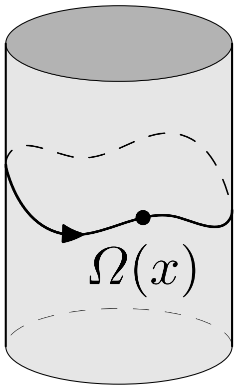
Applying Integrability in AdS and CFT



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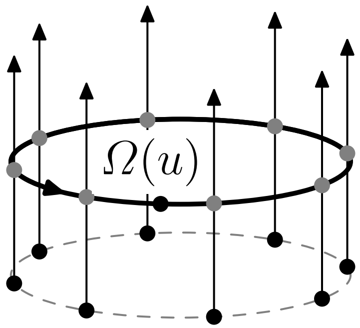


Strings 2005, Toronto

July 14, 2005

Introduction

Tests of AdS/CFT Conjecture:

- Spectra of string energies & gauge scaling dimensions should agree.
- Compute energies of near plane wave strings. [Callan, Lee, McLoughlin Schwarz, Swanson, Wu] [Swanson hep-th/0505028]
- Compute energies of classical spinning strings. [Gubser Klebanov Polyakov] [Frolov Tseytlin] [Tseytlin hep-th/0311139]
- Compute dimensions in near BMN limit. [Minahan Zarembo] [NB Kristjansen Staudacher] [NB hep-th/0407277]
- Compute dimensions in thermodynamic limit. [NB, Minahan Staudacher Zarembo] [Serban Staudacher] [NB hep-th/0407277]
- Agreement at $\mathcal{O}(\lambda^2/L^4)$. Mismatch at $\mathcal{O}(\lambda^3/L^6)$. [Callan, Lee, McLoughlin Schwarz, Swanson, Wu] [Serban Staudacher]

Conclude:

- AdS/CFT wrong?
- Strong/weak coupling problem. Cannot compare perturbatively!

Objective:

- How to compute *planar* energies & scaling dimensions?
- How to make use of apparent integrability in both models?

Outline

⋆ **Classical Strings**

- Derive & investigate spectral curve of a solution.
- Solve classical spectral problem.

⋆ **Higher-Loop Gauge Theory**

- Perform coordinate Bethe ansatz.
- Present Bethe equations to solve higher-loop spectrum.

⋆ **Deformations of $\mathcal{N} = 4$ gauge theory**

- Present several interesting deformations.

Overview Strings

⋆ **Classical Superstrings on $AdS_5 \times S^5$**

- Coset space sigma model,
- Integrability, Lax connection, monodromy,
- transformation to a spectral curve ($\rightsquigarrow$ mode decomposition)

⋆ **Spectral Curve**

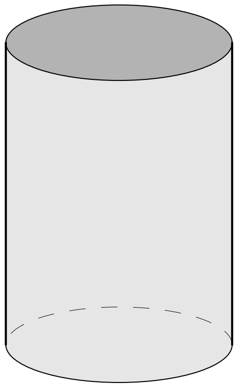
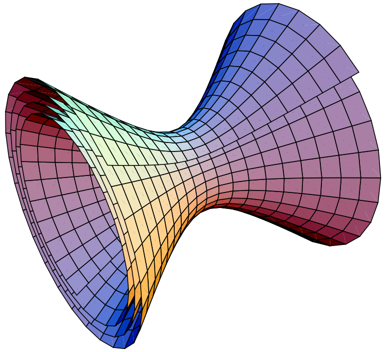
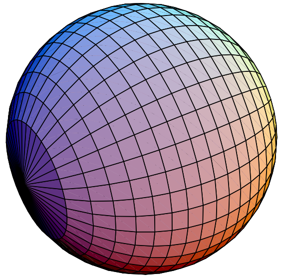
- Generic structure,
- singularities and
- string moduli.

⋆ **Quantisation**

- Spectral curve in quantum theory,
- integral equations, discretisation.

Strings on $AdS_5 \times S^5$

IIB superstrings propagate on the curved superspace $AdS_5 \times S^5$

 $\longrightarrow$  $\times$  $\times$ fermi

Coset space

$$AdS_5 \times S^5 \times \text{fermi} = \frac{\mathrm{PSU}(2,2|4)}{\mathrm{Sp}(1,1) \times \mathrm{Sp}(2)}\,.$$

Decomposition of the algebra $\mathfrak{u}(2,2|4)$ to $\mathfrak{sp}(1,1) \times \mathfrak{sp}(2)$

$$j \in \mathfrak{psu}(2,2|4), \qquad j = h + q_1 + p + q_2, \qquad h \in \mathfrak{sp}(1,1) \times \mathfrak{sp}(2).$$

Algebra $j = [j_1, j_2]$ respects $\mathbb{Z}_4$-grading $h\colon 0$, $q_1 : 1$, $p\colon 2$, $q_2 : 3$ [Berkovits Bershadsky, Hauer Zhukov, Zwiebach]

Supersymmetric Sigma Model

Field $g(\sigma,\tau) \in \mathrm{U}(2,2|4)$ (8×8 supermatrix) with flat connection J

$$J = -g^{-1} dg = H + Q_1 + P + Q_2, \qquad dJ - J \wedge J = 0.$$

Coset $g \simeq gh$ with $h(\sigma,\tau) \in \mathrm{Sp}(1,1) \times \mathrm{Sp}(2)$. Action [Metsaev Tseytlin][Roiban Siegel]

$$S_\sigma = \frac{\sqrt{\lambda}}{2\pi} \int \left(\tfrac{1}{2} \operatorname{str} P \wedge {*P} - \tfrac{1}{2} \operatorname{str} Q_1 \wedge Q_2 + \Lambda \wedge \operatorname{str} P\right).$$

$\mathfrak{psu}(2,2|4)$ Noether current K and equation of motion

$$K = P + \tfrac{1}{2}{*Q_1} - \tfrac{1}{2}{*Q_2} - {*\Lambda}, \qquad d{*K} - J \wedge {*K} - {*K} \wedge J = 0.$$

Virasoro constraints

$$\operatorname{str} P_+^2 = \operatorname{str} P_-^2 = 0.$$

Lax Connection

Integrability $\rightsquigarrow$ Lax pair: Family of connections [Bena Polchinski Roiban]

$$A(z) = H + \tfrac{1}{2}(z^{-2} + z^2)P + \tfrac{1}{2}(z^{-2} - z^2)(*P - \Lambda) + z^{-1}Q_1 + z\,Q_2.$$

Connection $A(z)$ flat for all values of the spectral parameter z

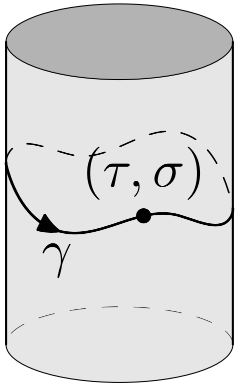
$$dA(z) - A(z) \wedge A(z) = 0.$$

Equivalent to flatness of J and conservation of K.

- Analytic for all $z \in \bar{\mathbb{C}}$.
- Poles at $z = 0, \infty$.
- Point $z = 1$ related to global symmetry: $A(1+\epsilon) = J - 2\epsilon *K + \ldots$.

Monodromy

Monodromy of Lax connection around closed string [Kazakov, Marshakov Minahan, Zarembo]



$$\Omega(z) = \left(\mathrm{P}\exp\oint_{-\gamma} J\right)\left(\mathrm{P}\exp\oint_{\gamma} A(z)\right).$$

Eigenvalues invariant under deformations of γ [NB, Kazakov Sakai, Zarembo]

$$\Omega(z) \simeq \operatorname{diag}\left(e^{i\hat{p}_1(z)},\ldots,e^{i\hat{p}_4(z)}\big|\big|\, e^{i\tilde{p}_1(z)},\ldots,e^{i\tilde{p}_4(z)}\right).$$

Transformation of solution $g(\sigma,\tau)$ to set of quasi-momenta $\{p_k(z)\}$.

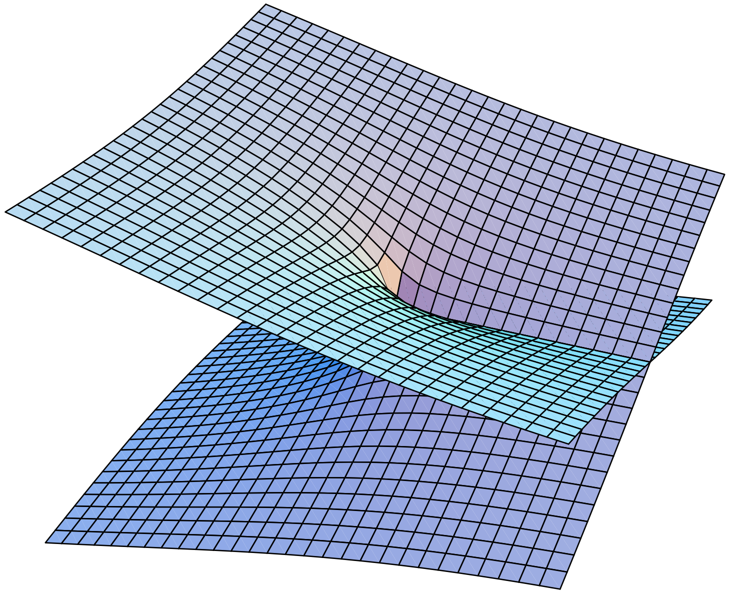
- The $p_k(z)$ are conserved, gauge-invariant quantities.
- No (conformal/kappa) gauge fixing required.
- Analytic functions of z: Much physical information in $\{p_k(z)\}$.
- $\{p_k(z)\}$ contains all (?) action variables in Hamilton-Jacobi formalism.
- Diagonalising $\Omega(z)$ introduces (solution-dependent) singular points.

Bosonic Branch Points

Eigenvalue crossing: Consider 2×2 bosonic submatrix Γ of $\Omega(z)$ [NB, Kazakov, Sakai, Zarembo]

$$\Gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \gamma_{1,2} = \tfrac{1}{2}\left(a + d \pm \sqrt{(a-d)^2 + 4bc}\right).$$

Generic behaviour at degenerate eigenvalues $e^{ip_k(z_a)} = e^{ip_l(z_a)}$:



$$e^{ip_k(z_a)}\left(1 \pm \alpha_a \sqrt{z - z_a} + \mathcal{O}(z - z_a)\right).$$

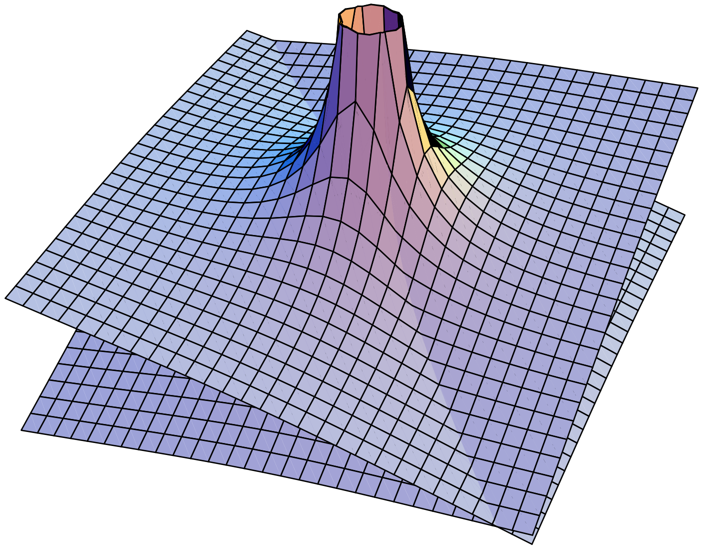
Full turn around z_a interchanges eigenvalues (labelling): Branch cut.

Fermionic Singularities

Mixed eigenvalue crossing: Consider $(1|1) \times (1|1)$ submatrix Γ of $\Omega(z)$

$$\Gamma = \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right), \qquad \hat{\gamma} = \frac{bc}{d-a} + a, \quad \tilde{\gamma} = \frac{bc}{d-a} + d\,.$$

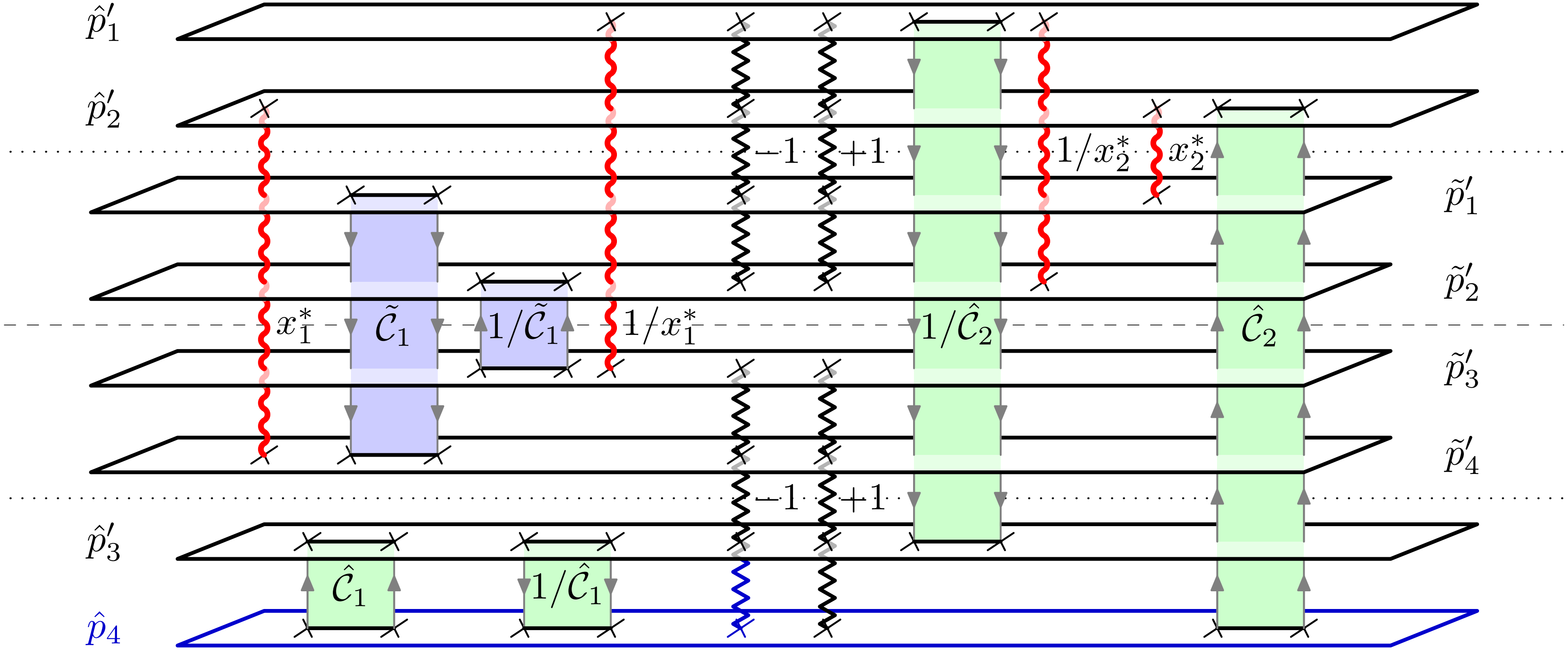
Generic behaviour at degenerate eigenvalues $e^{i\hat{p}_k(z_a^*)} = e^{i\tilde{p}_l(z_a^*)}$:



$$e^{i\hat{\not{p}}_k(z_a^*)} \left(\frac{\alpha_a^*}{z - z_a^*} + 1 + \mathcal{O}(z - z_a^*)\right).$$

Residue of fermionic singularity $\alpha_a^* \sim bc$ is nilpotent.

Spectral Curve



- More standard spectral parameter x: $z^2 = (x-1)/(x+1)$.
- Derivative $p'(x)$ is a curve of degree $4+4$. [NB, Kazakov Sakai, Zarembo]
- Singularities at $x = \pm 1$; asymptotics at $x = 0, \infty$; symmetry $x \mapsto 1/x$.
- Bosonic modes: Square-roots, branch cuts (Bose condensates).
- Fermionic excitations: Poles (Pauli principle).

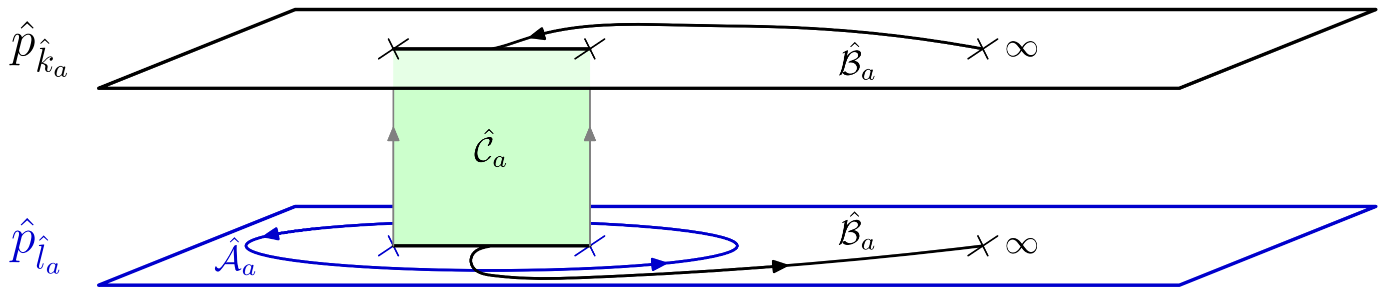
String Moduli

Single-valuedness of e^{ip}: All closed cycles must be integer

$$\oint dp \in 2\pi\mathbb{Z}.$$

Cuts/singularities: "mode number" $n_a \in \mathbb{Z}$ and "amplitude" $K_a \in \mathbb{R}$

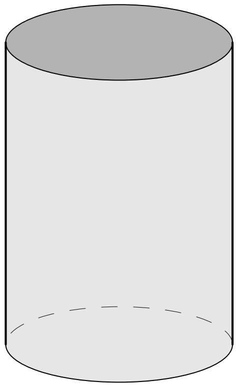
$$\int_{\mathcal{A}_a} dp = 0, \quad n_a = \frac{1}{2\pi}\int_{\mathcal{B}_a} dp, \quad K_a = -\frac{1}{2\pi i}\oint_{\mathcal{A}_a}\left(1-\frac{1}{x^2}\right)p(x)\,dx.$$



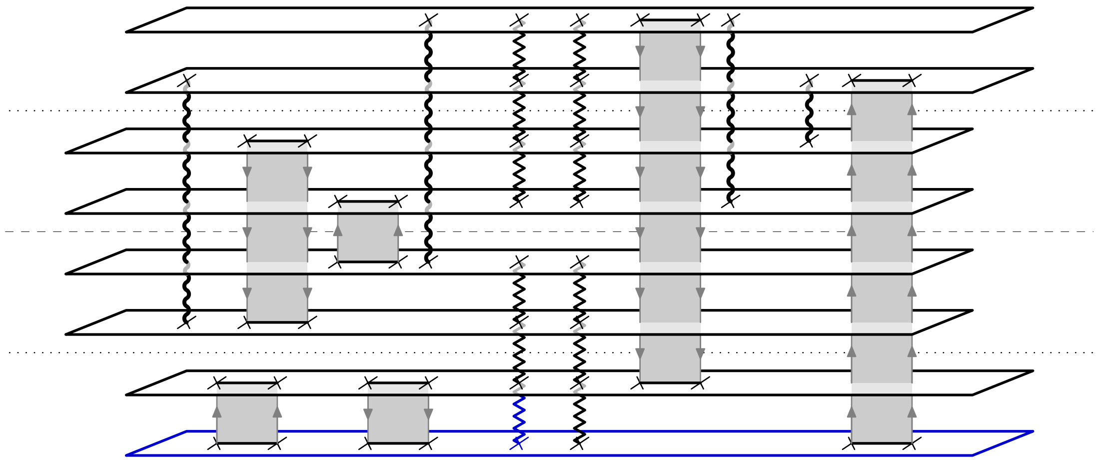
Solutions classified by: connection of sheets, mode numbers, amplitudes.

Spectral Transformation

From embedding of world-sheet $g(\sigma, \tau)$ to spectral curve $p'(x)$.



$\longmapsto$



Spectral curve encodes conserved charges of a string solution.

What about angle variables?

- Points on Jacobian of the spectral curve?
- Important classically, but
- obscured by Heisenberg uncertainty in quantum mechanics.

Towards Quantisation

Classical solutions understood.

Fix gauge, introduce ghosts, Poisson brackets, quantise. . . Some results:

- Uniform gauge & compatibility with Lax pair. [Arutyunov Frolov][Alday Arutyunov Tseytlin]
- Some quantum commuting local charges for near plane waves. [Swanson hep-th/0410282]
- BRST consistent bi-local generator (pure spinors). [Berkovits hep-th/0409159]

Some open problems:

- Current algebra, quantum anomalies?
- Algebra of monodromy (classical/quantum)?
- Proof of Serre relations for bi-local Yangian generators?

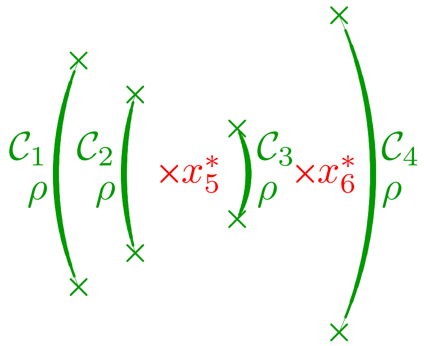
Directly quantise spectral curve? Qualitative picture.

Riemann-Hilbert Problem

Parameterise spectral curve through its branch cuts and singularities, e.g.

$$p(x) \sim \int_{\mathcal{C}} \frac{dy\,\rho(y)}{1-1/y^2}\,\frac{1}{y-x} + \text{sing.}$$

Function $\rho(x)$ parametrises discontinuity of $p(x)$ across the cuts $\mathcal{C}$.



Singular integral equations from integrality of cycles

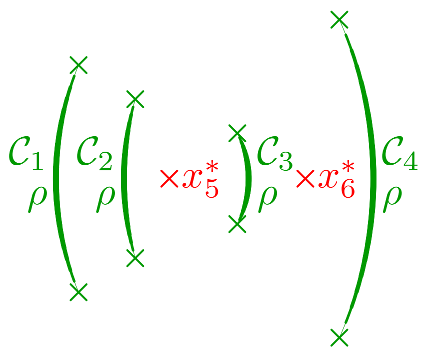
$$2\pi n_a = \not{p}_{l_a}(x) - \not{p}_{k_a}(x) \qquad \text{for } x \in \mathcal{C}_a, x_a^*$$

Quantising the Curve

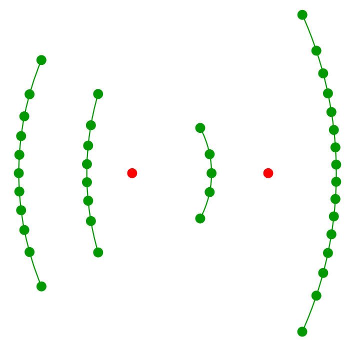
Classical solution is good approximation for large spins $\hbar \sim 1/S$ [Frolov, Tseytlin hep-th/0304255]

$$S = \int_{\mathcal{C}} dx\, \rho(x).$$

Spins of compact symmetries are quantised $S \in \mathbb{Z}$.
Interpret $\rho(x)$ as density of discrete contributions x_k: (c.f. matrix models)



$$\int_{\mathcal{C}} dx\, \rho(x) f(x) \quad \mapsto \quad \sum_{k=1}^{S} f(x_k, \hbar)$$



- To do: Find a suitable quantum extrapolation $f(x, \hbar)$ of $f(x)$.
- Gather inspiration from gauge theory: Finite S natural.

Overview Gauge Theory

⋆ **Dilatation Operator and Spin Chains**

- States, Hamiltonian, perturbation theory.
- Summarise results of construction.

⋆ **Asymptotic Bethe Ansatz**

- Diagonalise the Hamiltonian
- S-matrix, Factorised scattering.
- Asymptotic Bethe equations for $\mathcal{N} = 4$.

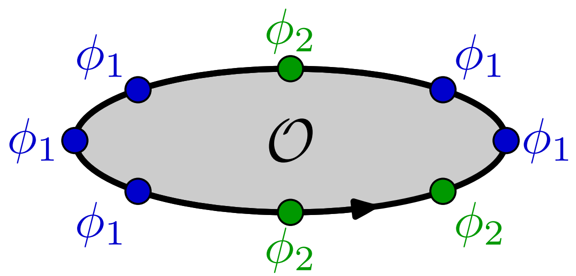
⋆ **Thermodynamic Limit**

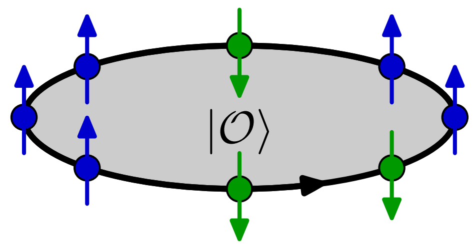
- Spectral curves and comparison to classical string theory.
- Bethe equations for quantum strings, stringy spin chain.
- $\mathcal{O}(1/L)$ finite size/quantum effects.

Gauge Theory and Spin Chains

Single trace operator, complex scalars ϕ_1, ϕ_2 equivalent to spins $|\uparrow\rangle$, $|\downarrow\rangle$

$$\mathcal{O} = \operatorname{Tr} \phi_1\phi_1\phi_2\phi_1\phi_1\phi_1\phi_2\phi_2 \qquad |\mathcal{O}\rangle = |\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\downarrow\downarrow\rangle$$





Length L: # of fields, # of sites.

More general: all fields $\mathcal{W} = \{\mathcal{D}^n\Phi, \mathcal{D}^n\Psi, \mathcal{D}^n\mathcal{F}\}$ instead of just ϕ_1, ϕ_2.
Infinite-dimensional module $\mathcal{W} \in \mathbb{V}_\mathrm{F}$ of $\mathfrak{psu}(2,2|4)$ symmetry.

Operator mixing, quantum superposition: $|\mathcal{O}\rangle = *|\ldots\rangle + *|\ldots\rangle + \ldots$

- Conserved charges: 2+3 integral spins of $\mathfrak{su}(2,2) \times \mathfrak{su}(4)$.
- 3rd conserved charge of $\mathfrak{su}(2,2)$: Continuous conformal dimension D.

Dilatation Generator

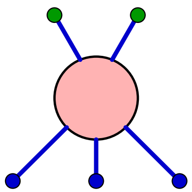
Scaling dimensions $D_{\mathcal{O}}(g)$ as eigenvalues of the dilatation generator $\mathfrak{D}(g)$

$$\mathfrak{D}(g)\,\mathcal{O} = D_{\mathcal{O}}(g)\,\mathcal{O}.$$

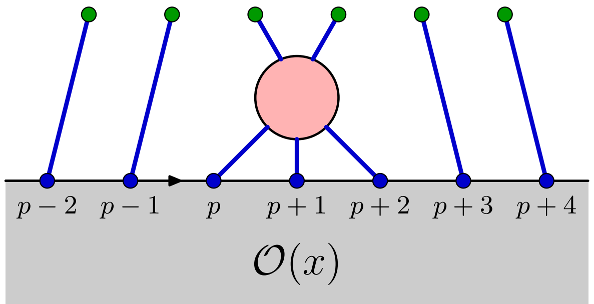
Dilatation generator/dimensions as spin chain Hamiltonian/energies.
Quantum corrections: $\mathcal{O}(g^{I+O-2})$ for I/O in/out legs

$$\mathfrak{D}(g) = \mathfrak{D}_0 + g^2\,\mathfrak{D}_2 + g^3\mathfrak{D}_3 + g^4\mathfrak{D}_4 + \ldots = \mathfrak{D}_0 + g^2\mathcal{H}(g).$$

Local action along spin chain (homogeneous, long-ranged, dynamic)



$$= \sum_{p=1}^{L}$$



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Construction

Hamiltonian/dilatation operator constructed (without field theory):

- completely at leading order: nearest-neighbour interactions, [NB hep-th/0407277]
- in $\mathfrak{su}(2|3)$ sector ($\phi_{1,2,3}, \psi_{1,2}$) at $\mathcal{O}(g^6)$ (dynamic), [NB hep-th/0310252]
- in $\mathfrak{su}(2)$ sector ($\phi_{1,2}$) at $\mathcal{O}(g^{10})$. [NB, Dippel Staudacher]

Integrability

Spin chain model apparently integrable:

- One-loop: R-matrix formalism, [Minahan Zarembo] [NB, Staudacher hep-th/0307042]
- some commuting charges constructed at higher loops, [NB Kristjansen Staudacher]
- degeneracies in the spectrum: planar parity pairs, [NB Kristjansen Staudacher] [NB hep-th/0310252]
- conserved bi-local Yangian generators. [Dolan Nappi Witten] [Agarwal Rageev]

Proof of Serre relations for perturbative bi-local generators?

Proof of integrability at all loops? Assume integrability. . .

Coordinate Space Bethe Ansatz

Solve spectrum for 0,1,2,many excitations. [Staudacher hep-th/0412188]

Vacuum of an *infinite* chain: half-BPS state ($\mathcal{Z}$: complex scalar)

$$|0\rangle = |\ldots\mathcal{Z}\mathcal{Z}\mathcal{Z}\ldots\rangle, \qquad \mathcal{H}|0\rangle = 0.$$

One-excitation states with excitation $\mathcal{A}$ at position a, momentum p.
Excitations $\mathcal{A} = \phi_k, \mathcal{D}_\mu\mathcal{Z}, \psi_\alpha$: $4+4|8$ flavours, stringy spectrum. [Berenstein Maldacena Nastase]

$$|\mathcal{A}, p\rangle = \sum\nolimits_a e^{ipa}|\ldots\mathcal{A}^a\ldots\rangle, \qquad \mathcal{H}|\mathcal{A},p\rangle = e_{\mathcal{A}}(p)|\mathcal{A},p\rangle.$$

Obtain dispersion relation $e_{\mathcal{A}}(p)$.

- $e_{\mathcal{A}}(p)$ derived at three loops, [Staudacher hep-th/0412188]
- equal for ϕ, ψ (and $\mathcal{D}\mathcal{Z}$) and
- asymptotic form conjectured (plane waves limit). [NB, Dippel Staudacher]

Elastic Scattering

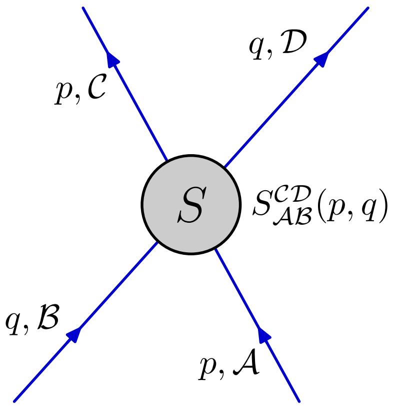
Two-excitation scattering state [Staudacher hep-th/0412188]

$$|\mathcal{A},p;\mathcal{B},q\rangle = \sum\nolimits_{a,b,\mathcal{C},\mathcal{D}} \Psi^{\mathcal{C}\mathcal{D}}_{\mathcal{A}\mathcal{B},ab}(p,q)|\ldots\mathcal{C}^a\ldots\mathcal{D}^b\ldots\rangle.$$

Demand eigenstate of elastic scattering process

$$\mathcal{H}|\mathcal{A},p;\mathcal{B},q\rangle = \big(e(p)+e(q)\big)|\mathcal{A},p;\mathcal{B},q\rangle$$

with asymptotics of the wave function Ψ

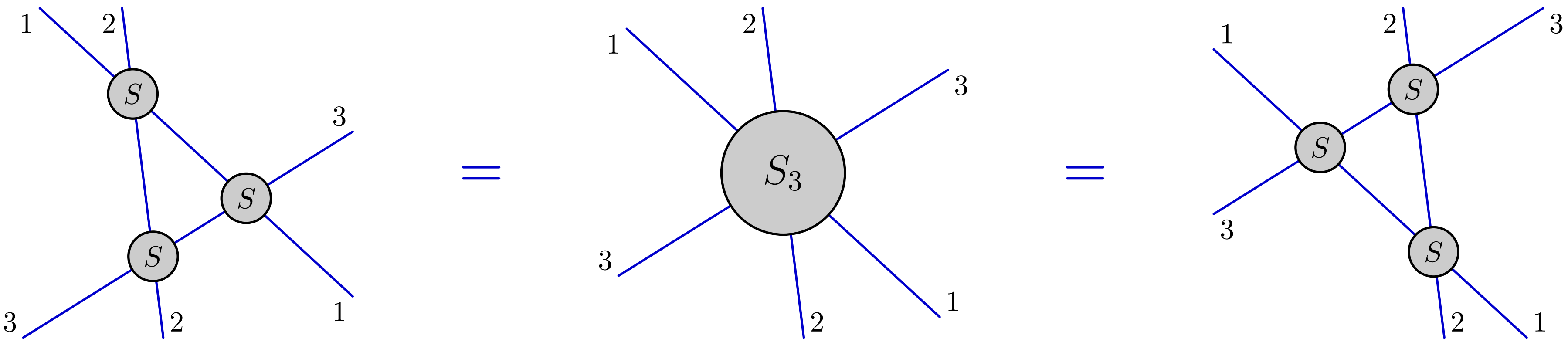


$$\Psi^{\mathcal{C}\mathcal{D}}_{\mathcal{A}\mathcal{B},a\ll b}(p,q) = e^{ipa+iqb}\,\delta^{\mathcal{C}}_{\mathcal{A}}\delta^{\mathcal{D}}_{\mathcal{B}}\,,$$
$$\Psi^{\mathcal{C}\mathcal{D}}_{\mathcal{A}\mathcal{B},a\gg b}(p,q) = e^{ipa+iqb}\,S^{\mathcal{C}\mathcal{D}}_{\mathcal{A}\mathcal{B}}(p,q).$$

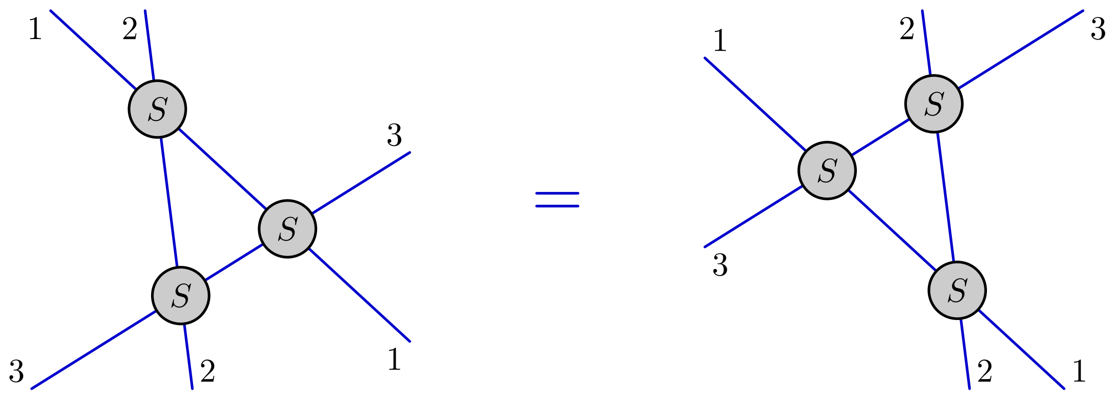
- S-matrix derived for $\mathfrak{su}(1|2)$ sector at $\mathcal{O}(g^6)$ and [Staudacher hep-th/0412188][NB, Staudacher hep-th/0504190]
- simple all-loop form conjectured (apparently new). [NB, Staudacher hep-th/0504190]

Factorised Scattering

Integrability: Factorised S-matrix, e.g. three excitations:



Yang-Baxter equation: Self-consistency condition for two-body S-matrix:

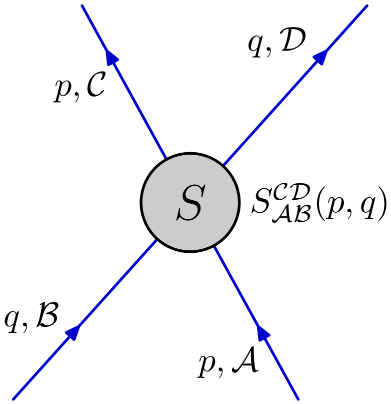


Computed and extrapolated S-matrices satisfy YBE.

[NB, Staudacher hep-th/0504190]

Nested Bethe Ansatz

Momentum flow diagonal, flavour flow non-diagonal.



YBE: S-matrix diagonalisable

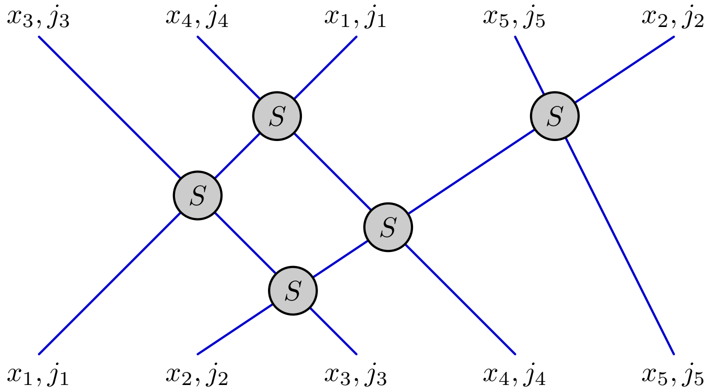
$$S^{\bar{\jmath}\bar{\jmath}'}_{jj'}(x,x') = \delta^{\bar{\jmath}}_{j}\delta^{\bar{\jmath}'}_{j'}\, S_{jj'}(x,x').$$

- Use new types of excitations $j = 1, 2, \ldots$.
- E.g.: $\phi : 1$, $\psi : 1 + 2$. Then 1: main spin wave; 2: nested flavour wave.
- Replace momenta p, p' by spectral parameters x, x'.

Asymptotic Eigenstates

Eigenstate composed from partial asymptotic wave functions.

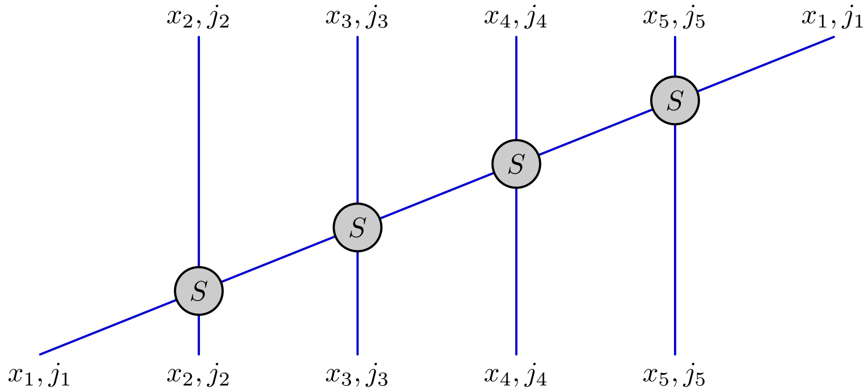
Phase between partial wave functions determined by S-matrix:



$$E = \sum_{k=1}^{K} e_k .$$

Note: Eigenstate on an infinite spin chain. Periodicity: Bethe equation

$$1 =$$



Bethe Equations

$\mathcal{N}=4$ SYM/$\mathfrak{psu}(2,2|4)$ has 7 types of diagonalised excitations.
Expected structure of Bethe equations:

$$1 = S_{j,0}(x_{j,k})^L \prod_{\substack{j'=1 \\ (j',k')\neq(j,k)}}^{7} \prod_{k'=1}^{K_{j'}} S_{j,j'}(x_{j,k}, x_{j',k'}).$$

Scattering phases $S_{jj'}(x,x')$ tightly constrained by desired properties:

- At $g=0$: One-loop equations; should explain multiplet splitting.
- should be compatible with Bethe equations in subsectors.
- Aspects of dynamic spin chain should be reflected.
- Restriction to cyclic states (trace) should be crucial.
- Structure of the algebra $\mathfrak{psu}(2,2|4)$ should be important.
- Spectrum should agree with known anomalous dimensions.

Full Bethe Equations for Higher-Loop $\mathcal{N} = 4$ SYM

Found solution which has all desired properties. [NB, Staudacher hep-th/0504190]

coupling constant

$$g^2 = \frac{\lambda}{8\pi^2}$$

transformation between u and x

$$x(u) = \tfrac{1}{2}u + \tfrac{1}{2}u\sqrt{1 - 2g^2/u^2}\,, \qquad u(x) = x + \frac{g^2}{2x}$$

$x^\pm$ parameters

$$x^\pm = x(u \pm \tfrac{i}{2})$$

spin chain momentum

$$e^{ip} = \frac{x^+}{x^-}$$

local charge eigenvalues

$$Q_r = \frac{1}{r-1}\sum_{j=1}^{K_4}\left(\frac{i}{(x_{4,j}^+)^{r-1}} - \frac{i}{(x_{4,j}^-)^{r-1}}\right)$$

anomalous dimension

$$\delta D = g^2 Q_2 = g^2 \sum_{j=1}^{K_4}\left(\frac{i}{x_{4,j}^+} - \frac{i}{x_{4,j}^-}\right)$$

some free parameters in $\sigma(x_1, x_2)$, for $\mathcal{N} = 4$ SYM: $\sigma = 1$

Bethe equations

$$1 = \prod_{j=1}^{K_4} \frac{x_{4,j}^+}{x_{4,j}^-}$$

$$1 = \prod_{j=1}^{K_2} \frac{u_{1,k} - u_{2,j} + \frac{i}{2}}{u_{1,k} - u_{2,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{1 - g^2/2x_{1,k}x_{4,j}^+}{1 - g^2/2x_{1,k}x_{4,j}^-}$$

$$1 = \prod_{\substack{j=1\\ j\neq k}}^{K_2} \frac{u_{2,k} - u_{2,j} - i}{u_{2,k} - u_{2,j} + i} \prod_{j=1}^{K_3} \frac{u_{2,k} - u_{3,j} + \frac{i}{2}}{u_{2,k} - u_{3,j} - \frac{i}{2}} \prod_{j=1}^{K_1} \frac{u_{2,k} - u_{1,j} + \frac{i}{2}}{u_{2,k} - u_{1,j} - \frac{i}{2}}$$

$$1 = \prod_{j=1}^{K_2} \frac{u_{3,k} - u_{2,j} + \frac{i}{2}}{u_{3,k} - u_{2,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{3,k} - x_{4,j}^+}{x_{3,k} - x_{4,j}^-}$$

$$1 = \left(\frac{x_{4,k}^-}{x_{4,k}^+}\right)^L \prod_{\substack{j=1\\ j\neq k}}^{K_4} \left(\frac{x_{4,k}^+ - x_{4,j}^-}{x_{4,k}^- - x_{4,j}^+}\,\frac{1 - g^2/2x_{4,k}^+x_{4,j}^-}{1 - g^2/2x_{4,k}^-x_{4,j}^+}\,\sigma^2(x_{4,k}, x_{4,j})\right)$$
$$\times \prod_{j=1}^{K_1} \frac{1 - g^2/2x_{4,k}^-x_{1,j}}{1 - g^2/2x_{4,k}^+x_{1,j}} \prod_{j=1}^{K_3} \frac{x_{4,k}^- - x_{3,j}}{x_{4,k}^+ - x_{3,j}} \prod_{j=1}^{K_5} \frac{x_{4,k}^- - x_{5,j}}{x_{4,k}^+ - x_{5,j}} \prod_{j=1}^{K_7} \frac{1 - g^2/2x_{4,k}^-x_{7,j}}{1 - g^2/2x_{4,k}^+x_{7,j}}$$

$$1 = \prod_{j=1}^{K_6} \frac{u_{5,k} - u_{6,j} + \frac{i}{2}}{u_{5,k} - u_{6,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{5,k} - x_{4,j}^+}{x_{5,k} - x_{4,j}^-}$$

$$1 = \prod_{\substack{j=1\\ j\neq k}}^{K_6} \frac{u_{6,k} - u_{6,j} - i}{u_{6,k} - u_{6,j} + i} \prod_{j=1}^{K_5} \frac{u_{6,k} - u_{5,j} + \frac{i}{2}}{u_{6,k} - u_{5,j} - \frac{i}{2}} \prod_{j=1}^{K_7} \frac{u_{6,k} - u_{7,j} + \frac{i}{2}}{u_{6,k} - u_{7,j} - \frac{i}{2}}$$

$$1 = \prod_{j=1}^{K_6} \frac{u_{7,k} - u_{6,j} + \frac{i}{2}}{u_{7,k} - u_{6,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{1 - g^2/2x_{7,k}x_{4,j}^+}{1 - g^2/2x_{7,k}x_{4,j}^-}$$

Results on Scaling Dimensions

Explicit (more or less) computations of higher-loop scaling dimensions:

- Various anomalous dimensions from constructed dilatation generator.
- Twist-two $\mathrm{Tr}\,\mathcal{W}\mathcal{W}$ at $\mathcal{O}(g^6)$ extrapolated from QCD [Moch Vermaseren Vogt][Kotikov, Lipatov Onishchenko Velizhanin]

$$\delta D = \frac{\lambda}{2\pi^2}h_1 - \frac{\lambda^2}{16\pi^4}\big(h_3 + h_{-3} - 2h_{-2,1} + 2h_1 h_2 + 2h_1 h_{-2}\big) + \ldots\,,$$

  comfirmed by unitarity of four-point amplitude, [Bern Dixon Smirnov]
- Konishi $\mathrm{Tr}\,\Phi_m\Phi_m$ and operator $\mathrm{Tr}\,\Phi_n\Phi_m\Phi_m$ at $\mathcal{O}(g^6)$ by EOM, [Eden Jarczak Sokatchev]
- operators $\mathrm{Tr}\,\mathcal{D}^3\phi^3$ at $\mathcal{O}(g^4)$. [Eden hep-th/0501234]

Complete agreement!

- Almost certainly true at three loops. Predictions for higher loops.
- Asymptotic: Designed for L loops (L: length). Wrappings?
- Proof? Analytic properties? Transfer matrices?

Dynamic Transformation

Curious identity of $S_{jj'}$:

$$S_{3j} = S_{1j}\,S_{0j} \quad \text{when} \quad x_3 = \frac{g^2}{2x_1}\,.$$

E.g. for $j = 4$ we have

$$S_{34} = \frac{x_3 - x_4^+}{x_3 - x_4^-}\,, \qquad S_{14} = \frac{1 - g^2/2x_1x_4^+}{1 - g^2/2x_1x_4^-}\,, \qquad S_{04} = \frac{x_4^+}{x_4^-}\,.$$

In words: An excitation of type 3 is equivalent to type $0 + 1$.

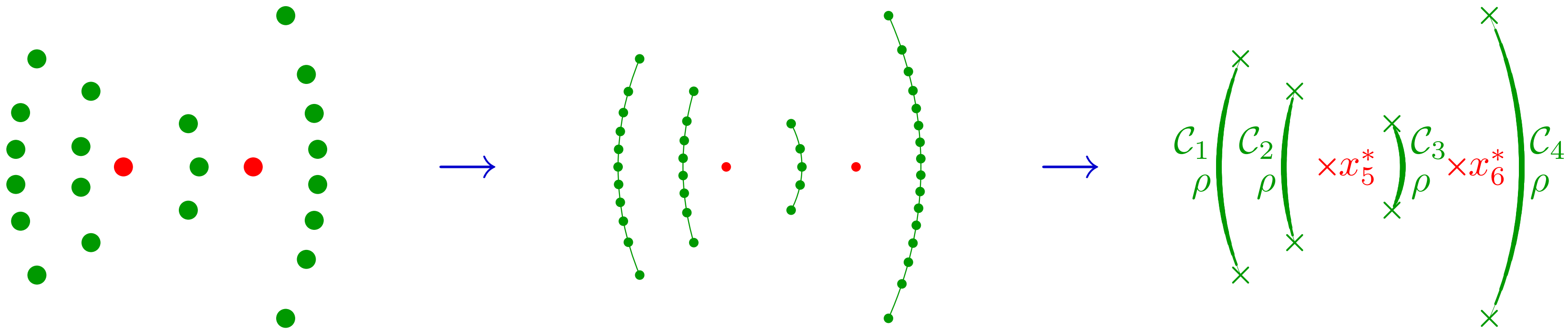
- Excitations $j' = 1, 3$ are flavour spin waves.
- "Excitation" $j' = 0$ adds a field $\mathcal{Z}$.

Dynamic spin chain effect!

Thermodynamic Limit

- Long spin chains, $L \to \infty$.
- Long wavelengths, $\lambda \sim L$, $p \sim 1/L$.
- Large number of excitations (spins) $K \sim L$, coherent states.

Roots x_k condense on curves $\mathcal{C}_a$ with density ρ. [Sutherland PRL 74,816] [NB, Minahan Staudacher Zarembo]



Roots can also remain single x_a^* and/or form stacks. [NB, Kazakov Sakai, Zarembo]

Discrete sums/products turn into integrals with density ρ

$$\sum_{k=1}^{K} f(x_k) \to \int_{\mathcal{C}} \frac{dx\,\rho(x)}{1 - g^2/2x^2}\, f(x) + \sum\nolimits_a f(x_a^*).$$

Comparison to Strings

Integral equations for thermodynamic limit similar to classical strings:

- Agreement at $\mathcal{O}(\lambda^2/L^4)$.
- Disagreement at $\mathcal{O}(\lambda^3/L^6)$: But $\lambda \ll 1$ vs. $\lambda \gg 1$!
- Bethe equations can be adjusted to classical strings.

Stringy Choice of Parameters

- Can extrapolate Bethe equations to finite L (quantum strings). [Arutyunov Frolov Staudacher]
  Reproduce $\mathcal{O}(1/L)$ results for near-plane wave strings.
  Reproduce generic $\sqrt[4]{\lambda}$ behaviour at large λ.
  What about quantum effects for spinning strings?
- Can also extrapolate to small λ. [NB hep-th/0409054]
  Bethe equations actually describe a spin chain: "string chain".
  Does the extrapolation have anything to with strings or is it lunatic?

Bethe Equations for Quantum Strings & String Chain

Bethe equations

$$1=\prod_{j=1}^{K_4}\frac{x_{4,j}^+}{x_{4,j}^-}$$

$$1=\prod_{j=1}^{K_2}\frac{u_{1,k}-u_{2,j}+\frac{i}{2}}{u_{1,k}-u_{2,j}-\frac{i}{2}}\prod_{j=1}^{K_4}\frac{1-g^2/2x_{1,k}x_{4,j}^+}{1-g^2/2x_{1,k}x_{4,j}^-}$$

$$1=\prod_{\substack{j=1\\j\neq k}}^{K_2}\frac{u_{2,k}-u_{2,j}-i}{u_{2,k}-u_{2,j}+i}\prod_{j=1}^{K_3}\frac{u_{2,k}-u_{3,j}+\frac{i}{2}}{u_{2,k}-u_{3,j}-\frac{i}{2}}\prod_{j=1}^{K_1}\frac{u_{2,k}-u_{1,j}+\frac{i}{2}}{u_{2,k}-u_{1,j}-\frac{i}{2}}$$

$$1=\prod_{j=1}^{K_2}\frac{u_{3,k}-u_{2,j}+\frac{i}{2}}{u_{3,k}-u_{2,j}-\frac{i}{2}}\prod_{j=1}^{K_4}\frac{x_{3,k}-x_{4,j}^+}{x_{3,k}-x_{4,j}^-}$$

$$\begin{aligned}1=\left(\frac{x_{4,k}^-}{x_{4,k}^+}\right)^L\prod_{\substack{j=1\\j\neq k}}^{K_4}\left(\frac{x_{4,k}^+-x_{4,j}^-}{x_{4,k}^--x_{4,j}^+}\,\frac{1-g^2/2x_{4,k}^+x_{4,j}^-}{1-g^2/2x_{4,k}^-x_{4,j}^+}\,\color{red}{\sigma^2(x_{4,k},x_{4,j})}\right)\\ \times\prod_{j=1}^{K_1}\frac{1-g^2/2x_{4,k}^-x_{1,j}}{1-g^2/2x_{4,k}^+x_{1,j}}\prod_{j=1}^{K_3}\frac{x_{4,k}^--x_{3,j}}{x_{4,k}^+-x_{3,j}}\prod_{j=1}^{K_5}\frac{x_{4,k}^--x_{5,j}}{x_{4,k}^+-x_{5,j}}\prod_{j=1}^{K_7}\frac{1-g^2/2x_{4,k}^-x_{7,j}}{1-g^2/2x_{4,k}^+x_{7,j}}\end{aligned}$$

$$1=\prod_{j=1}^{K_6}\frac{u_{5,k}-u_{6,j}+\frac{i}{2}}{u_{5,k}-u_{6,j}-\frac{i}{2}}\prod_{j=1}^{K_4}\frac{x_{5,k}-x_{4,j}^+}{x_{5,k}-x_{4,j}^-}$$

$$1=\prod_{\substack{j=1\\j\neq k}}^{K_6}\frac{u_{6,k}-u_{6,j}-i}{u_{6,k}-u_{6,j}+i}\prod_{j=1}^{K_5}\frac{u_{6,k}-u_{5,j}+\frac{i}{2}}{u_{6,k}-u_{5,j}-\frac{i}{2}}\prod_{j=1}^{K_7}\frac{u_{6,k}-u_{7,j}+\frac{i}{2}}{u_{6,k}-u_{7,j}-\frac{i}{2}}$$

$$1=\prod_{j=1}^{K_6}\frac{u_{7,k}-u_{6,j}+\frac{i}{2}}{u_{7,k}-u_{6,j}-\frac{i}{2}}\prod_{j=1}^{K_4}\frac{1-g^2/2x_{7,k}x_{4,j}^+}{1-g^2/2x_{7,k}x_{4,j}^-}$$

local charge contributions

$$q_r(x)=\frac{1}{r-1}\left(\frac{i}{(x^+)^{r-1}}-\frac{i}{(x^-)^{r-1}}\right)$$

auxiliary scattering phase

$$\theta_{r,s}(x_1,x_2)=(\tfrac{1}{2}g^2)^{(r+s-1)/2}\,q_r(x_1)\,q_s(x_2)$$

function $\sigma(x_1,x_2)$ for quantum strings

$$\sigma(x_1,x_2)=\exp\left(i\sum_{r=2}^{\infty}\big(\theta_{r,r+1}(x_1,x_2)-\theta_{r+1,r}(x_1,x_2)\big)\right)$$

resummed function $\sigma(x_1,x_2)$

$$\sigma(x_1,x_2)=\frac{1-\dfrac{g^2}{2x_1^-x_j^+}}{1-\dfrac{g^2}{2x_1^+x_2^-}}\left(\frac{1-\dfrac{g^2}{2x_1^-x_2^+}}{1-\dfrac{g^2}{2x_1^+x_2^+}}\,\frac{1-\dfrac{g^2}{2x_1^+x_2^-}}{1-\dfrac{g^2}{2x_1^-x_2^-}}\right)^{i(u_1-u_2)}$$

Challenge: Derive equations from strings? first principles, constraints?

Finite Size & Quantum Effects

Two types of effects at $\mathcal{O}(1/L)$. [Frolov, Tseytlin hep-th/0306130]

Spectrum of fluctuations around a classical solution:

⋆ Strings: Expand around classical solution to second order.

⋆ Bethe ansatz: Add one Bethe root. [NB, Minahan Staudacher, Zarembo][Freyhult hep-th/0405167]

- Find set of allowed positions (discrete).
- Compute backreaction on condensates of Bethe roots.
- Relate to certain cycles on the curve? (c.f. matrix models)

Quantum energy shift of classical solutions:

⋆ Strings: Sum of fluctuation energies $\delta E = \frac{1}{2}\sum_k e_k$. [Frolov Park Tseytlin][Tirziu Park Tseytlin]

⋆ Bethe ansatz: Careful expansion in $1/L$, anomalies. [NB, Tseytlin Zarembo][Hernández, López Periáñez, Sierra]

- Using analyticity, can derive $\delta E = \frac{1}{2}\sum_k \tilde{e}_k$ for $\mathfrak{su}(2)$ sector. [NB, Freyhult hep-th/0506243]
- Generalize to complete $\mathfrak{psu}(2,2|4)$ at higher loops?

Overview Deformations

Many results for $\mathcal{N} = 4$ SYM (few proofs). More general models:

$\star$ **Plane Wave Matrix Quantum Mechanics**

- Similar spin chain model.
- Perturbative computations somewhat simpler.

$\star$ **Twist Deformations**

- Large class of integrable deformations.

$\star$ **Large-N_c QCD**

- Open chains.
- Towards phenomenology?

$\star$ **Orbifolds of $\mathcal{N} = 4$ SYM**

- May preserve all integrability.

Plane Wave Matrix Quantum Mechanics

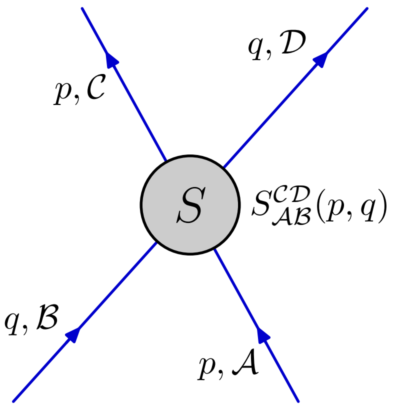
Truncation of $\mathcal{N} = 4$ SYM to finitely many spin orientations. [Berenstein Maldacena Nastase] [Kim Klose Plefka]

- Spin chain with $\mathfrak{su}(2|4)$ symmetry.
- Hamiltonian in $\mathfrak{su}(2)$ sector computed at four loops. [Klose Plefka] [Fischbacher Klose, Plefka]
- Bethe equations derived.
- Violation of BMN scaling at two/four loops.
- Wrapping effect computed, not yet compatible with Bethe equation.
- Complete higher-loop Bethe equations? Similar to $\mathcal{N} = 4$ SYM.
- Count states. Partition functions. Integrability? [Spradlin Van Raamsdonk Volovich] [Hadizadeh Ramadanovic Semenoff, Young]

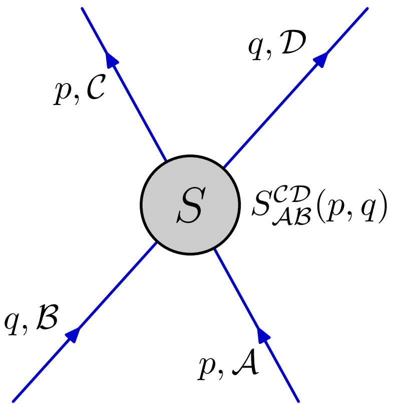
Twist Deformations

See also Frolov's & Lunin's talk.

Introduce phases when two charged excitations cross.



$$\mapsto \exp\left(\tfrac{i}{2}\vec{q}_{\mathcal{B}} \times \vec{q}_{\mathcal{A}} - \tfrac{i}{2}\vec{q}_{\mathcal{C}} \times \vec{q}_{\mathcal{D}}\right)$$



Antisymmetric product $\vec{q} \times \vec{q}' = C_{ab} q_a q'_b$, antisymmetric twist matrix C_{ab}.

- Similar twist for Hamiltonian: β-deformed $\mathcal{N} = 4$. [Leigh Strassler][Berenstein Cherkis][Lunin Maldacena]
- Twist of flavor symmetry: $*$-product.
- Can twist spacetime symmetry: Sort of noncommutative theory. [NB Roiban]
- YBE preserved by generic twist.
- Bethe equations for complete twisted model proposed. [Frolov Roiban Tseytlin][NB Roiban]

Large-N_{c} QCD

- One-loop integrability found several years ago. [Lipatov ICTP 1997][Belitsky, Braun Gorsky, Korchemsky]
- Not completely integrable, only in sectors.
- One-loop Bethe equations for maximal integrable sector. [NB, Ferretti Heise, Zarembo]
- Antiferromagnetic ground state. [Ferretti, Heise Zarembo]
- Open spin chains.
- Higher-loop integrability?
- Why integrability? Inherited from $\mathcal{N} = 4$ SYM?

Orbifolds of $\mathcal{N} = 4$ SYM

- Interesting & rich types of spin chains.
- Good chances for complete & higher-loop integrability. When?
- Some results & Bethe equations. [Wang, Wu hep-th/0311037][Chen, Wang, Wu hep-th/0401016]
- No conclusive picture yet.

Conclusions

- Exciting subject.
- A lot of progress in the last few years. This talk:
  - ⋆ Solved the spectral problem of classical strings.
  - ⋆ Proposed Bethe equations for $\mathcal{N} = 4$ SYM (and quantum strings).
- Many questions remain: Hard problems, easy problems.
- What about three-loop disagreement? Wrappings? (hard problem)
  Better question: Why two-loop agreement? (some understanding)
- Integrability: when and why? Inherited from $\mathcal{N} = 4$ SYM?!
- *If Bethe equations is an answer, how general can the question be?*