Geometry of Calabi-Yau Moduli Space and Flux Vacua

T. Eguchi and Y. Tachikawa
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♣ Moduli Problem

We consider string theory compactified on CY manifolds. CY manifolds in general have a number of moduli associated with the freedom of changing their complex and Kähler structures. These degrees of freedom appear as massless scalar fields in 4 dimensions.

Existence of massless scalars is in direct conflict with phenomenology. One has to generates a potential V for moduli fields so that they are fixed at the extremum of the potential.

In the following we consider type IIB theory and concentrate on stabilizing the complex structure moduli z_a $(a = 1, \ldots, h_{2,1})$.

A superpotential becomes generated when RR or NS fluxes are turned on,

$$H^{RR} \equiv dB^{RR},\ H^{NSNS} \equiv dB^{NSNS},\ \tau = C_0 + ie^{-\phi}$$

$$W(z_a) = \int_M (H^{RR} - \tau H^{NSNS}) \wedge \Omega(z_a)$$
$$= \sum N_I X_I(z_a) - \sum M_I F_I(z_a)$$

where

$$M_I = \int_{A_I} (H^{RR} - \tau H^{NSNS}),\ N_I = \int_{B_I} (H^{RR} - \tau H^{NSNS})$$
$$I = 0, 1, \cdots, h_{2,1}$$

are fluxes through A_I and B_I cycles and

$$X_I = \int_{A_I} \Omega, \quad F_I = \int_{B_I} \Omega = \frac{\partial F}{\partial X_I}$$

are their periods. Gukov-Vafa-Witten

It is then possible to fix all complex structure moduli.

$$\frac{DW}{Dz_a} = 0 \Longrightarrow \{z_a\} \text{ all fixed}$$

• Kähler potential on Calabi-Yau moduli space is given by

$$K = -\log i \int_M \Omega \wedge \bar{\Omega} = -\log i \sum_I \left(X_I \bar{F}_I - \bar{X}_I F_I\right)$$

Freedom of Kähler transformation:

$$K(z_a, \bar{z}_a) \to K(z_a, \bar{z}_a) + f(z_a) + \bar{f}(\bar{z}_a),$$

$$\Omega(z_a) \to e^{-f(z_a)}\Omega(z_a), \quad W(z_a) \to e^{-f(z_a)}W(z_a)$$

Periods (X_I, F_I) are holomorphic sections of a line bundle L. Metric is invariant under Kähler transformation

$$g_{a\bar{b}} = \partial_a \partial_{\bar{b}} K$$

♣ Number of vacua in string theory

- Fix CY mfd M
- number of 3-cycles: 100∼200
- Upper bound on fluxes:

$$\int H_1 \wedge H_2 \leq \text{const. depending on geometry of M}$$
$$\approx 1000 - 5000 \text{ (tadpole condition)}$$

- **possible choice of fluxes:**

$$10^{100} \sim 10^{200}$$

Altogether there exist an enormous number of string vacua $\mathcal{O}(10^{100})$

♣ Statistical treatment

Douglas, Ashok, Denef, ...

Vacua distribution function on moduli space $\mathcal{M}$

$$\rho(z) = \delta(D_a W)\delta(D_{\bar{b}} W^*) \times \left| \det \begin{pmatrix} \partial_a D_b W & \partial_a D_{\bar{b}} W^* \\ \partial_{\bar{a}} D_b W & \partial_{\bar{a}} D_{\bar{b}} W^* \end{pmatrix} \right|$$

Simplify $\Downarrow$

$$\tilde{\rho}(z) = \delta(D_a W)\delta(D_{\bar{b}} W^*) \times \det \begin{pmatrix} \partial_a D_b W & \partial_a D_{\bar{b}} W^* \\ \partial_{\bar{a}} D_b W & \partial_{\bar{a}} D_{\bar{b}} W^* \end{pmatrix}$$

This is an index counting the number of vacua with $\pm$ signs.

Further simplifying assumption:
Fluxes obey Gaussian distribution $\Longrightarrow W$ itself obeys Gaussian distribution.

It follows

$$\tilde{\rho}(z) \prod dz^a \wedge d\bar{z}^{\bar{a}} = \det \frac{1}{2\pi} \underbrace{\left(R^a_{\ b} + \delta^a_{\ b}\, \omega\right)}_{\text{curvature and Kähler form on } \mathcal{M}}$$

This depends only on the geometry of CY moduli space and is the Euler number of the bundle $T\mathcal{M} \otimes L$. Flux vacua should be concentrated around singular points in CY moduli space where the curvature $R^a_{\ b}$ is peaked.

♣ Singular loci in Calabi-Yau moduli space

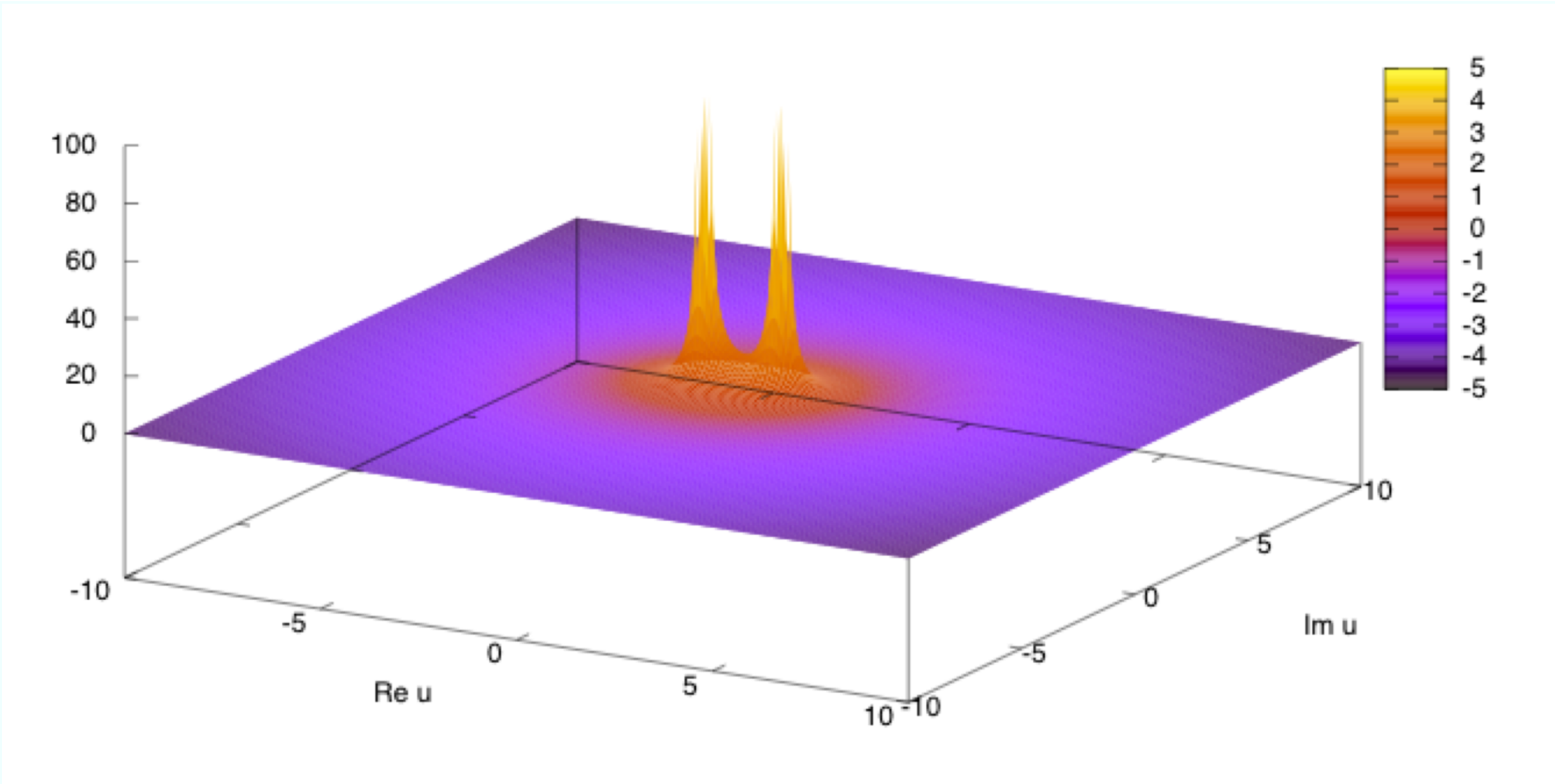
String vacua are not distributed uniformly but concentrated around singular locus of $\mathcal{M}$. We study the distribution of vacua around singular loci in CY moduli space where interesting non-perturbative phenomena take place.

Types of singularities:

- conifolds: generation of massless matter multiplets.
- ADE singularities and rigid limit: gauge symmetry enhancements and decoupling of gravity
- Argyres-Douglas points: massless electrons and monopoles.
- Large complex structure limit: Mirror of the large radius limit.
- $\cdots$

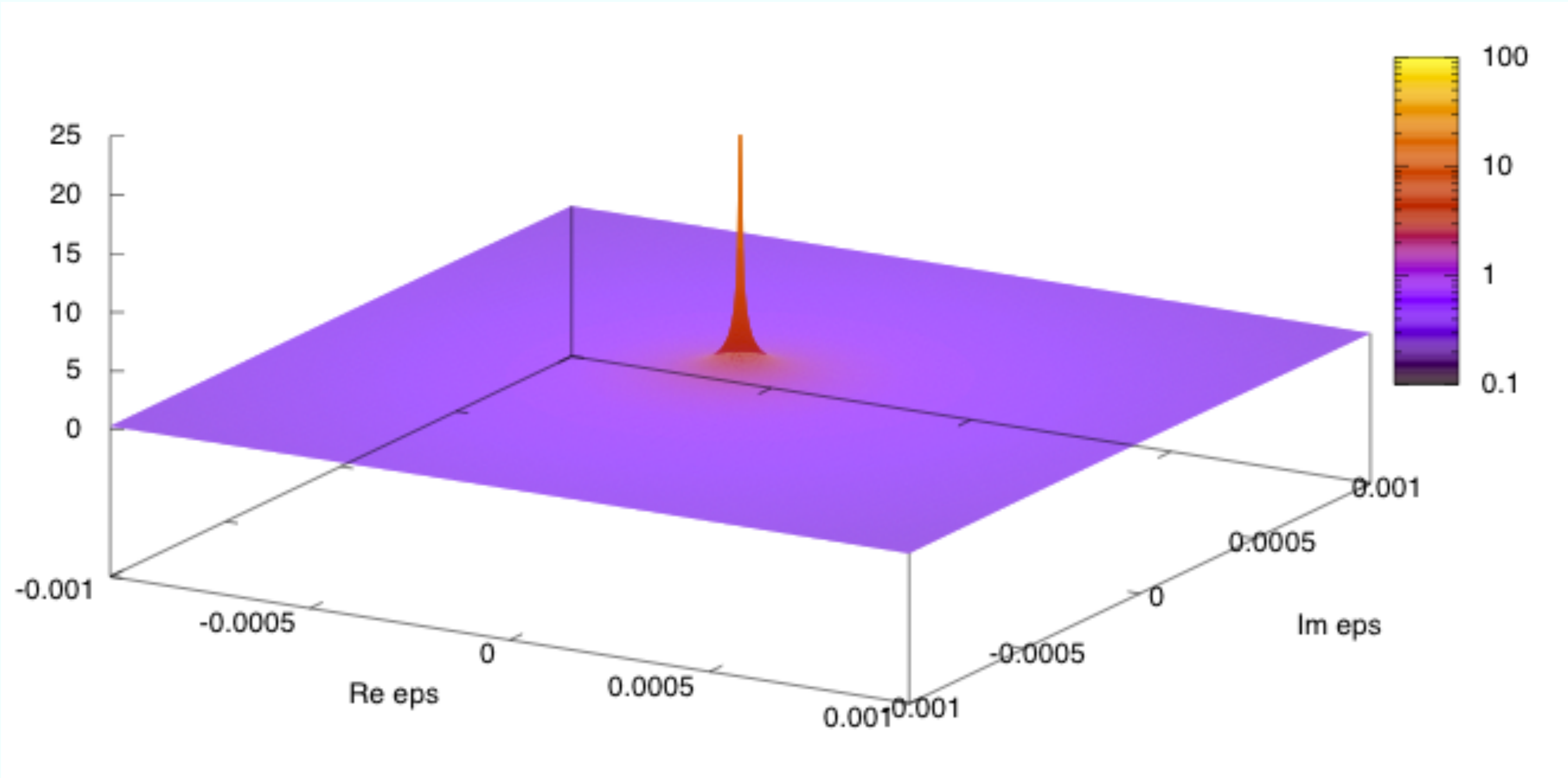
Examples: on the Seiberg-Witten u plane.

$$\tilde{\rho} \sim \frac{1}{|u \mp 1|^2 (\log |u \mp 1|)^2} \quad \textbf{near} \quad u \sim \pm 1$$



Near the rigid limit.

$$\tilde{\rho} \sim \frac{1}{|\epsilon|^2 (\log|\epsilon|)^2} \qquad \text{near} \quad \epsilon \sim 0$$



We claim that the vacuum density behaves as

$$\text{vacuum density} \approx \frac{dz d\bar{z}}{|z|^2 \log |z|^2}$$

around each of these singular points. Note that the integral around $z = 0$ is finite

$$\int d^2 z \frac{dz d\bar{z}}{|z|^2 \log |z|^2} < \infty$$

so that there exist a finite number of vacua around these singular loci.

♣ Special geometry relations

- metric
  $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$

- **Yukawa coupling**
  $$F_{ijk} = \sum_I X_I \partial_i \partial_j \partial_k F_I - (X \leftrightarrow F)$$

- **curvature**
  $$R_{i\bar{j}k\bar{\ell}} = g_{i\bar{j}} g_{k\bar{\ell}} + g_{i\bar{\ell}} g_{k\bar{j}} - e^{2K} g^{m\bar{n}} F_{ikm} \bar{F}_{\bar{j}\bar{\ell}\bar{n}}$$

♣ Nilpotent Orbit Theorem

Assemble periods $(X_I, F_I) \Longrightarrow \Omega_I \;\; (I = 1, \cdots, h_{2,1} + 2)$
Under monodromy transformation

$$\Omega_I \to M \Omega_I$$

eigenvalues of M **are roots of unity (**$1/k$**-th power,say). Then** $N \equiv M^k - 1$ **becomes a nilpotent matrix and after a change of variable** $a = z^k$ **one**

finds

$$\Omega_I = e^{\frac{N}{2\pi i}\log a}\left[\Omega_I^{(0)} + a\Omega_I^{(1)} + a^2\Omega_I^{(2)} + \cdots\right]$$

There is an integer p so that

$$N^p\Omega_I \neq 0 \qquad \text{but} \qquad N^{p+1}\Omega_I = 0$$

Then we find $X_I, F_I \approx \log^p a$.

♣ Conifolds

singularity at $z = 0$

$$z = \int_A \Omega, \qquad \partial_z F = \int_B \Omega$$

Under monodromy transformation

$$A \to A,\ \ B \to B + A$$

Thus

$$\begin{cases} \Omega_1 = z \\ \Omega_2 = \dfrac{1}{2\pi i} z \log z \\ \cdot \\ \cdot \end{cases}$$

Hence

$$K \approx \log(\textbf{const} + |z|^2 \log|z|) \approx \textbf{const}' + |z|^2 \log|z|$$

$$\Downarrow$$

$$g_{z\bar{z}} \approx \log|z|$$

$$\Downarrow$$

$$R_{z\bar{z}} \approx -\partial_z \partial_{\bar{z}} \log\det g_{z\bar{z}} = \frac{1}{|z|^2 \log^2|z|}$$

One can generalize the discussion to many variable cases and present a general analysis. Instead we would like to present more specific examples in the following.

♣ Large Complex Structure Limit

$$\begin{cases} \Omega_1 = \text{const} \\ \Omega_2 \approx \log z \\ \Omega_3 \approx (\log z)^2 \\ \Omega_4 \approx (\log z)^3 \\ \cdot \\ \cdot \end{cases}$$

$$K \approx \log(\log |z|^3) \Longrightarrow g_{z\bar{z}} \approx \frac{1}{|z|^2 \log |z|^2} \Longrightarrow R_{z\bar{z}} \approx \frac{1}{|z|^2 \log |z|^2}$$

We again find the same distribution.

Probably the most interesting cases are the non-compact limit of Calabi-Yau manifolds where K_3 fibration develops ADE singularities. In this limit gravitational degrees of freedom become decoupled and string theory is reduced to SUSY gauge theories.

♣ Decoupling (Rigid) Limit: SU(2) Example

$$X_8[1,1,2,2,2]:$$
$$W = \frac{B}{8}x_1^8 + \frac{B}{8}x_2^8 + \frac{1}{4}x_3^4 + \frac{1}{4}x_4^4 + \frac{1}{4}x_5^4 - \psi_0 x_1 x_2 x_3 x_4 x_5 - \frac{1}{4}\psi_2 (x_1 x_2)^4$$

By a change of variable $x_0 = x_1 x_2, \zeta = (x_1/x_2)^4$, W may be written as

$$W(x; B', \psi_0) = \frac{1}{4}(B' x_0^4 + x_3^4 + x_4^4 + x_5^4) - \psi_0 x_0 x_3 x_4 x_5$$

with

$$B' = \frac{1}{2}(B\zeta + \frac{B}{\zeta} - 2\psi_2)$$

This is a K3 fibration over P^1. Decoupling limit is given by

$$B \longrightarrow 0$$

When we parametrize

$$B = \epsilon \Lambda^2, \qquad \psi_2 + \psi_0^4 = \epsilon u,$$

and make a suitable redefinition of variables we obtain an A_1 singularity fibered over P^1:

$$W = \frac{\epsilon}{2}\left[\frac{1}{2}(\zeta + \frac{\Lambda^4}{\zeta}) + y_1^2 + y_2^2 + y_3^2 - u\right]$$

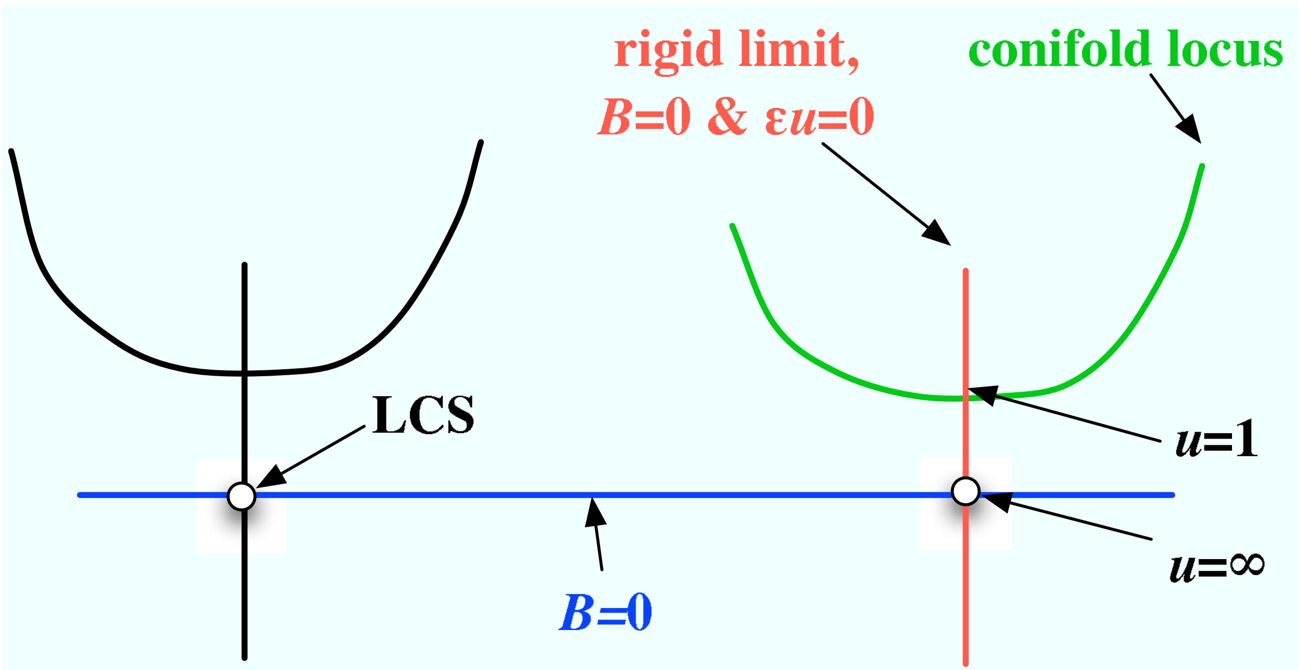
Discriminant of CY manifold is given by

$$\Delta_{CY} \propto B^2 (B^2 - \psi_2^2)(B^2 - (\psi_2 + \psi_0^4))$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$

decoupling LCS $SU(2)$

where LCS is the large complex structure limit.



♣ Decoupling (Rigid) Limit: $SU(3)$ Example

$$X_{24}[1,1,2,8,12]:$$

$$W = \frac{B}{24}(x_1^{24} + x_2^{24}) - \frac{\psi_2}{12}(x_1 x_2)^{12} + \frac{1}{12}x_3^{12} + \frac{1}{3}x_4^3 + \frac{1}{2}x_5^2$$
$$-\psi_0 x_1 x_2 x_3 x_4 x_5 - \frac{1}{6}\psi_1 (x_1 x_2 x_3)^6$$

This space again has a K3 fibration. By a change of variable $x_0 = x_1 x_2,\ \zeta = (x_1/x_2)^{12}$ it is rewritten as

$$W = \frac{B'}{12}x_0^{12} + \frac{1}{12}x_3^{12} + \frac{1}{3}x_4^3 + \frac{1}{2}x_5^2 - \psi_0 x_0 x_3 x_4 x_5 - \frac{1}{6}\psi_1 (x_0 x_3)^6$$
$$B' = \frac{1}{2}(B\zeta + \frac{B}{\zeta}) - \psi_2$$

Discriminant is given by

$$\Delta_{CY} = B^2(B^2 - (\psi_1^2 + \psi_2)^2)(B^2 - ((\psi_1 + \psi_0^6)^2 + \psi_2)^2)(B^2 - \psi_2^2)$$

$$\begin{array}{ccc} \Downarrow & \searrow \quad \swarrow & \Downarrow \\ \text{Decoupling} & \text{SU(3)} & \text{LCS} \end{array}$$

Decoupling limit is taken as

$$B = \epsilon\Lambda^3, \quad \psi_0^6\psi_1 = \epsilon u^{3/2}, \quad \psi_1^2 + \psi_2 = \epsilon(v - u^{3/2}), \qquad \epsilon \longrightarrow 0$$

By a suitable redefinition of variables we obtain an A_2 singularity fibered over P^1

$$W = \epsilon\left[\frac{1}{12}(\zeta + \frac{\Lambda^6}{\zeta}) + \frac{y_3^2}{2} + \frac{y_4^2}{2} + \frac{y_5^3}{3} - \frac{u}{4}y_5 - \frac{v}{12}\right]$$

Billó-Denef-Frè-Pesando-Troost-Van Proyen and Zanon, hep-th/9803228 made a detailed analysis of these models: they explicitly constructed

3-cycles and evaluated the behavior of the periods in the decoupling limit.

In the $X_{24}[1,1,2,8,12]$ model there exist 8 cycles

$$\left(V_{v_a}, V_{v_b}, V_{t_a}, V_{t_b}, T_{v_a}, T_{v_b}, T_{t_a}, T_{t_b}\right)$$

out of which $V_{v_a}, V_{v_b}, T_{v_a}, T_{v_b}$ are the periods of $SU(3)$ gauge theory. Other cycles are needed when one embeds gauge theory into super-gravity. Periods behave as

$$V_{v_a}, V_{v_b}, T_{v_a}, T_{v_b} \sim \epsilon^{1/3} : \qquad \text{gauge theory periods}$$

$$V_{t_a}, V_{t_b}, T_{t_a}, T_{t_b} \sim \text{const} + \text{const}' \cdot \log \epsilon : \qquad \text{gravity periods}$$

In the case of $SU(2)$ example, gauge theory periods behave as $\epsilon^{1/2}$ and gravity periods as const+const'$\cdot \log \epsilon$.

• Vacuum density near decoupling point

We have the behavior of periods

$$\begin{cases} \Omega_1 \approx \log \epsilon \\ \Omega_2 \approx \epsilon^{1/N} \\ \cdot \\ \cdot \end{cases}$$

and the Kähler potential

$$K \approx \log \left[\log |\epsilon| + |\epsilon|^{2/N} K(u, u^*, \cdots) + \cdots \right]$$

Here $K(u, u^*, \cdots)$ denotes the Kähler potential of the gauge theory. We then have the behavior

$$g_{\epsilon\bar{\epsilon}} \approx \frac{1}{|\epsilon|^2 \log |\epsilon|^2}, \qquad R_{\epsilon\bar{\epsilon}} \approx \frac{1}{|\epsilon|^2 \log |\epsilon|^2}$$

We again find the same enhancement of vacuum concentration near decoupling point.

• Heterotic Duals and RG Flow

Presence of two length scales $|\epsilon|^{2/N}$, $\log 1/|\epsilon|$ suggest a ratio of mass scales Λ of gauge theory and that of ambient supergravity

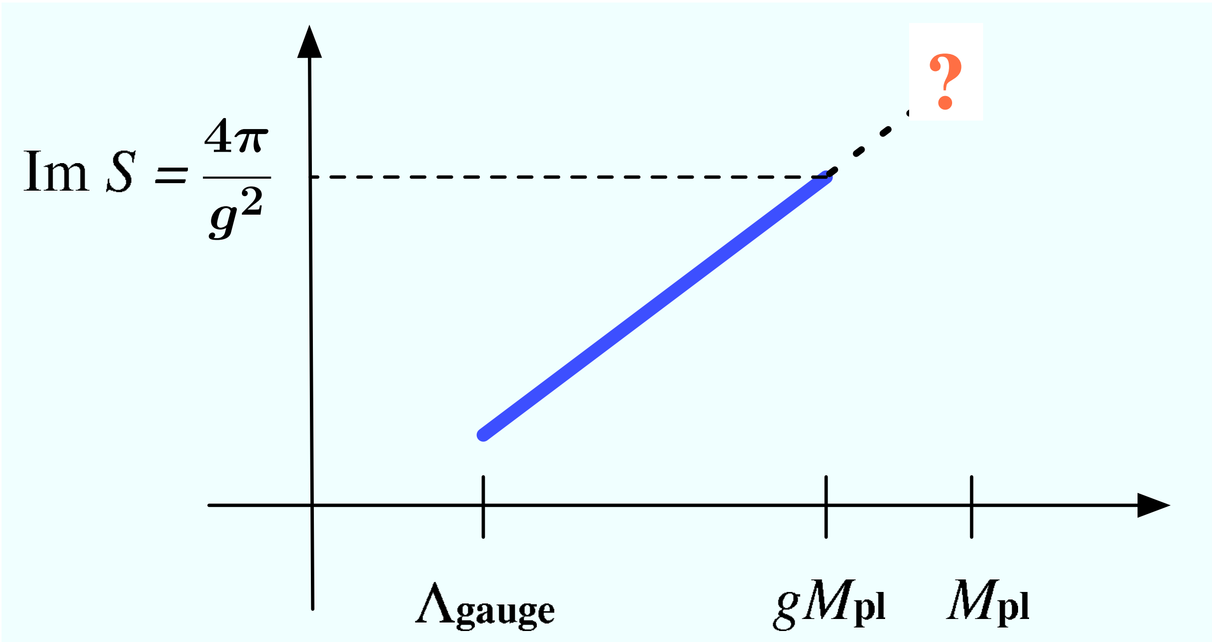
$$\frac{|\epsilon|^{2/N}}{\log 1/|\epsilon|} \approx \left(\frac{\Lambda_{gauge}}{M_{pl}}\right)^2$$

On the other hand, it is well-known that the above models have a dual heterotic description: the model $X_{24}[1,1,2,8,12]$, for instance, coincides with the (S,T,U) model of heterotic string compactified on $K_3 \times T^2$.

Here $\log 1/\epsilon$ corresponds to the variable S where $S = \dfrac{4\pi}{g^2}$ is the heterotic dilaton and $\epsilon \approx e^{-8\pi^2/g^2}$. Then the above mass ratio can be written as

$$\Lambda_{gauge} \approx e^{-8\pi^2/Ng^2}\; gM_{pl}$$

As compared with the standard (one-loop) RG formula, there exists an extra factor of g in front of the right-hand-side. This is in fact the length scale of heterotic string theory and has the form of the recent proposal by Arkani-Hamed-Motl-Nicolis-Vafa of an anomalously small mass scale $\Lambda = gM$ in field theory embedded in gravity (swampland conjecture).



♣ Summary

We have seen a universal behavior for the curvature

$$R_{z\bar{z}} \approx \frac{1}{|z|^2 \log|z|^2}$$

around various singular loci of CY moduli space. Although our analysis is not exhaustive, we may assume its general validity and suppose that there do not occur singularities with a worse behavior in CY moduli space. If this is the case, the number of flux vacua is finite in type IIB string theory. It is curious to see that the function $(|z|\log|z|)^{-2}$ gives a maximum amount of enhancement under the condition of integrability. Thus by choosing a point $z \approx 0$ in the moduli space we do not lose much statistical weights in the distribution of vacua and this fact may have interesting implications on the issue of vacuum selection in string theory.

- **Mathematical treatments**

As we see, the metric of moduli space at LCS limit and decoupling limit has the form of the Poincare metric. Metric near conifold has a weak logarithmic singularity. Thus it seems fairly clear that the volume of the moduli space should be finite. Rigorous derivation of the finiteness of the volume of moduli space has been given by Todorov, Lu-Sun.
Discussion of finiteness of the integral of Chern classes is more complex and is recently given by Douglas-Lu.