CLASSICAL/QUANTUM INTEGRABILITY IN AdS/CFT

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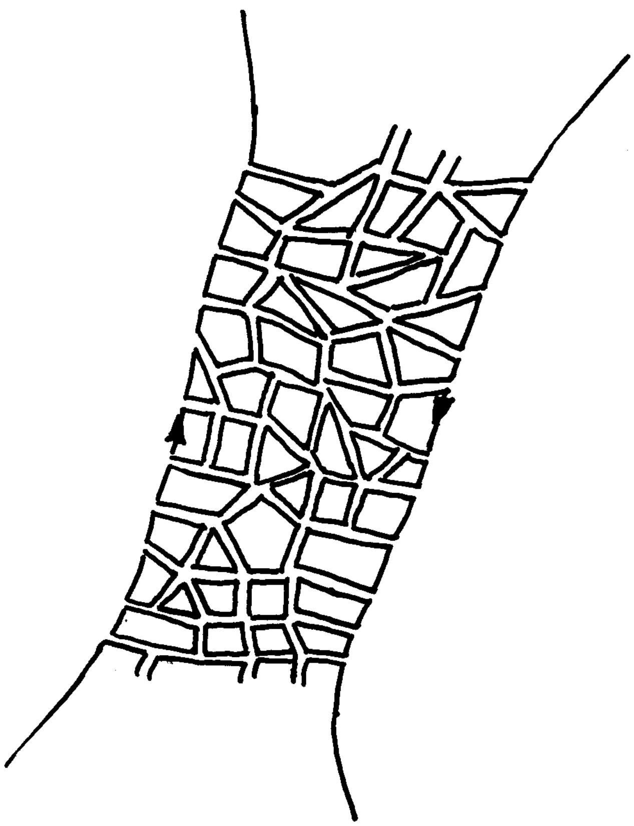
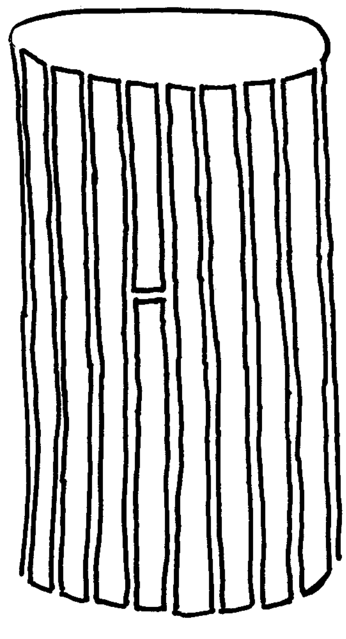
STRINGS 2004, PARIS, 1/7/04

LARGE-N LIMIT:

$$N \to \infty \,, \qquad \underbrace{\lambda = g_{YM}^2 N}_{\text{FIXED}}$$

STRINGS:





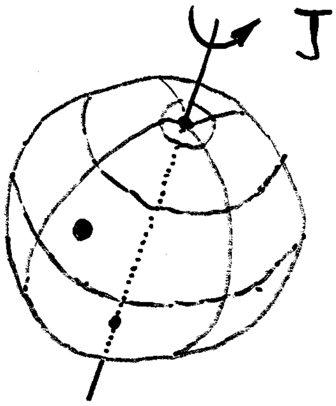
LARGE NUMBER OF VERTICES AND PROPAGATORS ($\lambda \gg 1$)

OR

LARGE NUMBER OF CONSTITUENTS

$$Z = \Phi_1 + i\,\Phi_2$$

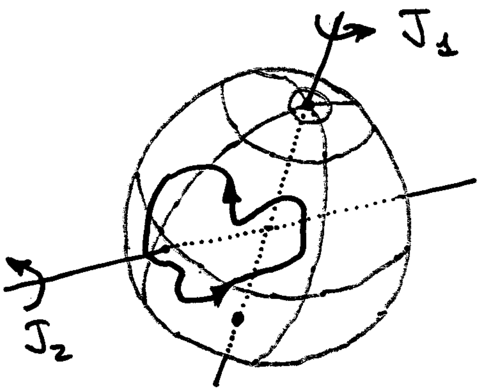
$$W = \Phi_3 + i\,\Phi_4$$



$$\mathrm{tr}\, Z^J \quad (J \gg 1) \qquad \Longleftrightarrow \qquad |0, J\rangle$$

$$\sum_{l=0}^{J} e^{\frac{(2l+1)\pi i n}{J+1}}\, \mathrm{tr}\, W Z^l W Z^{J-l} \qquad \Longleftrightarrow \qquad a_n^\dagger a_{-n}^\dagger |0, J\rangle$$

Berenstein, Maldacena, Nastase '02
Gubser, Klebanov, Polyakov '02



$$\mathrm{tr}\left(Z^{J_1} W^{J_2} + \text{perm.}\right)$$

$$J_1 \sim J_2 \gg 1$$

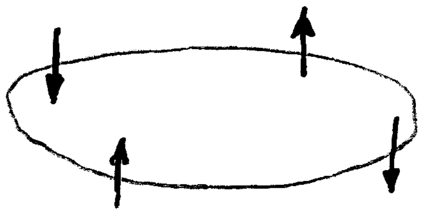
Frolov, Tseytlin '03
Beisert, Minahan, Staudacher, Z. '03

NOTATION:

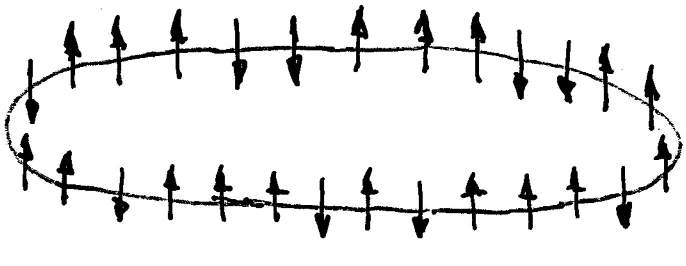
$Z_{ij} \longleftrightarrow$ 

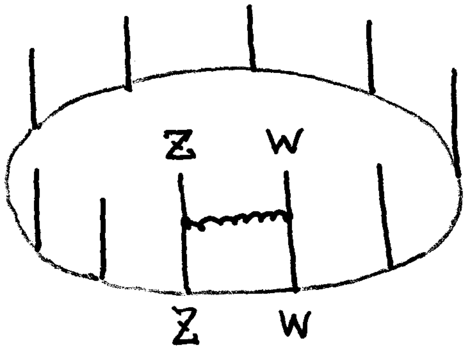
$W_{ij} \longleftrightarrow$ 

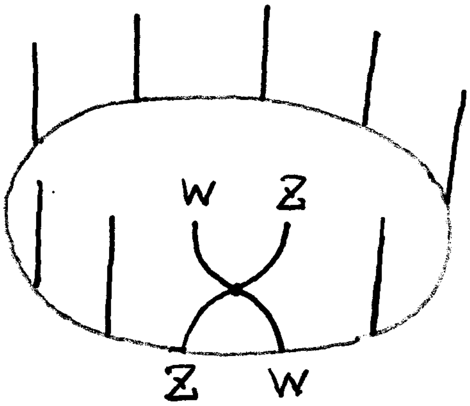
tr ZWZW:



tr ZZWZZZWZWZZZWZZZWWZZZWWZZZW:







$$P\, a \otimes b = b \otimes a$$

MIXING MATRIX:

$$\Gamma = \frac{\lambda}{8\pi^2} \sum_{l=1}^{L} \left(1 - P_{l,l+1}\right) = \frac{\lambda}{16\pi^2} \sum_{l=1}^{L} \left(1 - \vec{\sigma}_l \cdot \vec{\sigma}_{l+1}\right)$$

Minahan, Z. '02

THIS IS THE HAMILTONIAN OF THE HEISENBERG SPIN CHAIN

$$\Gamma |\mathcal{O}_n\rangle = \gamma_n |\mathcal{O}_n\rangle$$

$\mathcal{O}_n$ – CONFORMAL OPERATORS

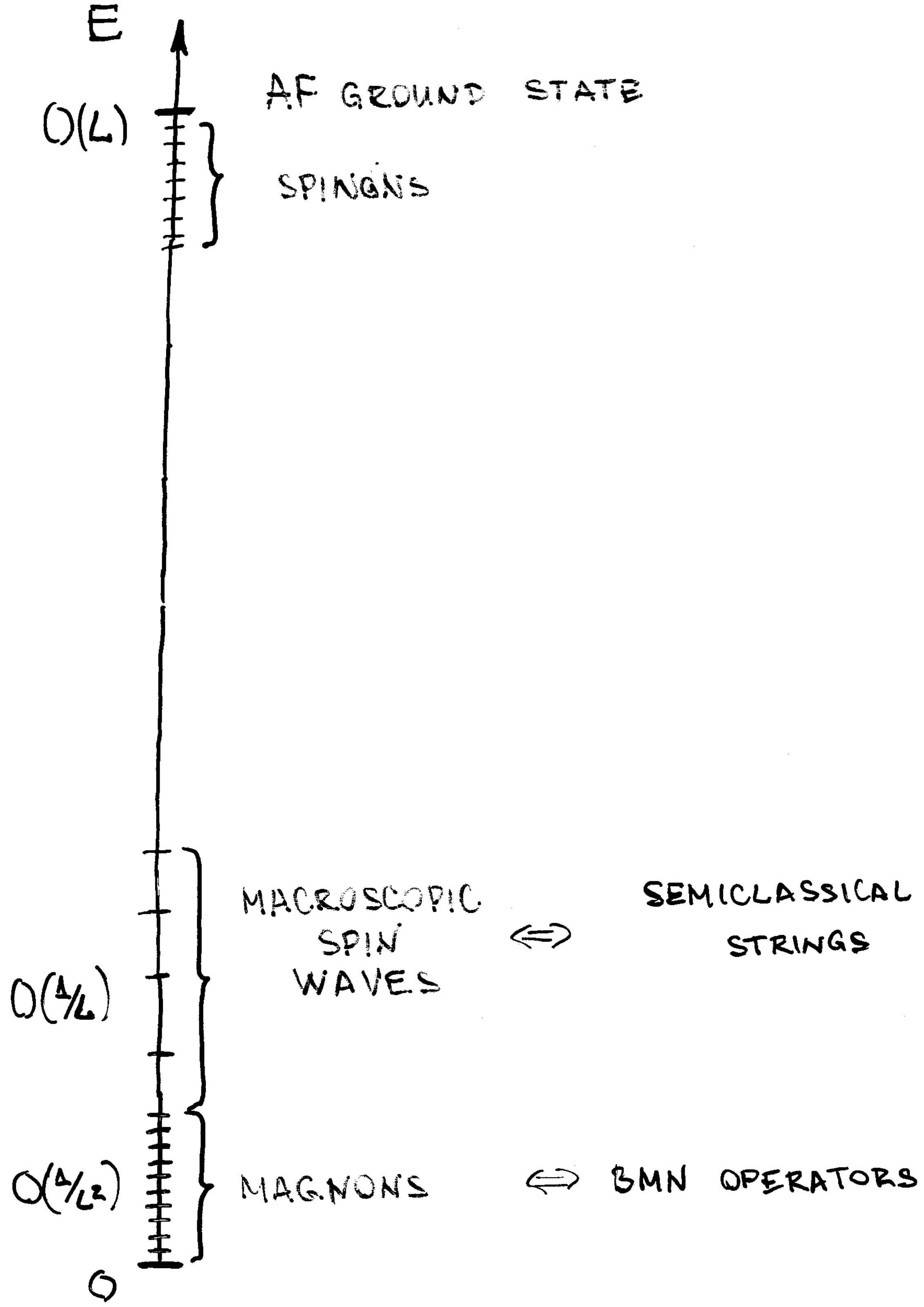
γ_n – ANOMALOUS DIMENSIONS

$$\Delta_n = L + \gamma_n + \mathcal{O}(\lambda^2)$$

- 2- AND 3-LOOP MIXING MATRICES ARE KNOWN AND ARE ALSO INTEGRABLE

Beisert, Kristjansen, Staudacher '03
Beisert '03
Serban, Staudacher '04

SPECTRUM OF HEISENBERG MODEL:



BETHE ANSATZ:

$$\mathcal{O} = \mathrm{tr}\left(Z^{L-M} W^{M} + \dots\right)$$

$$\Gamma \,|u_1 \dots u_M\rangle = \gamma \,|u_1 \dots u_M\rangle$$

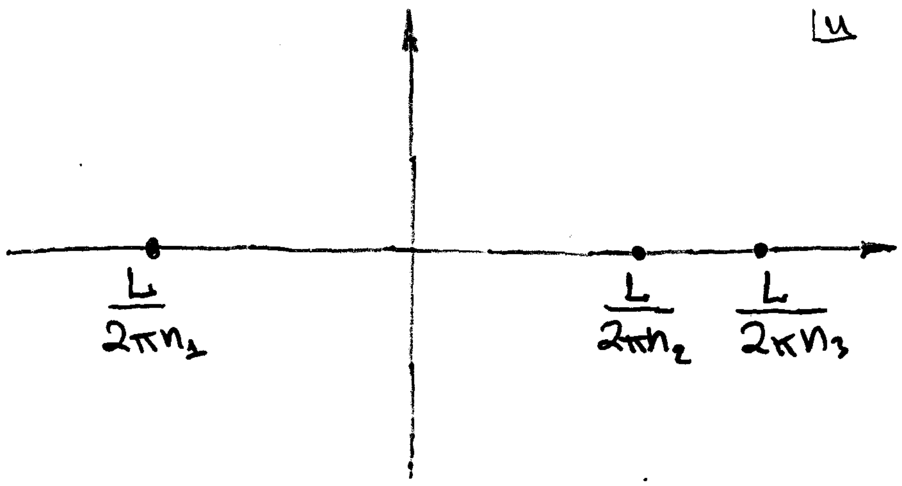
$$\boxed{\begin{gathered} \gamma = \frac{\lambda}{8\pi^2} \sum_j \frac{1}{u_j^2 + 1/4} \\ \left(\frac{u_j + i/2}{u_j - i/2}\right)^L = \prod_{k \neq j} \frac{u_j - u_k + i}{u_j - u_k - i} \\ \prod_j \frac{u_j + i/2}{u_j - i/2} = 1 \end{gathered}}$$

MOMENTUM

$$e^{ip} = \frac{u + i/2}{u - i/2}$$

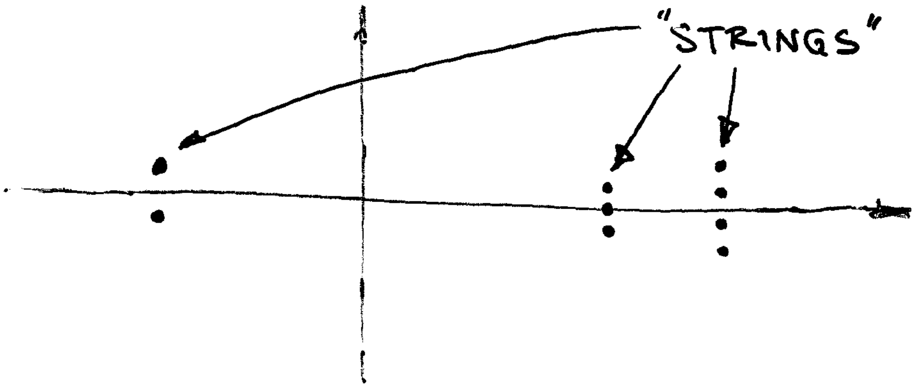
RAPIDITY

$\underline{L \to \infty}$: $u \approx 1/p \sim L$



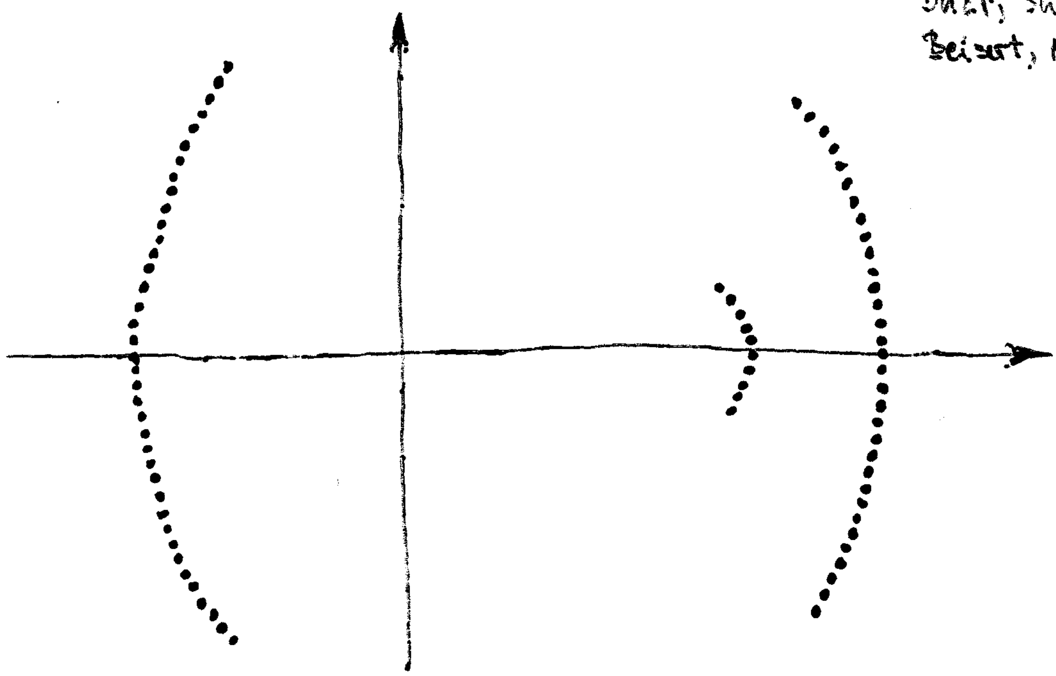
$$\sum n_i = 0$$

BOUND STATES:



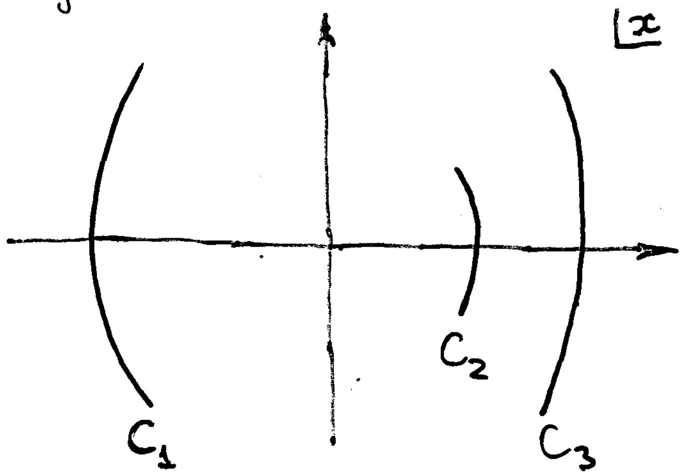
SCALING LIMIT ($M \to \infty$, M/L - FIXED):

Sutherland '95
Dhar, Shastry '00
Beisert, Minahan, Staudacher, Z. '03



$$u_j = L x_j$$

$$\rho(x) = \frac{1}{L} \sum_{j=1}^{M} \delta(x - x_j)$$



SCALING LIMIT OF BETHE EQUATIONS:

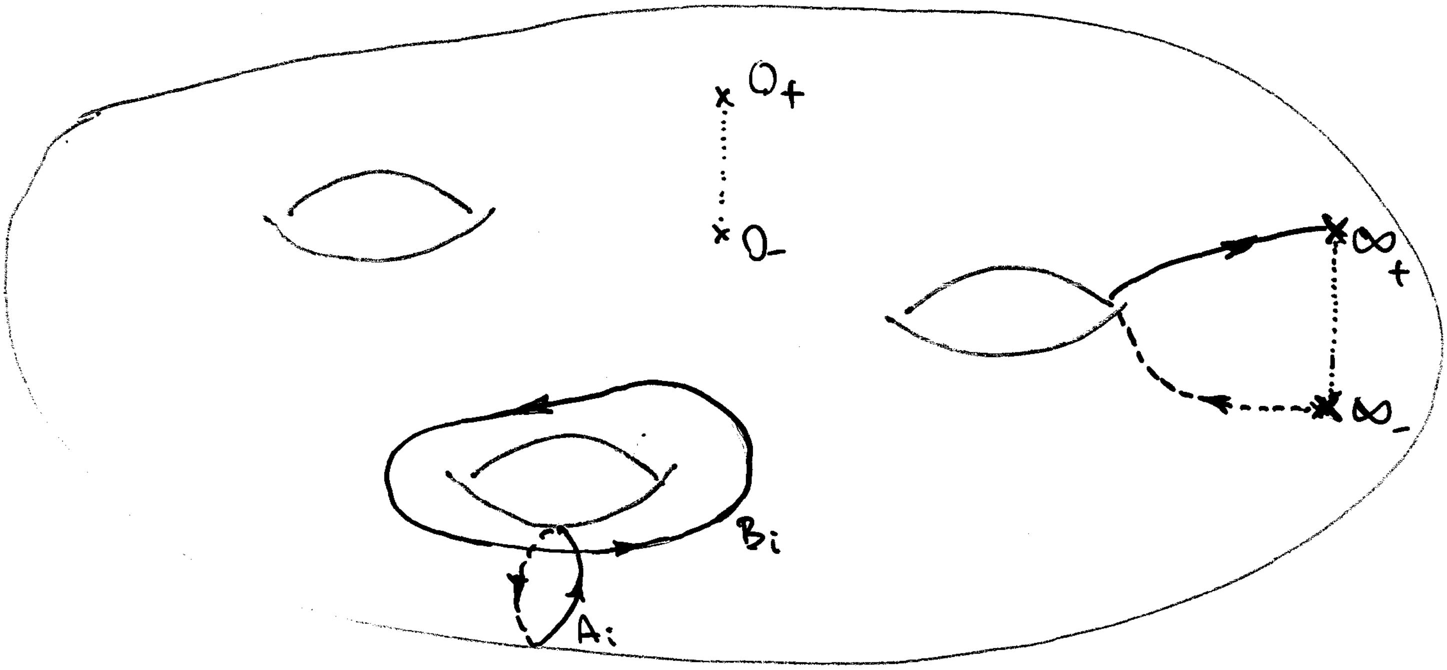
$$2 \fint dy \, \frac{\rho(y)}{x-y} = \frac{1}{x} + 2\pi n_k , \qquad x \in C_k$$

$$\int dx \, \rho(x) = \frac{M}{L}$$

$$\int dx \, \rho(x) \frac{1}{x} = 2\pi m$$

$$\Delta = L + \frac{\lambda}{8\pi^2} \int dx \, \rho(x) \frac{1}{x^2}$$

GENERAL SOLUTION:



$$\int_{A_i} dp = 2\pi m_i \qquad\qquad i = 1, \dots, K-1$$

$$\int_{B_i} dp = 2\pi (n_i - n_K)$$

$$\int_{\infty_-}^{\infty_+} dp = 2\pi n_K$$

$x \to 0_+$:

$$p(x) = -\frac{1}{2x} - 2\pi m - \frac{8\pi^2 L}{\lambda} \gamma x + O(x^2)$$

$x \to \infty$:

$$p(x) = \left(\frac{M}{L} - \frac{1}{2}\right)\frac{1}{x} + \dots$$

STRING ON $S^3 \times \mathbb{R}^1$:

$$S = -\frac{\sqrt{\lambda}}{4\pi} \int d\sigma\, d\tau \left[\frac{1}{2} \operatorname{tr} j_a^2 + (\partial_a X^0)^2 \right]$$

X^0 – GLOBAL TIME IN AdS_5

$$j_a = g^{-1} \partial_a g$$

$$g \in SU(2) = S^3 \subset S^5$$

CAN BE COMPARED TO THE HEISENBERG SPIN CHAIN

- AT THE LEVEL OF EFFECTIVE ACTIONS

  Kruczenski '03
  Kruczenski, Ryzhov, Tseytlin '0[illegible]
  ...

- AT THE LEVEL OF SOLUTIONS

  (BETHE ANSATZ VS. CLASSICAL SOLUTIONS OF THE SIGMA-MODEL)

EQUATIONS OF MOTION:

$$\partial_+ \partial_- X^0 = 0 \quad \Rightarrow \quad X^0 = \varkappa \tau$$

$$(*) \quad \partial_+ j_- + \partial_- j_+ = 0$$

$$(**) \quad \partial_+ j_- - \partial_- j_+ + [j_+, j_-] = 0$$

$$\Updownarrow$$

$$j_\pm = g^{-1} \partial_\pm g$$

$$J_\pm = \frac{j_\pm}{1 \mp x}$$

$$\boxed{\partial_+ J_- - \partial_- J_+ + [J_+, J_-] = 0 \quad \forall x \quad \Leftrightarrow (*), (**)}$$

CLASSICAL INTEGRABILITY
Pohlmeyer '76
Zakharov, Mikhailov '78

- THE FULL SUPERSYMMETRIC SIGMA-MODEL ON $AdS_5 \times S^5$ IS ALSO INTEGRABLE!

Bena, Polchinski, Roiban '03

$$j_\pm = i\,\frac{\Delta}{\sqrt{\lambda}}\;\vec{S}_\pm\cdot\vec{\sigma}$$

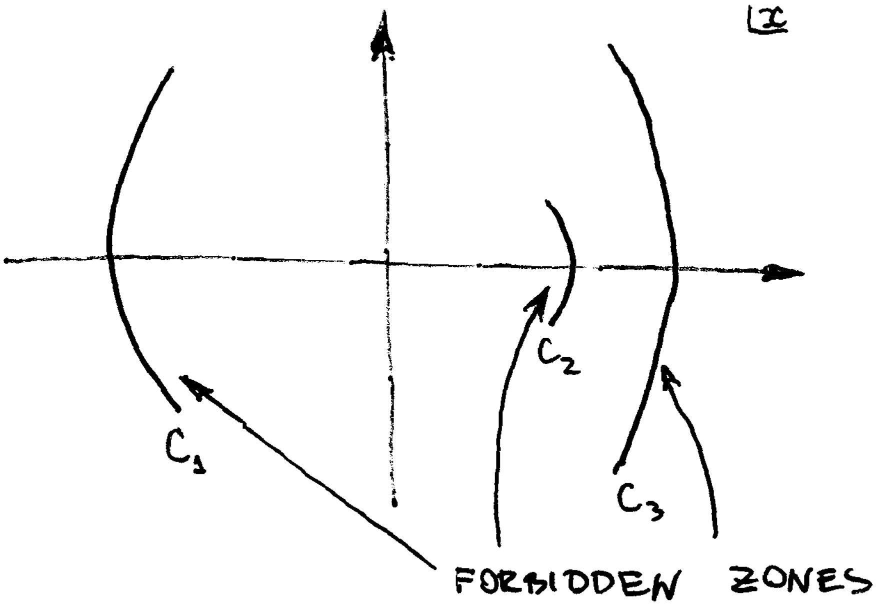
(VIRASORO CONSTRAINS: $\vec{S}_\pm^{\,2} = 1$)

AUXILIARY LINEAR PROBLEM:

$$\left[\partial_\sigma + \frac{i\Delta}{2\sqrt{\lambda}}\left(\frac{\vec{S}_+\cdot\vec{\sigma}}{1-x} - \frac{\vec{S}_-\cdot\vec{\sigma}}{1+x}\right)\right]\Psi = 0$$

1D DIRAC EQN. WITH A PERIODIC POTENTIAL

x – SPECTRAL PARAMETER:



$$\boxed{\begin{aligned}
&2\,⨍ dy\,\frac{\rho(y)}{x-y} = \frac{x}{x^2-\frac{\lambda}{16\pi^2L^2}}\cdot\frac{\Delta}{L} + 2\pi n_k\,, \qquad x\in C_k\\
&\int dx\,\rho(x) = \frac{M}{L} + \frac{\Delta - L}{2L}\\
&\int dx\,\rho(x)\,\frac{1}{x} = 2\pi m\\
&\Delta - L = \frac{\lambda}{8\pi^2 L}\int dx\,\rho(x)\,\frac{1}{x^2}
\end{aligned}}$$

ONE-LOOP SYM

$$\boxed{\left(\frac{u_j + i/2}{u_j - i/2}\right)^L = \prod_{k\neq j} \frac{u_j - u_k + i}{u_j - u_k - i}}$$

QUANTUM BETHE EQS.

$L \to \infty$

$$\boxed{2 \fint dy \, \frac{\rho(y)}{x-y} = \frac{1}{x} + 2\pi n_k}$$

CLASSICAL BETHE EQN.

$\frac{\lambda}{L^2} \to 0$

$$\boxed{2 \fint dy \, \frac{\rho(y)}{x-y} = \frac{x}{x^2 - \frac{\lambda}{16\pi^2 L^2}} \cdot \frac{\Delta}{L} + 2\pi n_k}$$

CLASSICAL BETHE EQN.

$L \to \infty$

? ? ? ? ?

QUANTUM BETHE EQN. ?

CANDIDATE DISCRETE EQS. PROPOSED BY Arutyunov, Frolov, Staudacher (two days ago)

SIGMA - MODEL

$$S = -\frac{\sqrt{\lambda}}{8\pi}\int d^2\sigma \, \mathrm{tr}\left(g^{-1}\partial_a g\right)^2$$

$$\Downarrow$$

$$S = \int d^2\sigma \left[i\,\bar{\Psi}_A \gamma^a \partial_a \Psi_A + \frac{2\pi}{\sqrt{\lambda}} \left(\bar{\Psi}_A \gamma^a \tau^i \Psi_A\right)^2 \right]$$

$A = 1 \dots N \qquad N \to \infty$

$$\Downarrow$$

$$\left(\frac{u_j + \frac{\sqrt{\lambda}}{2\pi} + \frac{iN}{2}}{u_j + \frac{\sqrt{\lambda}}{2\pi} - \frac{iN}{2}} \cdot \frac{u_j - \frac{\sqrt{\lambda}}{2\pi} + \frac{iN}{2}}{u_j - \frac{\sqrt{\lambda}}{2\pi} - \frac{iN}{2}} \right)^L = \prod_{k \neq j} \frac{u_j - u_k + i}{u_j - u_k - i}$$

Polyakov, Wiegmann '83