = 1, \dots, 6)$$

Extra condition: $\quad \boxed{V_m^2 = 0}$

$$\boxed{L \to \infty, \quad \tilde{\lambda} = \frac{\lambda}{L^2} = \text{fixed}}$$ limit of spin chain

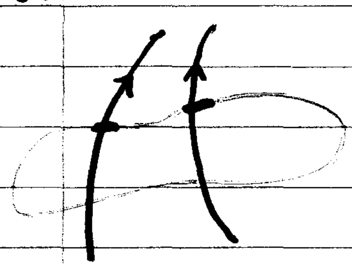
$$\boxed{\mathcal{L} = -i\, V^* \partial_0 V - \frac{\tilde{\lambda}}{2} |D_1 V|^2}$$ LL $\sigma$-model on $G_{2,6}$

How to relate 8-dim LL model to string $\sigma$-model on $R \times S^5$ ?!

Spin chain produces phase-space action!

$(1+5)\times 2 - 2\times 2 = 8$ dim of string $(R\times S^5)$ phase space $= 4+4$

Need to go to phase space to isolate "fast" coordinate $\alpha$ of generic string motion



to define $\alpha$ need to know position and velocity of string bit

$$\mathcal{L} = -(\partial t)^2 + \partial X_m \partial X_m \,, \qquad \dot{X}^2 + X'^2 = \varkappa^2 \,, \quad \dot{X}X' = 0$$

geodesics: $\boxed{X_m(\tau) = a_m \cos\alpha + b_m \sin\alpha}$

$\alpha = \varkappa\tau$, $a_m^2 = 1$, $b_m^2 = 1$, $a\cdot b = 0$

$$\boxed{X_m = \frac{1}{\sqrt{2}}\left(e^{i\alpha} V_m + e^{-i\alpha} V_m^*\right)}$$

$$V_m = \frac{a_m - i b_m}{\sqrt{2}} : \qquad V_m V_m^* = 1, \quad V_m V_m = 0$$

$$\boxed{V_m \in G_{2,6}}$$

General fast motion: $V_m$ slowly changes with $\sigma, \tau$ due to interaction with neighbours (due to string tension)

General case: $\quad X_m X_m = 1$

$$\mathcal{L} = P_m \dot{X}_m - \frac{1}{2} P_m P_m - \frac{1}{2} X'_m X'_m$$

$$\left[\begin{array}{l} X_m = \frac{1}{\sqrt{2}} \left( e^{i\alpha} \underline{V_m} + e^{-i\alpha} \underline{V_m^*} \right) \\ P_m = \frac{i}{\sqrt{2}} P_\alpha \left( e^{i\alpha} \underline{V_m} - e^{-i\alpha} \underline{V_m^*} \right) \end{array}\right.$$

(cf. coherent state parametrization for harm. oscill.)

$$V_m V_m^* = 1 , \quad V_m V_m = 0 , \quad V_m \in G_{2,6}$$

$$\alpha \to \alpha - \beta , \quad V_m \to e^{i\beta} V_m$$

Gauge fixing: $\quad t \sim \tau , \quad P_\alpha \sim L = \text{const}$ or $\tilde{\lambda}$ ~

$$\mathcal{L} = \underset{\downarrow}{P_\alpha D_0 \alpha} - \underline{\frac{1}{2} |D_1 V|^2} - \underset{\downarrow}{\frac{1}{4} \left( e^{2i\alpha} (D_1 V)^2 + c.c. \right)^2}$$

$$C_0 = -i V_m^* \partial_0 V_m$$

averages to zero
$(\alpha = \varkappa \tau + \ldots , \ \varkappa \to \infty)$

Local phase-space action but no local coordinate-space action!

$$\boxed{\mathcal{L} = -i V_m^* \partial_0 V_m - \frac{\tilde{\lambda}}{2} |D_1 V_m|^2 + O(\tilde{\lambda}^2)}$$

$$\tilde{\lambda} = \frac{\lambda}{L^2}$$

Matches spin chain action

SU(3) sector: special case $\to \quad CP^2 \subset G_{2,6}$

$|u_i|^2 = 1 \qquad V = (u_1, iu_1, u_2, iu_2, u_3, iu_3)$

# Gauge theory side: SO(6) spin chain

$$\mathcal{O} = C_{m_1 \dots m_L} \, \mathrm{Tr}\,(\phi_{m_1} \dots \phi_{m_L})$$

1-loop <u>planar</u> dilatation operator

Minahan Zarembo

$$H^{n_1 \dots n_L}_{m_1 \dots m_L} = \frac{\lambda}{(4\pi)^2} \sum_{\ell=1}^{L} \left( \delta_{m_\ell m_{\ell+1}} \delta^{n_\ell n_{\ell+1}} + 2 \delta^{n_\ell}_{m_\ell} \delta^{n_{\ell+1}}_{m_{\ell+1}} - 2 \delta^{n_{\ell+1}}_{m_\ell} \delta^{n_\ell}_{m_{\ell+1}} \right)$$

<u>Coherent state</u> / Semiclassical limit $(L \to \infty)$

Instead of general attempt to minimize

$$S = \int dt \left( i \, C^*_{m_1 \dots m_L} \frac{d}{dt} C_{m_1 \dots m_L} + C^*_{m_1 \dots m_L} H^{m_1 \dots m_L}_{n_1 \dots n_L} C^{n_1 \dots n_L} \right)$$

( $C_{\dots} = C_{\dots}(t)$, RG evolution of couplings)

Consider factorized ansatz: $|V_m|^2 = 1$ KT

$$\boxed{C_{m_1 \dots m_L} = V_{1\, m_1} \dots V_{L\, m_L}} \qquad \{V_{\ell\, m}\} \quad \ell = 1, \dots, L \quad m = 1, \dots, 6$$

<u>BPS case</u>: $V_\ell = V^{(0)}$, $(V^{(0)})^2 = 0$

$\mathrm{Tr}\,(\phi_1 + i \phi_2)^L$ $\qquad V^{(0)} = (1, i, 0, 0, 0, 0)$

$$\boxed{S = \sum_{\ell=1}^{L} \int dt \left( i V^*_\ell \dot{V}_\ell + \frac{\lambda}{(4\pi)^2} \left[ (V^*_\ell V^*_{\ell+1})(V_\ell V_{\ell+1}) + 2 - 2 (V^*_\ell V_{\ell+1})(V_\ell V^*_{\ell+1}) \right] \right)}$$

If $V_\ell$ changes slowly: <u>potential term</u> $(V^* V^*)(VV)$

vanishes in BPS case but spoils low-energy expansion in gen[eral]

**Demand**

$$\boxed{V_\ell^2 = 0} \qquad \ell = 1, \dots, L$$

Minimizes potential energy

Equivalent to "locally BPS" condition:

$$\delta \left( v_m \phi_m \right) = i \; \bar{\varepsilon} \left( V_m \Gamma_m \right) \psi$$

$$v^2 = 0 \quad \to \quad \left( V_m \Gamma_m \right)^2 = 0 \quad : \quad \tfrac{1}{2} \text{ BPS}$$

$$\boxed{\mathcal{O} = \mathrm{Tr} \left( \prod_{\ell=1}^{L} V_{\ell m} \phi_m \right) = \mathrm{Tr} \left( \hat{\phi}_1 \dots \hat{\phi}_L \right)}$$

$$\hat{\phi}_\ell = V_{\ell m} \phi_m = R(v) \, \phi_{(0)}$$

coherent operators $V \in G_{2,6}$

Then continuum limit $\boxed{L \to \infty, \; \tilde{\lambda} = \frac{\lambda}{L^2} = \text{fixed}}$

is well-defined:

gives $G_{2,6}$ LL $\sigma$-model : $V(t,\sigma) = V(t, \frac{2\pi}{L}\ell)$, $\ell = 1, \dots, L$

$$\boxed{S = -L \int dt \int \frac{d\sigma}{2\pi} \left( i \, V_m^* \partial_t V_m + \frac{\tilde{\lambda}}{2} \left| D_1 V_m \right|^2 \right)}$$

$$D_1 V = \partial_1 V - i \, C_1 V \,, \qquad C_a = -i \, V^* \partial_a V$$

$$|V|^2 = 1 \,, \quad V^2 = 0$$

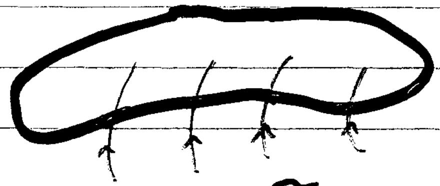
No potential term for $V^2 = 0$, higher deriv. $\sim O(\frac{1}{L})$

String solution $V_m(\tau, \sigma)$ $\longleftrightarrow$ coherent state / operator with $\hat{\phi}_\ell(t) = V_m(t,\sigma) \, \phi_m$

"pure" quantum string states $\longleftrightarrow$ eigenstates of $D$

semiclassical / coherent states $\longleftrightarrow$ coherent states of

time evolution $\longleftrightarrow$ RG evolution of coeffs.

 $\longleftrightarrow$ $\mathrm{Tr}\left(\prod_{\sigma} V_m(t,\sigma)\,\phi_m\right)$

$L\to\infty$, $\tilde{\lambda}\to 0$ (ultrarelativistic) ("locally BPS" $V^2=0$)

precise relation between string states and SYM (coherent) operators

Explicit examples: rotating + pulsating strings as solutions of LL $\sigma$-model  K.T.

Summary:

- each portion of string $\rightarrow$ linear combination $V_m(t,\sigma)\,\phi_m$

  $\sigma \longleftrightarrow \sigma_\ell = \frac{2\pi}{L}\ell$

- fast variable $\alpha$: "polar angle" in $(P,X)$ plane

  $P_\alpha$ large ; conserved after canonical transf to eliminate $\alpha$-dependence

- gauge: $P_\alpha$ = const or uniformly distributed $\rightarrow$ can set $= L$ (or 2-d duality $\alpha\to\tilde{\alpha}$ and $\tilde{\alpha}\sim L\sigma$)

- local action in phase space

  (non-local if attempt to eliminate 1/2 variables)

# Conclusions

- Remarkable generalization of near-BPS $\equiv$ BMN limit to non-BPS but "locally BPS" sector of states or "fast" strings:
  progress in quantitative understanding of gauge - string duality
- Explicit map between string / SYM:
  effective actions (in phase space) match $\Rightarrow$
  - string states $\longleftrightarrow$ SYM operators
  - $E(\lambda, L, \ldots) = \Delta(\lambda, L, \ldots)$
- Importance of phase space picture suggested by SYM / spin chain side:
  lessons for further exploration of spectrum
- Use string theory as a guide to SYM structure:
  use particular examples of matching to fix general structure of dilatation operator, exact Bethe ansatz, spectrum, etc.

Example: demand BMN limit   Ryzhov, A.T.

$\longrightarrow$ fix part of $D$ to all orders in $\lambda$

## SU(2) sector: $\mathrm{Tr}\,(\Phi_1 \dots \Phi_2 \dots)$

Motivated by results of Beisert, Kristjansen, Staudacher

Beisert 0308074

Serban, Staudacher

(and also Gross, Mikhailov, Roiban)

organize dil. op. as expansion in powers of

$$\boxed{Q_{k,l} = I - \vec{\sigma}_k \vec{\sigma}_l}$$ : in powers of "spin-spin" interactions

$$D = \sum_{r=0}^{\infty} \lambda^r \mathcal{D}_r$$ loop expansion

$$\mathcal{D}_r = \sum_{k=1}^{L} \sum_{l=1}^{L-1} a_{r,l}\, Q_{k,k+l} + \sum QQ + \sum QQQ + \dots$$

each loop order — starts at 3 loops — 4-loops

Impose regularity of $L \to \infty$, $\tilde{\lambda} = \frac{\lambda}{L^2}$ = fixed limit

(and BMN spectrum): **fix order Q-terms!**

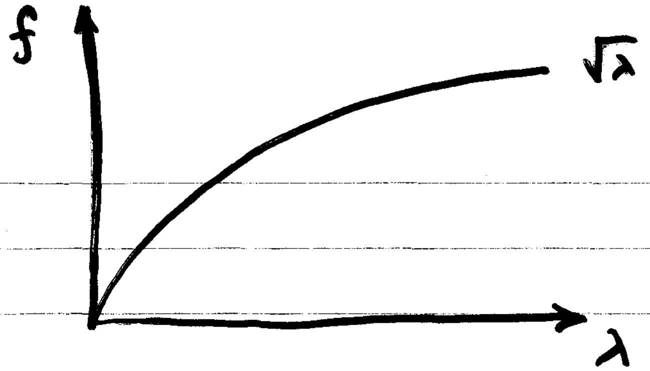
Asymptotic $L \to \infty$ limit:

$$\boxed{D = \sum_{l=1}^{\infty} f_l(\lambda) \sum_{k=1}^{L} Q_{k,k+l} + \sum QQ + \dots}$$

$$f_l(\lambda) = \left(\frac{\lambda}{4\pi^2}\right)^l \frac{\Gamma(l-\frac{1}{2})}{4\sqrt{\pi}\,\Gamma(l+1)}\; {}_2F_1\left(l-\tfrac{1}{2}, l+\tfrac{1}{2}; 2l+1; -\frac{\lambda}{\pi^2}\right)$$

$$f_l(\lambda \to 0) \simeq a_1 \lambda + a_2 \lambda^2 + \dots$$

$$f_l(\lambda \to \infty) \simeq l\sqrt{\lambda} + \dots$$ as expected for "string" operators




"interpolating function"

$L < \infty$ : need $QQ$ etc. terms $\rightarrow$ $\sqrt[4]{\lambda}$ scaling

Use comparison with string theory in other special cases to fix $QQ$, etc ?
( cf. fixing coeffs in effective action )

String - gauge theory duality:
get <u>much more</u> than put in !