Quantum Foam, Topological Strings and Black Holes

Cumrun Vafa

Strings '04

Based on:

Okounkov, Reshetikhin, V. hep-th/0309208

Iqbal, Nekrasov, Okounkov, V. /0312022

Neitzke, V. /0402128

Nekrasov, Ooguri, V. /0403167

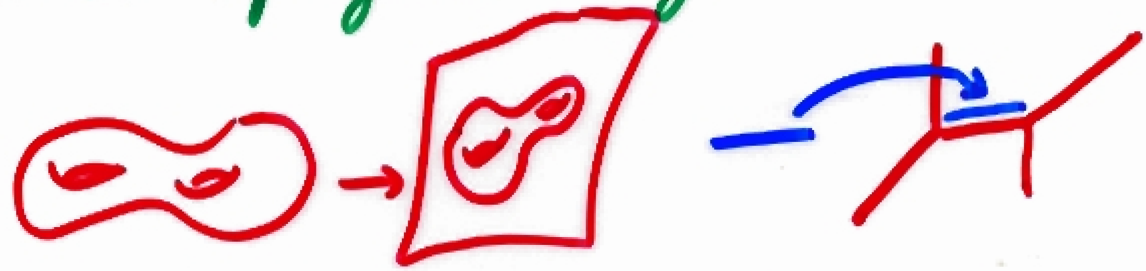
} See Dijkgraaf + Nekrasov's talks

Ooguri, Strominger, V. /0405146 } See Strominger's talk

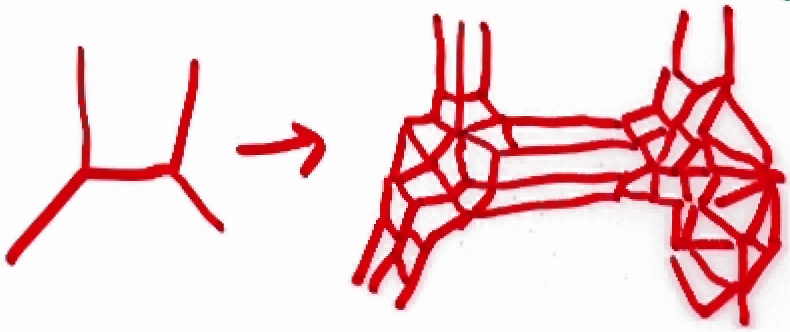
V. /0406058

Plan of my talk:

- Review of topological strings



- Quantum foam interpretation of topological strings



- 5d Black holes $\longleftrightarrow$ perturbative topological string

$$Z_{BH}^{(5)} = Z_{per}^{top}$$

- 4d Black holes $\longleftrightarrow$ Non-perturbative topological strings

$$Z^{(4)} = |Z_{top}|^2$$

- Example: (Elliptic 3-fold)

2d YM $\longleftrightarrow$ 4d Blackhole

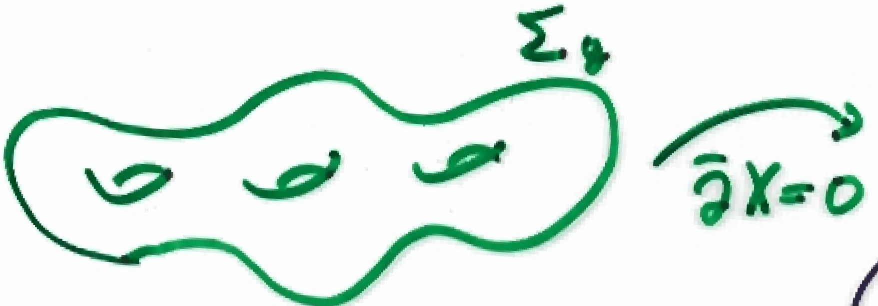
2d YM $\xrightarrow{\text{'t Hooft limit}}$ pert. top string

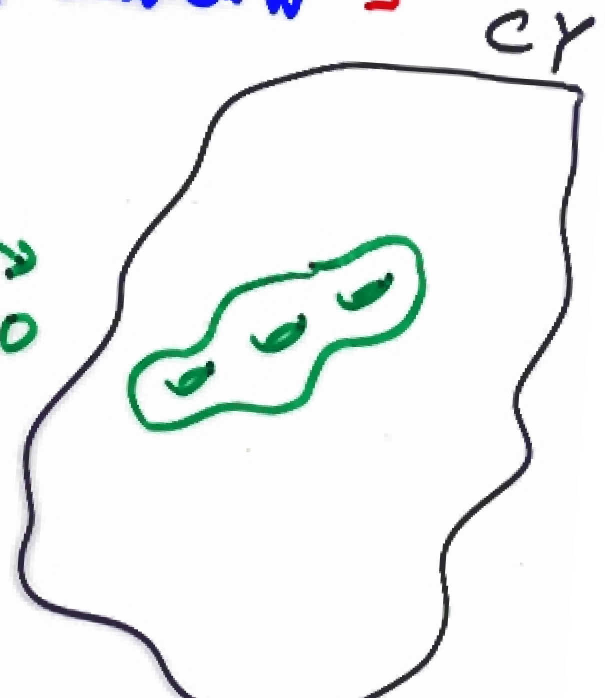
quantum foam $\cong$ pert. top string $\cong$ 5d Blackhole

Review of Topological Strings

Here we will mainly discuss A-model topological string on CY. [W]

$$\left[\begin{array}{ccc} \text{A-model}_M & \xleftrightarrow[\text{Sym.}]{\text{mirror}} & \text{B-model}_W \\ \text{S-duality} \updownarrow \lambda\to\frac{1}{\lambda} & & \text{S-duality} \updownarrow \lambda\to\frac{1}{\lambda} \\ \text{B-model}_M & \xleftrightarrow[\text{Sym.}]{\text{mirror}} & \text{A-model}_W \end{array}\right]$$





$$Z_{top}^{pert} = \exp\left[\sum_{g=0}^{\infty} \lambda^{2g-2} F_g(t)\right]$$

$$F_g(t) = \sum_{\substack{\text{hol.}\\ \text{maps}\\ \text{from genus } g}} e^{-A_{\Sigma_g}(t)}$$

t: Kähler moduli of CY

Despite tremendous reduction to holomorphic maps ⟶ Still hard to compute. For generic CY only low genus F_g has been computed (using mirror sym.)
[BCOV]

For non-compact (toric) CY ⟶ complete sol'n to perturbative topological string using: Topological Vertex formalism
[AKMV]

which in turn was arrived at using large N-duality of C.S. ⟶ top. string
[GV]

and recently rederived using mirror symmetry.
[ADKMV]

Toric CY

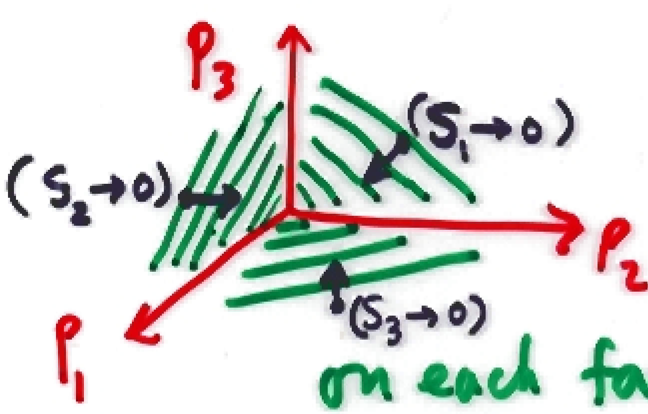
Toric CY: CY as the phase space of a particle on T^3 with restrictions on momenta:

$$\mathbb{C}^3 : (z_1, z_2, z_3) \longrightarrow (|z_1|^2, |z_2|^2, |z_3|^2, \theta_1, \theta_2, \theta_3)$$

$$(p_1, p_2, p_3, \theta_1, \theta_2, \theta_3)$$

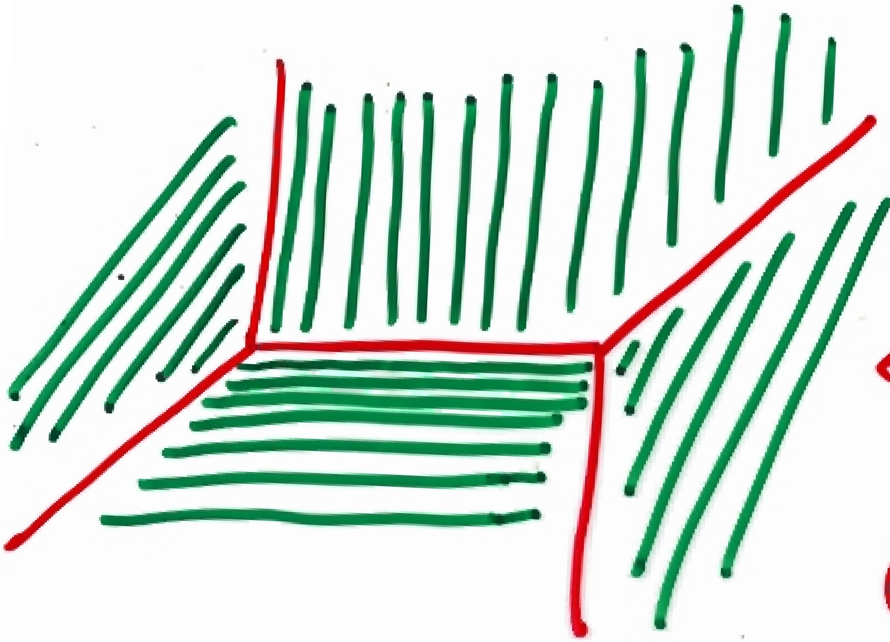
$$k = \sum_i dz_i \wedge d\bar{z}_i \longrightarrow \sum_i dp_i \wedge d\theta_i$$

Note: $\underline{p_i \geq 0}$ $\quad \{\theta_i\} \in T^3$



$\mathbb{C}^3 \rightarrow$ positive octant

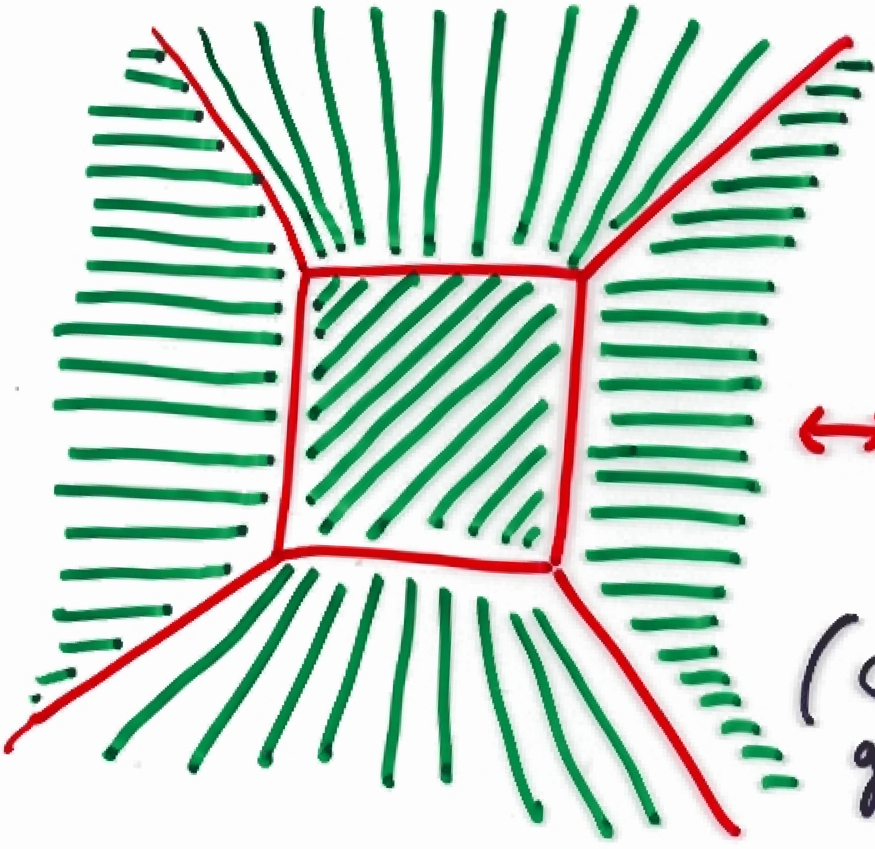
on each face $T^3 \rightarrow T^2$



$\longleftrightarrow$

$$\begin{array}{c}\mathcal{O}(-1)\oplus\mathcal{O}(-1)\\ \downarrow \\ \mathbb{P}^1\end{array}$$

(resolution of conifold)



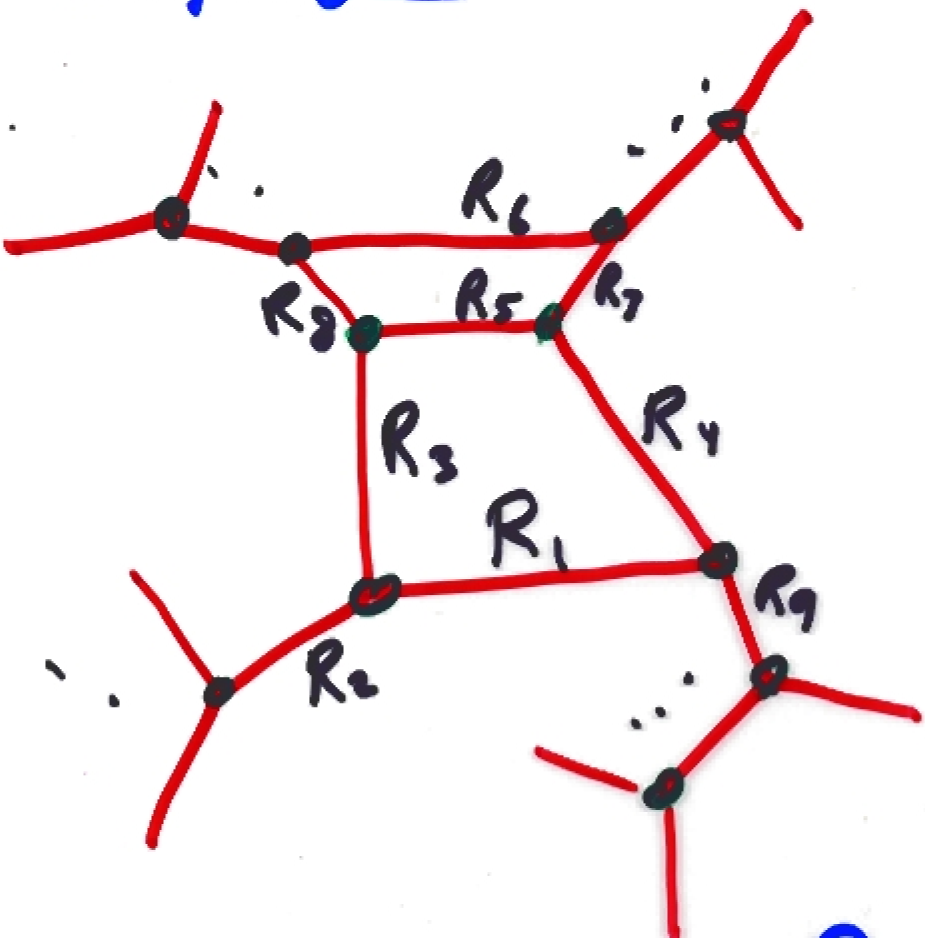
$\longleftrightarrow$

$$\begin{array}{c}\mathcal{O}(-2,-2)\\ \downarrow \\ \mathbb{P}^1\times\mathbb{P}^1\end{array}$$

(can be used for geometric engineering of $\mathcal{N}=2$, $SU(2)$ in $d=4$)

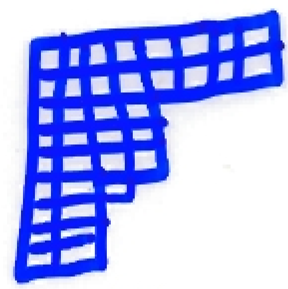
$\vdots$

Topological Vertex Formalism



trivalent
"Feynman"
diagrams

To each edge $\longrightarrow$ R_i



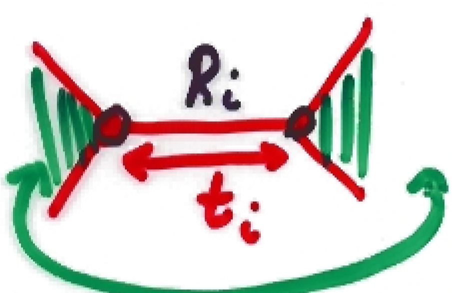
2d Young diagram

Propagator $\rightarrow e^{-t_i|R_i| - m_i \lambda K(R_i)}$

$|R_i|$ = # boxes, $K(R_i)$ Casimir

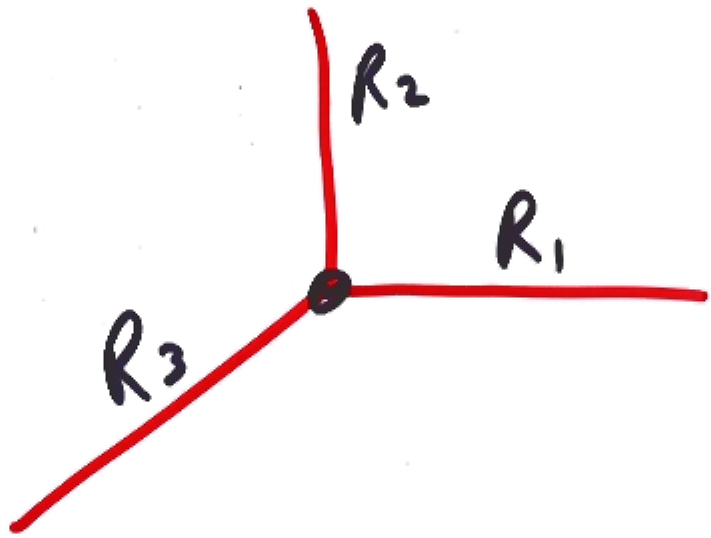
t_i : length

m_i : relative orientation of faces



λ: string coupling

relative orientation : m_i



To each vertex

$$C_{R_1 R_2 R_3}(q) \text{ : rational fn. of } q$$

$$q = e^{-\lambda}$$

$$\text{“}\sum_{R_i}\text{”} \rightarrow Z_{top}^{(pert.)}$$

Reinterpretation of Topological Vertex

as

Quantum Foam

Target space theory of A-model

$\downarrow$

Kähler gravity on CY

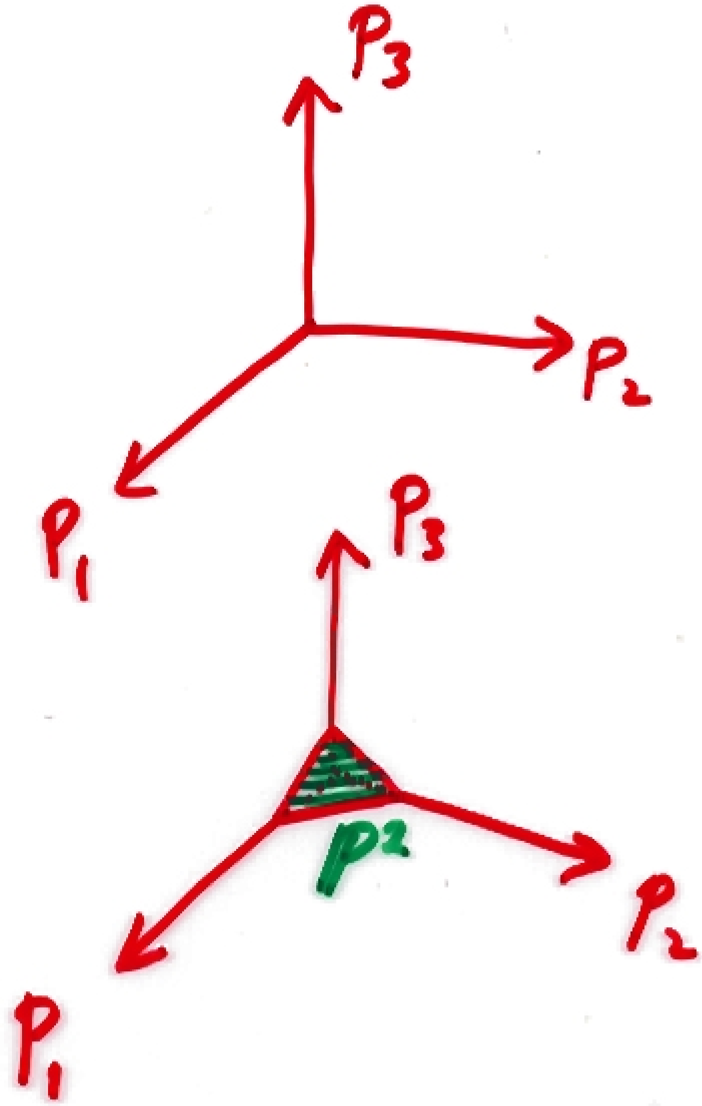
We expect

$$Z_{target} = \text{``}\sum_{\substack{\text{Kähler}\\ \text{geometries + topologies}}} e^{-S}\text{''}$$

$$S = \frac{1}{\lambda^2}\int k \wedge k \wedge k$$

For example:

$\mathbb{C}^3 \longrightarrow$



$\rightsquigarrow$ blowup

" $\mathcal{O}(-1)$ "

$\downarrow$

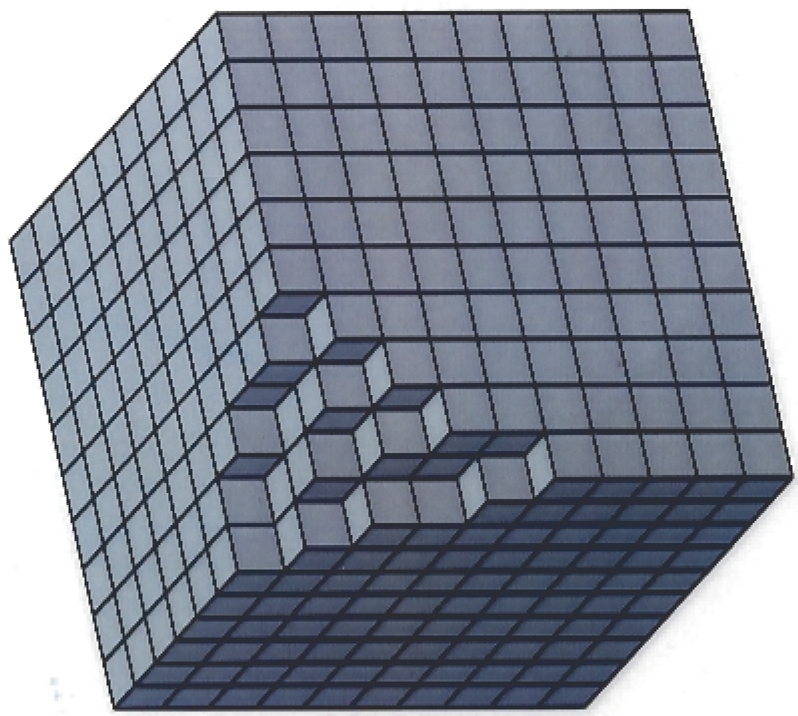
$\mathbb{P}^2$

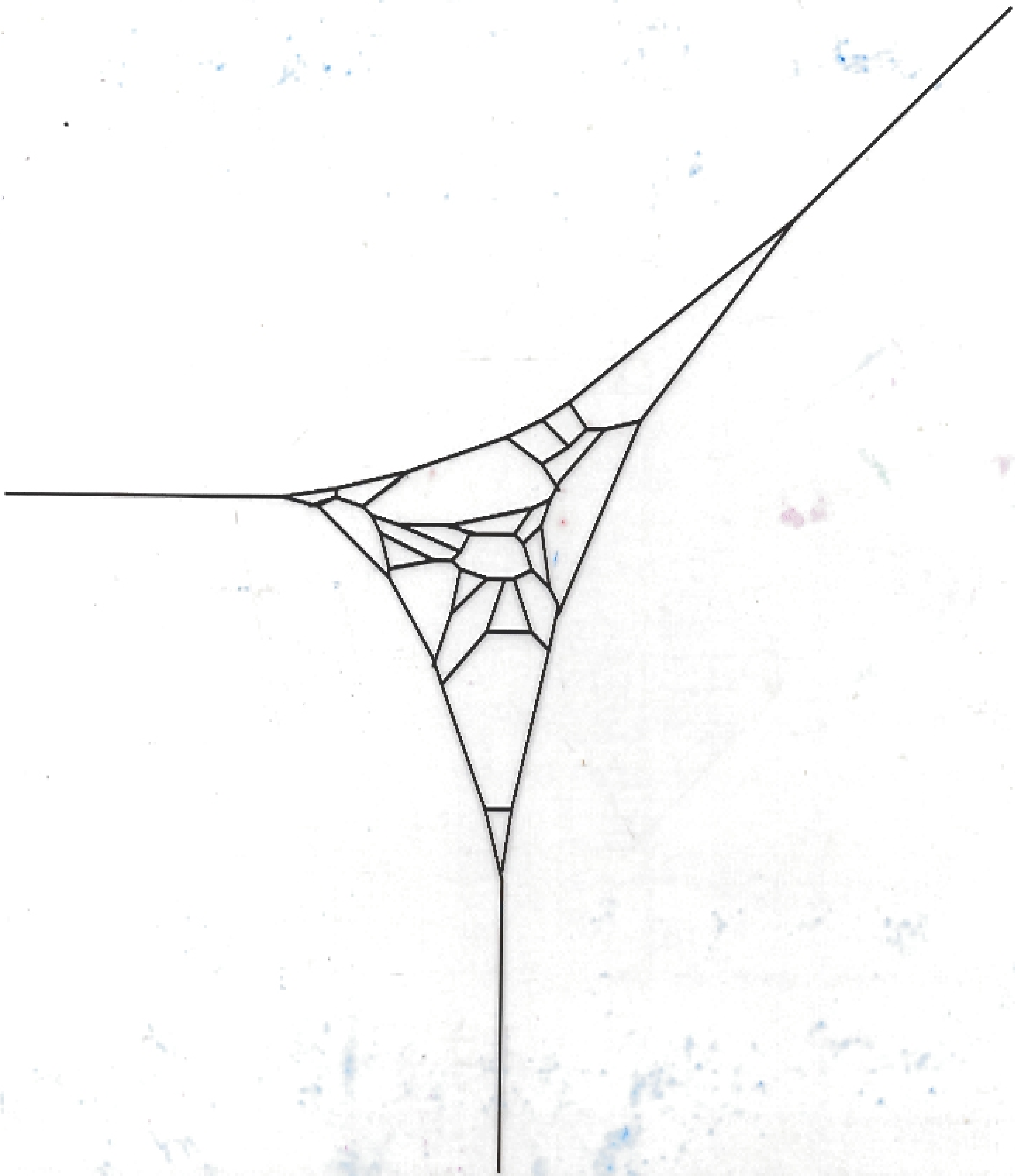
$$\Delta S = \text{Action cost: } \frac{1}{\lambda^3}\int k\wedge k\wedge k = \frac{1}{\lambda^3}\int_{\text{deleted region}} dp_1\, dp_2\, dp_3$$

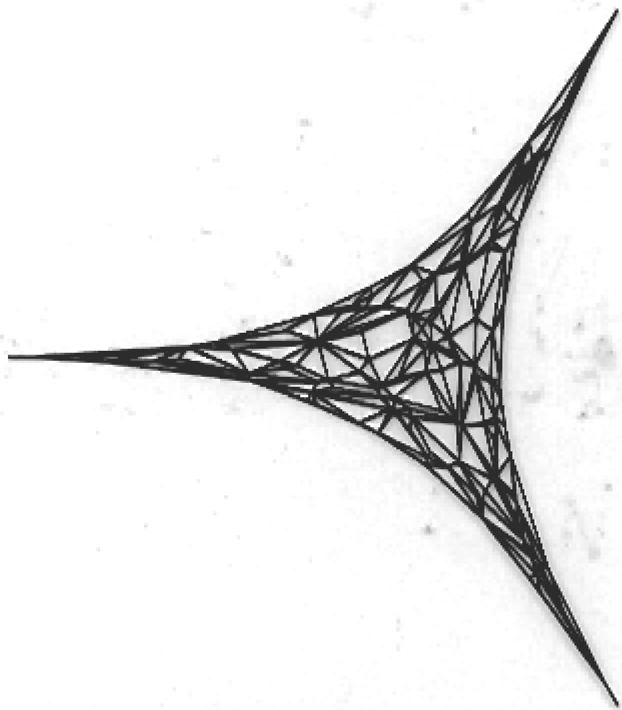
$\hat{p}_i = \dfrac{p_i}{\lambda}$ "Planck units"

$$\Rightarrow \Delta S = -\lambda \hat{V} \Rightarrow e^{\Delta S} = q^{\hat{V}} = q^N$$

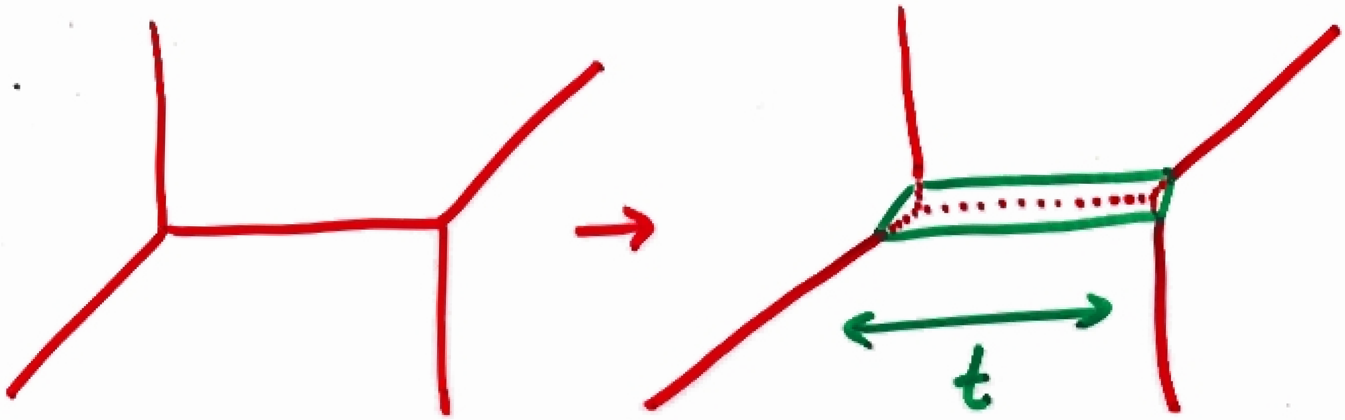
N: number of points deleted







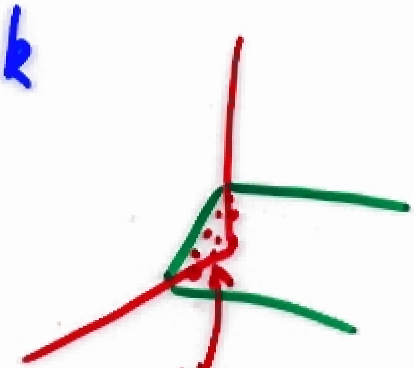
We can also blow up along curves:



Action cost:

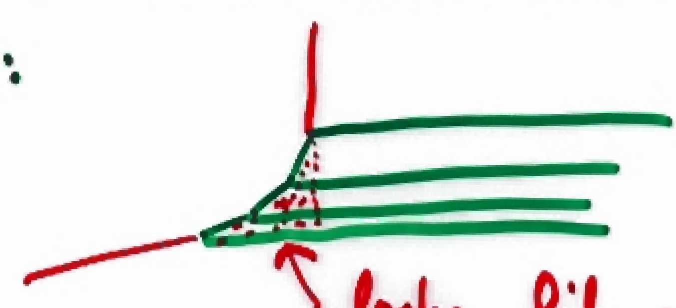
$$\Delta\left[\frac{1}{\lambda^2}\int k\wedge k\wedge k\right] = \int \frac{k}{\lambda}\wedge\frac{k}{\lambda}\wedge k$$

$$= N t$$



number of transverse deleted points $= N$

we can also have more complicated blowups:



looks like a 2D Young diagram

$$Z_{top}^{pert.} = \sum_{\substack{(\pm) \\ \text{toric blowups}}} q^{n_0} e^{-t_i n_2^i}$$

toric blow ups
along points (n_0)
+ curves (n_2^i)
subject to $\int_{C_2} k = m\lambda$

$\updownarrow$ Bohr - Sommerfeld quantization

$$\int \frac{dp\,dq}{2\pi} = m\hbar$$

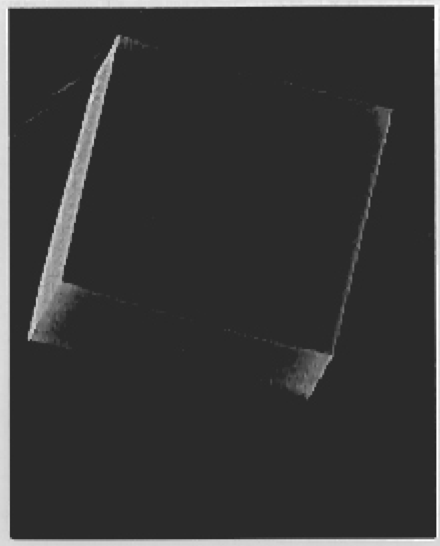
$$C_{R_1 R_2 R_3}(q) = \sum_{(\pm)} q^{n_0}$$

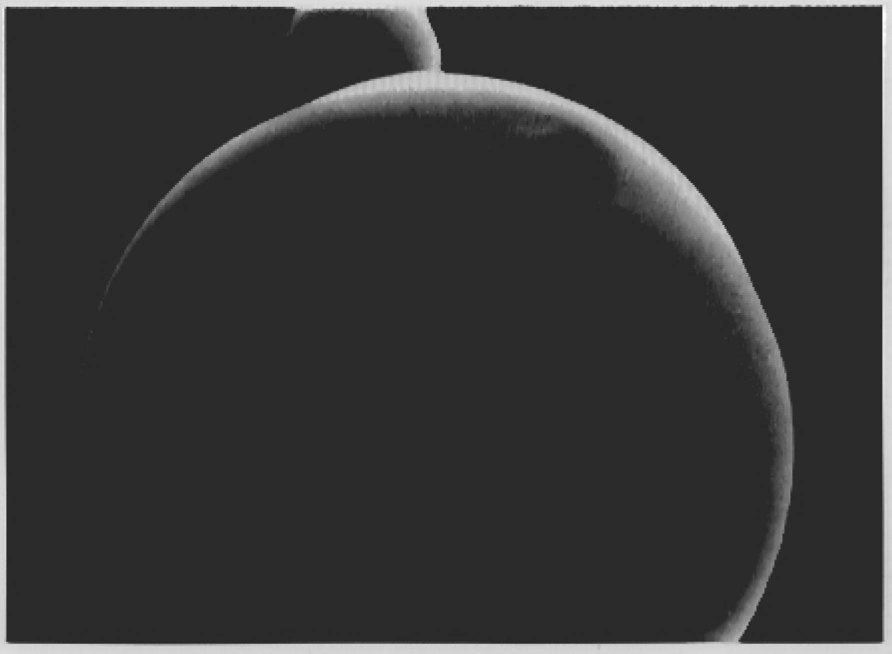
blow up of points in the geometry

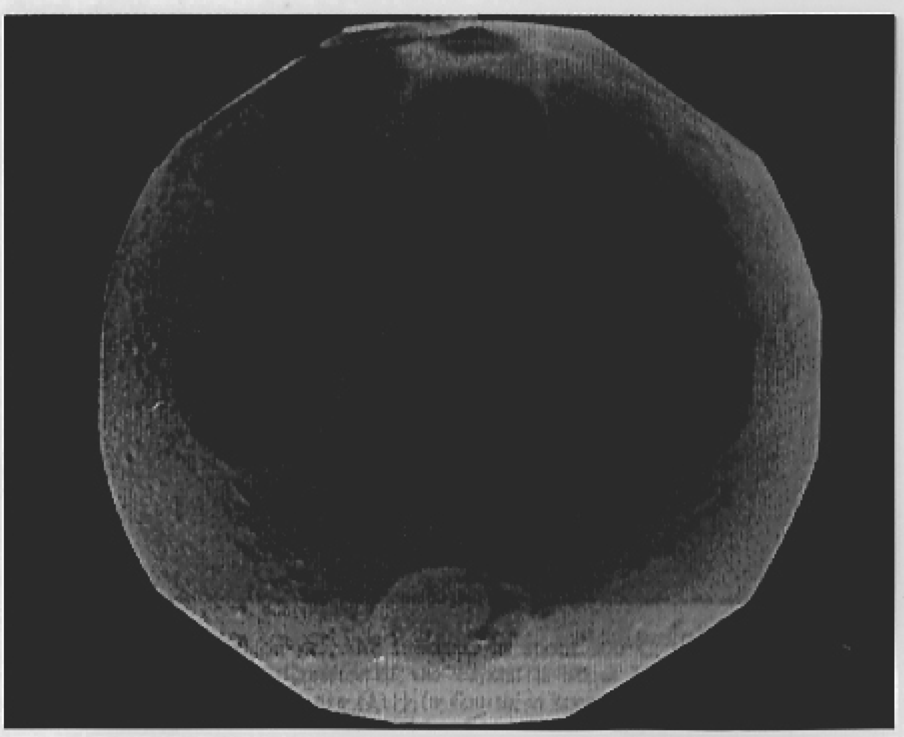
In the S-dual B-model

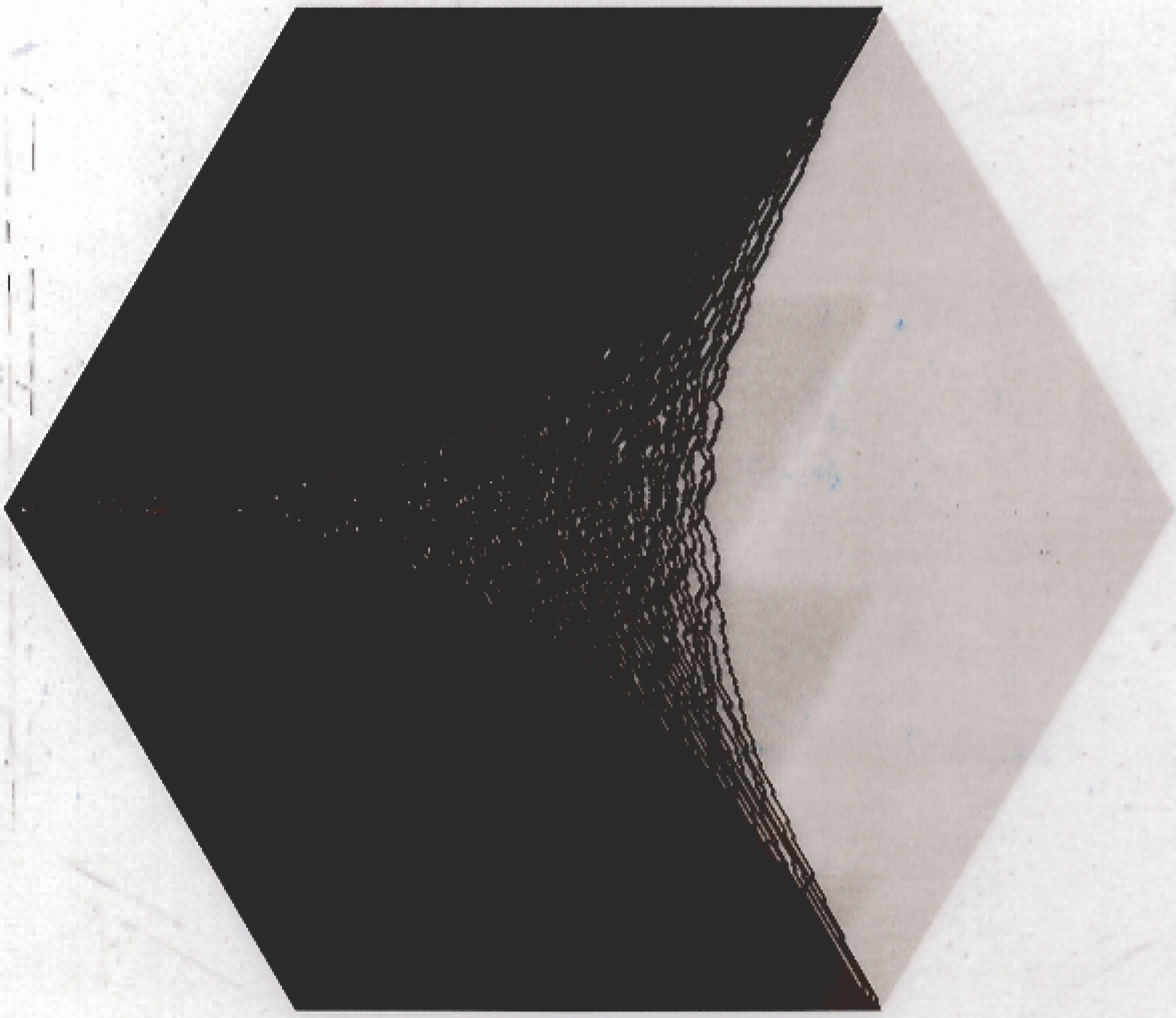
$$\begin{cases} \text{Blowup along points} \longleftrightarrow D_{-1}\text{-instanton} \\ \text{Blowup along curves} \longleftrightarrow \text{D1-brane instantons} \end{cases}$$

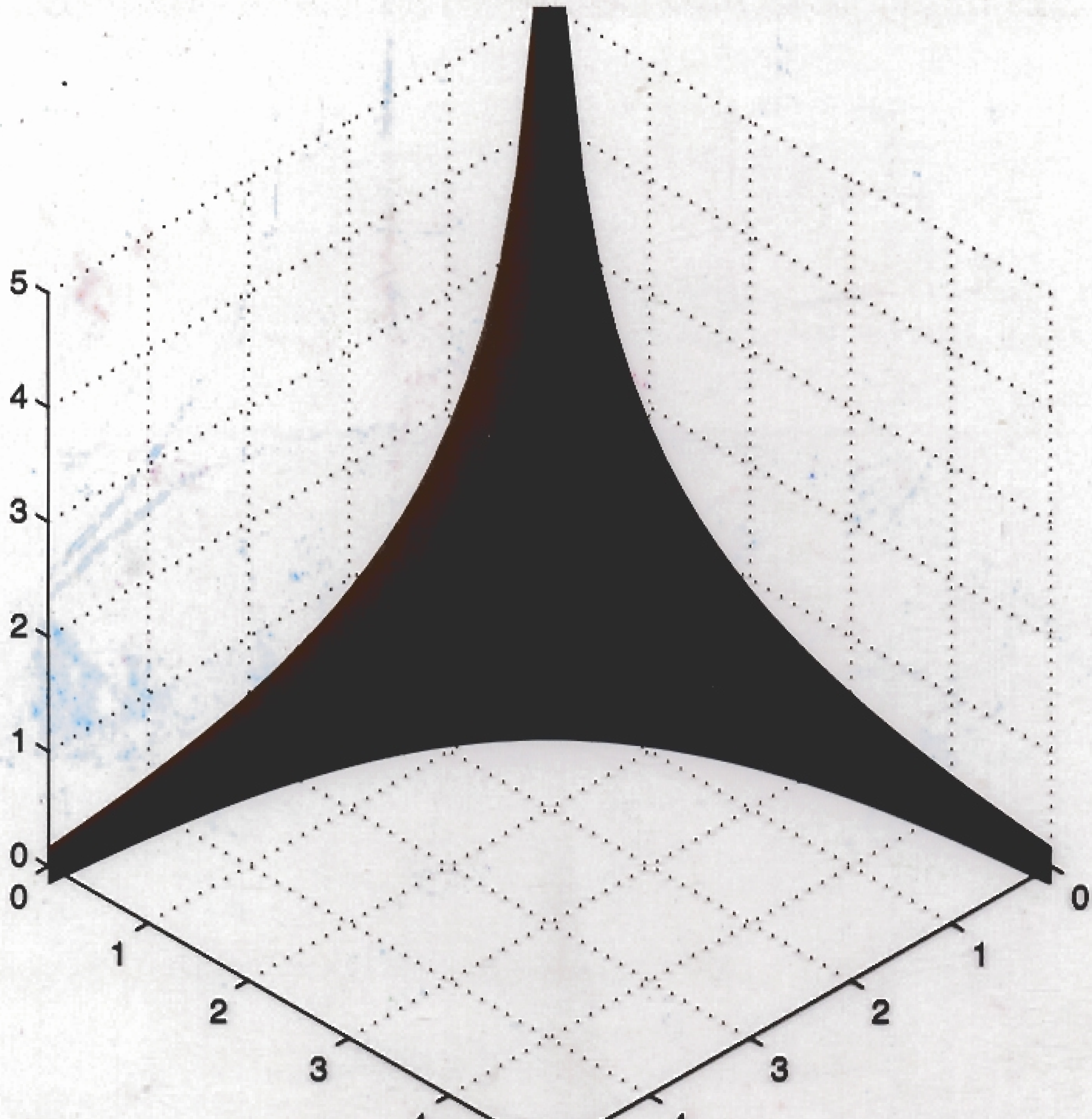
"explains" quantization of k.





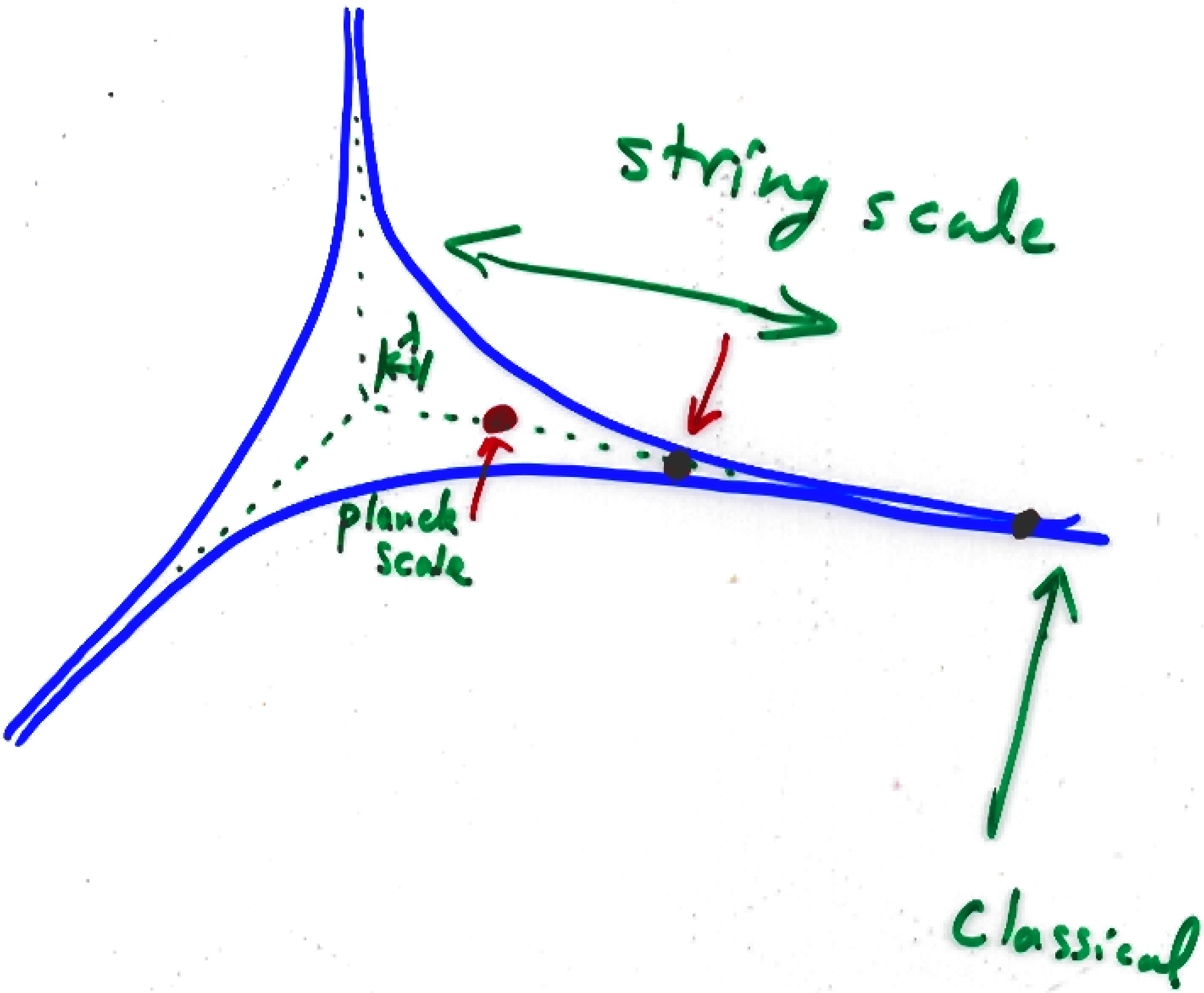






Kenyon et. al.

Okounkov, Reshetikhin



string scale
planck
scale
Classical

Topological Strings

vs.

Superstrings

Main link: [BCOV], [AGNT]

$$F_g(t) \longrightarrow \int d^4x\, d^4\theta\; F_g(T)(\mathcal{W}^2)^g$$

$\mathcal{W}$: graviphoton field

[$F_0(T) \rightarrow$ prepotential for $\mathcal{N}=2$ SUSY $d=4$]

[open string version $\sum F_{0,h} S^h \longrightarrow$ superpot. $\mathcal{N}=1$, $d=4$]

More interesting to give v.e.v. to $\mathcal{W}^2$

then $\sum F_g(\mathcal{W}^2)\partial$ sums up all genus.

(BPS) Black hole

[GV]	<u>5d</u>	<u>4d</u> [C.dWM]
	M-theory	IIA
	CY	CY
	$\times$	$\times$
	R^5	R^4

M2 branes wrapping 2-cycles of CY $\leftrightarrow$ Spinning B.H.

D0, D2, D4, D6 branes wrap CY $\leftrightarrow$ Charged B.H.

5d Case

[G.V]

M: $CY \times R^5$

$SO(4)$ = Rotation group = $SU(2)_L \times SU(2)_R$

$$\underbrace{\mathrm{Tr}}_{\text{second quantized trace}} (-)^F q^{J_L^3} \exp(-t_i \underbrace{Q_i}_{\text{charges of M2 brane}})$$

$$= Z_{top}^{pert}(\lambda, t_i)$$

$q = e^{-\lambda}$

t_i = Kähler parameters

Note: This sum has the same structure as quantum foam

$\Rightarrow$ Quantum foam top. string $\Leftrightarrow$ 5d B.h. degeneracies

4d Black Holes

$$Z^{\text{mixed ensemble}}_{BH,\,\text{index}} \equiv \sum_{n_0,\,n_2^i} (\pm) \exp\left[-2\pi^2 n_0\left(\frac{1}{\lambda}+\frac{1}{\bar\lambda}\right) - \frac{\pi}{i} n_2^i\left(\frac{t^i}{\lambda}-\frac{\bar t^i}{\bar\lambda}\right)\right] \cdot \Omega(n_0, n_2^i, n_4^i, n_6)$$

fix magnetic charges n_4^i, n_6
chemical potential for n_0, n_2^i electric charges

$$Z_{BH} = Z_{top}(t,\lambda)\,\bar Z_{\overline{top}}(\bar t,\bar\lambda)$$

$$n_6 = 2\pi i\left[\frac{1}{\lambda}-\frac{1}{\bar\lambda}\right]$$

$$n_4^i = \frac{t^i}{\lambda} + \frac{\bar t^i}{\bar\lambda}$$

Attractor mechanism
charge $\leftrightarrow$ moduli
[FKS]

$Z_{BH} = |Z_{top}|^2$ should hold to all order in $(n_4^i, n_6) \gg 1$

Defines

$\rightarrow$ there could be e^{-N} effects $\rightarrow |Z_{top}|^2$

How to check this?

$$Z_{BH} \underset{\uparrow}{=} Z_{Brane}$$

microcanonical def'n of BH

$$Z_{Brane} \xrightarrow[\text{'t Hooft limit}]{} |Z^{pert}_{top}|^2$$

$N \gg 1$

We will consider a case where

$n_6 = 0$ $(\lambda = \bar{\lambda})$ $\quad n_4 = \text{fixed}$ $\left(t = \frac{n_4 \lambda}{2} + i\theta\right)$ $\quad n_2, n_0$ induced

D4-brane wrapping 4-cycle

$$Z_{Brane} = Z^{\text{top. twisted}}_{\mathcal{N}=4,\ d=4}$$

[VW]

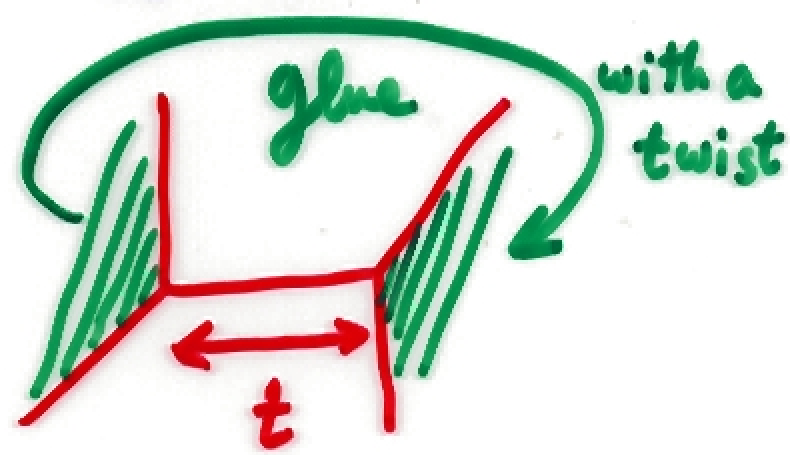
We then check if

$$Z^{\text{top. twised}}_{\mathcal{N}=4} \rightarrow \left| Z^{\text{pert.}}_{\text{top. string}} \right|^2$$

We consider local CY geometry

$$\mathcal{L} \oplus \mathcal{L}^{-1}$$
$$\downarrow$$
$$T^2$$

$\mathcal{L}$: Θ-line bundle



$$Z^{\text{pert.}}_{\text{top. string}} = \sum_R \exp\left(-\frac{\lambda}{2}\kappa(R) - |R|t\right)$$

twisting

propagator

consider N D4-branes

wrapped over C_4 : $\begin{array}{c}\mathcal{L}^{-1}\\ \downarrow\\ T^2\end{array}$

$$Z_{BH} = Z_{C_4}^{\mathcal{N}=4}\left(\begin{array}{c}\text{chemical potential}\\ \text{turned on}\end{array}\right)$$

induces D0 ⟷ ⟷ induces D2

$$\frac{1}{2\lambda}\int_{C_4} \mathrm{tr}\, F\wedge F + \frac{t-\bar{t}}{\lambda}\int_{C_4}\mathrm{tr}\, F\wedge\omega$$

where $\displaystyle\int_{T^2}\omega = 1$

$\displaystyle\int_{\text{fiber } \mathcal{L}^{-1}} F = \Phi$

reduces to $\mathcal{N}=1^{*}$, reduces to (2,2) top. twisted on T^2

$$\frac{1}{\lambda}\int_{T^2}\mathrm{tr}\, F\wedge\phi + \frac{t-\bar{t}}{\lambda}\int_{T^2}\mathrm{tr}\,\phi + \int_{T^2}\frac{1}{2}\mathrm{tr}\,\phi^2$$

$$Z_{BH} = Z^{\mathcal{N}=4}_{C_4} = Z^{(2,2)}_{T^2} = Z^{YM}_{T^2}$$
[W]

$$\rightarrow \frac{1}{\lambda} \int_{T^2} \frac{\mathrm{Tr} F^2}{2} + \theta \, \mathrm{Tr} F$$

large N limit studied

[G]
[GT]
⋮

$$Z^{YM}_{T^2} \sim |Z_{chiral}|^2$$

$$Z_{chiral} = \sum_R \exp\left(-\frac{\lambda}{2} C_2(R) + i\theta |R|\right)$$

$$= \sum_R \exp\left(-\frac{\lambda}{2} K(R) + \left(\frac{N\lambda}{2} + i\theta\right)|R|\right)$$

$$= Z^{pert}_{top}\Big(\lambda, \underbrace{t = \frac{N\lambda}{2} + i\theta}_{\text{exactly as predicted!}}\Big)$$

$$Z_{BH} = |Z^{pert}_{top}|^2$$

Note :

$\mathcal{N}=4$ YM on C_4 (in confining phase $\mathcal{N}=1^*$) $\longleftrightarrow$ $e^{-\frac{\frac{1}{2}\int F^2}{T^2\lambda}}$

4D ensemble = path-integral

Lag.

Olive-Montonen

$\lambda \to \frac{1}{\lambda}$ (+ Large N)

Hamiltonian

$\mathcal{N}=4$ YM on C_4 (in Higgs branch of $\mathcal{N}=1^*$)

$$\sum e^{-\frac{\lambda}{2} C_2(R)}$$

5D ensemble $\parallel$ quantum foam

e.g. $N=3 \to U(3)$

$$\text{Lag} \to Z^{4D}_{BH} = \frac{1}{6}\left[\theta^3(1/\lambda) - 3\theta(1/2\lambda)\theta(1/\lambda) + 2\theta(1/3\lambda)\right]$$

$$\sum e^{-\frac{\lambda}{2}C_2(R)} = \text{Hamiltonian} \qquad = \frac{1}{6}\left[\theta^3(\lambda) - 3\theta(2\lambda)\theta(\lambda) + 2\theta(3\lambda)\right]$$

The fact that

Z_{BH} is <u>not</u> exactly $|Z_{top}^{pert}|^2$

$\longrightarrow$ gives an interesting non-perturbative completion of top. string.

$\longrightarrow$ potentially interesting implications for AdS^2 $\rightarrow$

$Z_{top}\{\overline{}\,\underline{}\}\bar{Z}_{top}$ under study