From OSV and OSV to OSV

Frederik Denef

University of Leuven

Strings 2006, Beijing

with G. Moore, to appear

From Open String Vacua and Objects Splitting under moduli Variation to a conjecture by Ooguri, Strominger and Vafa

Frederik Denef

University of Leuven

Strings 2006, Beijing

with G. Moore, to appear

Outline

Introduction and overview

Fareytail expansion

The split character of polar states

Why OSV is right (?)

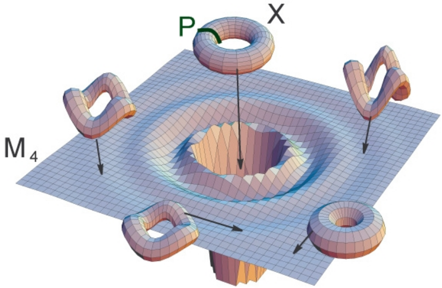
Why OSV is wrong (?)

Note: closely related work and talks at Strings 2006:

- Xi Yin [Gaiotto-Strominger-Yin]
- Andy Strominger [Gaiotto-Strominger-Yin]
- Erik Verlinde [Cheng-de Boer-Dijkgraaf-Manschot-E.Verlinde]

Introduction and overview

Setting



- IIA on Calabi-Yau X $\rightsquigarrow$ 4d $\mathcal{N}=2$ supergravity $+(h^{1,1}+1)$ gauge fields
- D6-D4-D2-D0 BPS bound st. (D-branes + gauge flux) $\rightsquigarrow$ BPS black holes with magn. and el. charges (p^0, p^A, q_A, q_0)

The OSV conjecture

Warning: In this and what follows, often $2 \equiv \pi \equiv i \equiv 1$.

Define

$$\mathcal{Z}_{osv}(\phi) \equiv \sum_q \Omega(p,q)\, e^{\phi \cdot q}$$

where $\Omega(p,q)$ is second helicity supertrace in charge sector (p,q):

$$\Omega(p,q) = -\tfrac{1}{2}\mathrm{Tr}_{p,q}\,(-)^f f^2 = Tr'_{p,q}\,(-)^{f'}$$

with $f = 2J_3$ and Tr', f' same but with universal center of mass half-hypermultiplet $(0,0,\frac{1}{2})$ factored out.

[Ooguri-Strominger-Vafa] conjectured:

$$\mathcal{Z}_{osv}(\phi) \sim \mathcal{Z}_{top}(g_{top},t)\,\overline{\mathcal{Z}_{top}(g_{top},t)}$$

with identifications:

$$g_{\mathrm{top}} = \frac{1}{\phi^0 + i\,p^0}, \qquad t^A = \frac{\phi^A + i\,p^A}{\phi^0 + i\,p^0}.$$

The OSV conjecture: some general remarks

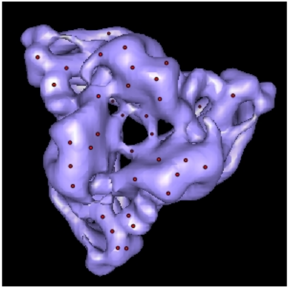
- Is example of open string ($\mathcal{Z}_{osv}$) - closed string ($\mathcal{Z}_{top}$) duality.
- Writing it as

$$\Omega(p, q) \sim \oint \mathcal{Z}_{osv}(\phi)\, e^{-\phi \cdot q}$$

  the conjecture can be seen as giving infinite series of corrections to Bekenstein-Hawking entropy formula.
- Some (well known) problems with $\mathcal{Z}_{osv} \sim |\mathcal{Z}_{top}|^2$:
  - Periodicity LHS and RHS do not match
    sol.: make RHS periodic by sum over integral shifts $\phi \to \phi + 2\pi n i$
  - Both sides badly divergent
    so-so sol.: regularize by considering more physical $\mathcal{Z}$
  - $\Omega(p, q)$ strongly depends on background Kähler moduli $B + iJ$: jumping phenomena at walls of marginal stability.
    so-so-so sol.: take infinite radius limit $B + iJ = i\infty$.

Specialize to zero D6-charge ($p^0 = 0$)

D4-D2-D0 bound states given by D4 wrapped on 4-cycle P + $U(1)$ fluxes F through nontrivial 2-cycles of P + pointlike D0-branes:



Susy configurations ("Open String Vacua"): [Mariño-Minasian-Moore-Strominger]:

$$P \text{ holomorphic}, \qquad F^{0,2} = 0.$$

Second condition constrains deformation moduli of P [Gaoitto-Guica-Huang-Simons-Strominger-Yin, Gomis-Marchesano-Mateos].

D4 partition function

D4 on euclidean $(S^1)_\beta \times$ (very ample) divisor $P = p^A D_A$ + $U(1)$ flux F + N bound (anti-)D0-branes:

$$\mathcal{Z}_{D4}(\beta, C) = \sum_{N,F} \mathrm{Tr}'_{N,F} \, (-)^{f'} \, e^{-\beta H + iC\cdot q} = \sum_q \Omega(p,q) \, e^{-\beta |H_{BPS}(p,q)| + iC\cdot q}$$

where summed over induced D0- and D2-charges are:

$$q_0 = \frac{\chi(P)}{24} - N + \int_P \frac{F^2}{2}, \qquad q_A = \int_P D_A \wedge F.$$

with $\chi(P) = P^3 + c_2 P$, $F \in \frac{c_1(P)}{2} + H^2(P, \mathbb{Z})$.

Then:

$$\mathcal{Z}_{D4}|_{\beta=0, C=i\phi} = \mathcal{Z}_{osv}(\phi)$$

(note: limit singular)

Deriving OSV: rough outline of strategy

1. Use $SL(2,\mathbb{Z})$-duality of partition sum to rewrite $\mathcal{Z}_{D4}$ as Fareytail/Rademacher series built on polar part $\mathcal{Z}_{D4}^{-}$.
2. Note that polar BPS states naturally split. In 4d sugra: not realized as single centered black hole, but as two-centered black hole "molecule" with first center (D6-D4-D2-D0) and second center anti-(D6-D4-D2-D0). Microscopically: described by quiver with a D6 and an anti-D6 node.
3. $\Rightarrow$ in suitable asymptotic limit:

$$\mathcal{Z}_{D4} \sim \mathcal{Z}_{\mathrm{D6-D4-D2-D0}} \, \mathcal{Z}_{\mathrm{anti-(D6-D4-D2-D0)}}.$$

4. Identify

$$\mathcal{Z}_{\mathrm{D6-D4-D2-D0}} \rightsquigarrow \mathcal{Z}_{DT}, \quad \mathcal{Z}_{\mathrm{anti-(D6-D4-D2-D0)}} \rightsquigarrow \overline{\mathcal{Z}_{DT}}.$$

5. Use $\mathcal{Z}_{DT} = \mathcal{Z}_{GW}$ to get

$$\mathcal{Z}_{D4} = |\mathcal{Z}_{top}|^2.$$

.

Fareytail expansion

Polar terms and Fareytail expansion of modular forms

Rademacher-Jacobi-Farey-Poincaré $\rightsquigarrow$ "Fareytail" [DMMV]

Let $f(\tau)$ be modular form of weight w:

$$f(A\cdot\tau) = j(A,\tau)\, f(\tau)$$

where $A \equiv \binom{ab}{cd} \in SL(2,\mathbb{Z})$, $A\cdot\tau \equiv \frac{a\tau+b}{c\tau+d}$, $j(A,\tau) \equiv \omega_A(c\tau+d)^w$, and ω_A is some phase.

Has Fourier expansion:

$$f(\tau) = \sum_{n=0}^{\infty} c_n\, e^{2\pi i(n-\Delta)\tau}$$

$\rightsquigarrow$ polar part $f^-(\tau)$ is just (finite) sum of terms with $n-\Delta<0$.

$\rightsquigarrow$ determines full $f(\tau)$:

$$f(\tau) = {\sum_A}' j(A,\tau)^{-1}\, f^-(A\cdot\tau).$$

[Divergent for $w<2$ but can be "renormalized" by adding polynomial in τ.]

Polar terms and Fareytail expansion of $\mathcal{Z}_{D4}$

With $\tau \equiv \oint C_1 + dt/g_{IIA} = C_0 + \beta/g_{IIA} = \tau_{IIB}$:

$$\mathcal{Z}_{D4}(\beta, C_1, C_3) = \mathcal{Z}_{D4}(\tau, \bar{\tau}, C_3) = \sum_{\gamma} \Psi_{\gamma}(\tau, \bar{\tau}, C_3)\, H_{\gamma}(\tau)$$

where $\gamma \in H^2(P,\mathbb{Z})/(H^2(X,\mathbb{Z})|_P \oplus H^2(X,\mathbb{Z})|_P{}^{\perp}) \rightsquigarrow$ "glue vector", Ψ_γ Siegel-Narain theta function for lattice $H^2(X,\mathbb{Z})|_P$ with shift $\gamma + P/2$.

Not just simple modular form but: TST duality $\Rightarrow$ H_γ modular vector $\Rightarrow$ $\mathcal{Z}_{D4}$ generalized multivariable Jacobi form transforming schematically as

$$\mathcal{Z}_{D4}(A \cdot \tau, \ldots) = j(A, \tau, \ldots)\, \mathcal{Z}_{D4}(\tau, \ldots)$$

$\rightsquigarrow$ Fareytail expansion:

$$\mathcal{Z}_{D4} = {\sum_{A}}' j(A, \tau, \ldots)^{-1}\, \mathcal{Z}_{D4}^{-}(A \cdot \tau, \ldots)$$

Here polar part $\mathcal{Z}_{D4}^{-}$ are terms with

$$\widehat{q}_0 \equiv q_0 - \tfrac{1}{2}(D_{ABC}\, p^C)^{-1} q_A q_B > 0.$$

The split character of polar states

Polar states split

In large radius approx, entropy of D4-D2-D0 black hole (with very ample P, i.e. $P > 0$) is

$$S = 2\pi\sqrt{-\widehat{q}_0\,\chi(P)/6}$$

Polar terms: $\widehat{q}_0 > 0 \rightsquigarrow$??? No BH solution!

Related to fact that central charge $Z(p, q; t)$ has zero locus in moduli space, distinct from discriminant locus. Attractor flow = gradient flow $\log|Z| \rightsquigarrow$ "crashes" on zero.

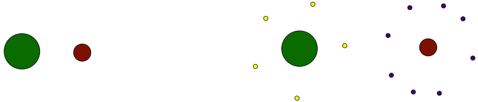
Polar BPS states cannot exist at zero locus (would be massless hence on discriminant locus, but is not), but exist by assumption at large $\mathrm{Im}\, t$.

$\Rightarrow$ when moving to zero of Z, must cross wall of marginal stability on which BPS state decays, i.e. splits in two constituent BPS states.

[is robust under α' corrections]

4d supergravity realization of polar states

Not single centered black hole, but two (or more) centered BPS bound state (equilibrium distance fixed) [FD]:



Clusters carry D6–charge $(r, -r)$, $r \neq 0$, and arbitrary D4-D2-D0 charges.

Simplest example: "most polar" state (largest $\widehat{q}_0$, namely $\widehat{q}_0 = \chi(P)/24$) = pure D4 + flux F in $H^2(X, \mathbb{Z})$ only = bound state of single pure D6 (+flux) and anti-D6 (+flux).

Approaching MS wall: two clusters get infinitely separated:

$\Rightarrow$ expect factorization of index of BPS states:

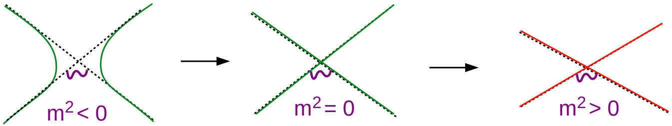
$$\Omega_{tot} = k\, \Omega_1\, \Omega_2$$

with k = electron-monopole type LLL degeneracy = intersection product betw. two clusters $\equiv \langle 1, 2 \rangle$.

Microscopic D-brane picture

Bound states are geometrical (bundles/slags), and always localized at one point in noncompact space.

E.g. decay at MS wall in mirror intersecting D3-brane picture:



k intersection points of same sign $\rightsquigarrow$ k light chiral multiplets Φ_i with D-term potential

$$V \sim (\sum_{i=1}^{k} |\phi_i|^2 - \xi)^2$$

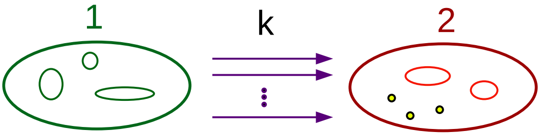
Vacua: $V = 0$ mod $U(1) = \mathbb{CP}^{k-1}$.

Relation to multicentered "molecular" bound states?

$\rightsquigarrow$ D-brane picture valid in limit $g_s \to 0$: multicenter equilibrium distance in string units $\sim g_s \to 0 \Rightarrow$ tachyon condensation collapses configuration into single D-brane [FD qq&hh].

Microscopic D-brane picture: moduli space near MS

Near MS (stable side):



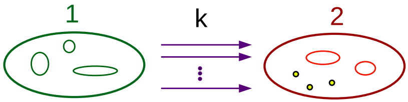
Here k = generic # light open stretched strings = $\langle 1, 2 \rangle$.

So expect moduli space $\mathcal{M}$ to be fibration over $\mathcal{M}_1 \times \mathcal{M}_2$ with generic fiber $\mathbb{CP}^{k-1}$. If no fiber degeneracies:

$$\chi(\mathcal{M}) = k\,\chi(\mathcal{M}_1)\,\chi(\mathcal{M}_2)$$

and corresponding factorization for index Ω. Note: agrees with sugra picture.

Microscopic D-brane picture: most polar states



"Most polar" terms (at large P roughly $\widehat{q}_0 > \frac{1}{4}\frac{\chi(P)}{24}$) are bound states of single $D6$ with flux and dilute $\overline{D2}$ gas, and $\overline{D6}$ with flux and dilute $D2$ gas, plus dilute $\overline{D0}$ gas in either D6 or anti-D6 (which one depends on background B).

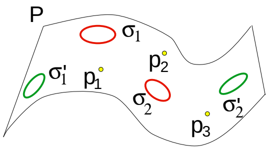
BPS states of D6-D4-D2-D0 system presumably counted by Donaldson-Thomas inv $N_{DT}(p,q)$ [Iqbal,Nekrasov,Okounkov,Vafa, Dijkgraaf-Verlinde-Vafa] (Subtlety: DT invariants do not care about background moduli, but BPS spectrum does.)

$\Rightarrow$ For above D-brane system and under above assumptions:

$$\Omega = k\, N_{DT}^1\, N_{DT}^2$$

Microscopic D-brane picture: moduli space at large radius

How do we see this structure directly at large radius?



Susy configurations: [Gaoitto-Guica-Huang-Simons-Strominger-Yin]

$$F^{2,0} = 0 \Leftrightarrow F = \iota_P^* S + \widehat{\sigma} - \widehat{\sigma}'$$

where $S \in H^2(X)$, σ, σ' are collection of holomorphic curves, hat denotes Poincaré dual 2-form in $H^2(P, \mathbb{Z})$.

$\Rightarrow$ susy config. parametrized by picking N points p_i in X and hol. curve collections σ, σ' in X and requiring P to pass through all of those. For large P / small N, σ, σ' (i.e. sufficiently polar states), this is good parametrization.

Gives moduli space $\mathcal{M}$ again as $\mathbb{CP}^{k-1}$ fibration over moduli of dilute $D2$ gas + $\overline{D2}$ gas + $\overline{D0}$-gas. Further nontrivial checks: charges and values k agree in two pictures! ✓

Why OSV is right (?)

Sketch of formal derivation OSV

We had (in OSV limit):

$$\mathcal{Z}_{osv}(\phi) = \sum_A{}' j(A,\phi)^{-1}\, \mathcal{Z}^-_{osv}(A\cdot\phi)$$

with

$$\mathcal{Z}^-_{osv}(\phi) = \sum_{\widehat{q}_0>0} \Omega(p,q)\, e^{\phi\cdot q}, \quad \Omega(p,q) = \sum_{(p,q)\to 1+2} \langle 1,2\rangle\, \Omega_1\, \Omega_2.$$

We want to use this to extract degeneracies

$$\Omega(P,Q) = \oint d\phi\, \mathcal{Z}_{D4}(\phi)\, e^{-\phi\cdot Q}$$

as saddle point series (so we take $\widehat{Q}_0 < 0$). Then:

- Dominant contributions come from $A = \left(\begin{smallmatrix} 0 & -1 \\ 1 & d \end{smallmatrix}\right)$.
- For ϕ^0 sufficiently small, the most polar terms dominate, and splits in single D6 / anti-D6 states dominate $\rightsquigarrow \Omega_i = N^i_{DT}$.

Sketch of formal derivation OSV

Putting all this together, using relation $\mathcal{Z}_{DT} = \mathcal{Z}_{top}$ [INOV, Maulik-Nekrasov-Okounkov-Pandharipande, DVV], ignoring variation of $\langle 1, 2 \rangle$ (and fact that correct "fiber contribution" to degeneracies might be more complicated than $\langle 1, 2 \rangle$), various exponentially suppressed terms and more critically thinking collaborators, this gives

$$\Omega(P, Q) \sim I_P \int d\phi \, \phi^0 \, \mathcal{Z}_{top}(p, \phi) \, \overline{\mathcal{Z}_{top}(p, \phi)}$$

where $I_P = \chi(\mathcal{M}_P) = P^3/6 + c_2 P/12$ and the residual $SL(2, \mathbb{Z})$ sum $\phi^0 \to \phi^0 + id$ and theta function sum $\phi \to \phi + iS$ are absorbed in extending the contours over the entire imaginary axis.

In other words, (essentially) OSV.

Why OSV is wrong (?)

A problem

- In this and other derivations, it is important for justifying approximations that ϕ^0 (or τ) is sufficiently small. But since $g_{top} \sim 1/\phi^0$, this means g_{top} sufficiently large $\rightsquigarrow$ opposite of a priori supposed regime of validity of OSV.
- For D4-D2-D0 system with $\mathcal{Z}_{top}$ in DT or GV form, this regime is not immediately nonsensical, since is effectively expansion in $g_{top}^n e^{-g_{top}\beta\cdot P}$, $\beta \neq 0$ and $e^{-g_{top}}$, so expansion parameters go to zero when $g_{top} \to \infty$.
- But what about small g_{top} regime? OSV valid? $\rightsquigarrow$ depends on growth polar degeneracies.
- Hard to compute directly, but if naively estimated from 2-centered black hole BH entropies, growth too strong to have valid derivation of OSV!

An explanation and a puzzle

In fact, if we take black hole entropy as estimate for index $\Omega(P, Q)$ (with $\Omega(P, Q)$ defined at $B + iJ = i\infty$), we can see directly that OSV fails at small g_{top}, even at leading order in saddle point approximation:

- Small g_{top} regime = large (P, Q) regime. More precisely, when $(P, Q) \to \Lambda(P, Q)$, at saddle point $g_{top} \to g_{top}/\Lambda$.
- But for sufficiently large Λ, there is always a 2-centered black hole solution whose BH entropy scales as Λ^3, while single centered scales as Λ^2! $\Rightarrow$ Two-centered entropy parametrically larger than single centered!
- Since leading order OSV prediction for Ω is precisely $e^{S_{BH}}$ for single centered BH, the conjecture already fails at leading order in the large Λ regime...

Possible resolutions

- BH entropy is maybe not a good estimate for index: since one must sum over different configurations, there may in principle be miraculous cancelations, leading to a much smaller index both in the latter considerations as well as for the growth of the polar degeneracies. Seems like a far stretch, but who knows...
- Maybe defining $\Omega(P,Q)$ at $t = i\infty$ is the wrong thing to do. Bothersome multicentered configurations are e.g. guaranteed to be absent at attractor point $t_*(P,Q)$, so maybe one should define $\Omega(P,Q)$ at $t_*(P,Q)$. Problem: brings closed string elements in open string $\mathcal{Z}_{D4}$, not natural from point of view of D4-D2-D0 partition function (no attractor points for polar terms!), spoils modular invariance, ...
- to get scaling $S \sim \Lambda^3$, one needs to keep P^0 of centers fixed $\rightsquigarrow$ sugra entropy formula not valid? This would imply $\log \Omega(P^0, 0, Q, 0)$ does not scale as $Q^{3/2}$ in limit $Q \to \infty$, P^0 fixed.
- Other way to define $\mathcal{Z}_{osv}$?

Conclusions

Better understanding made us understand better that we understand less than we thought we understood.