Surface Operators in Gauge Theory and Categorification

Sergei Gukov

based on:

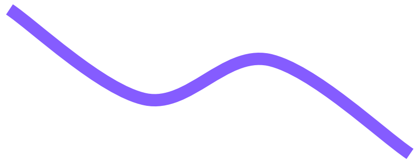
S.G., A.Schwarz, C.Vafa, hep-th/0412243

N.Dunfield, S.G., J.Rasmussen, math.GT/0505662

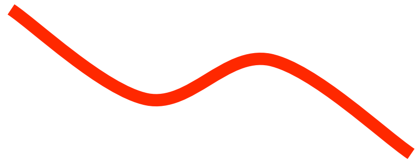
S.G., J.Walcher, hep-th/0512298

joint work with E.Witten

- Line operators:



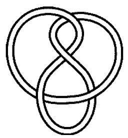
Wilson line



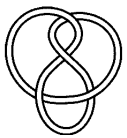
't Hooft line

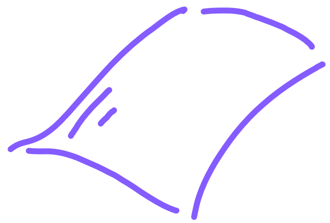
- in Chern-Simons theory

[E.Witten]

$\langle$  $\rangle = q^{-5} + q^{5}$

- Surface operators in 4D gauge theory

$\langle$  $\rangle =$ vector space



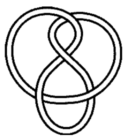
Categorification

categorification → / ← dimension

Number ⇄ Vector Space ⇄ Category

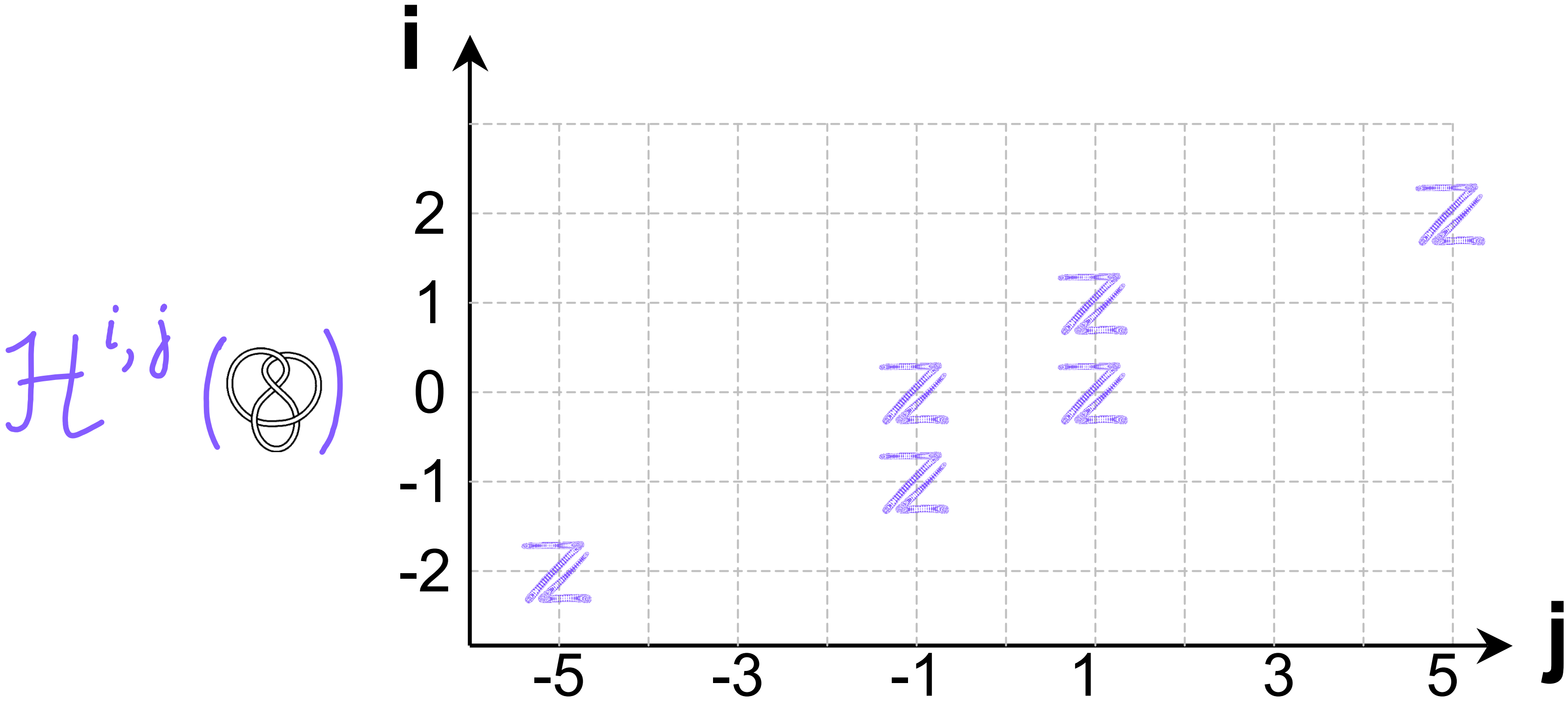
categorification → / ← Grothendieck group

- Knot homology



$$\left\langle \text{knot} \right\rangle = \sum_{i,j} (-1)^i \, q^j \, \dim \mathcal{H}^{i,j}$$

Euler characteristic = Chern-Simons invariant



$$\chi(\mathcal{H}) = q^{-5} + q^{5}$$

A general picture of knot homologies

G	Knot Polynomial	Knot Homology
$U(1\|1)$	Alexander	knot Floer homology $HFK(K)$
"SU(1)"	——	Lee's deformed theory $H'(K)$
SU(2)	Jones	Khovanov homology $H^{Kh}(K)$
SU(N)	$P_N(q)$	sl(N) homology $HKR^N(K)$

Physical Interpretation

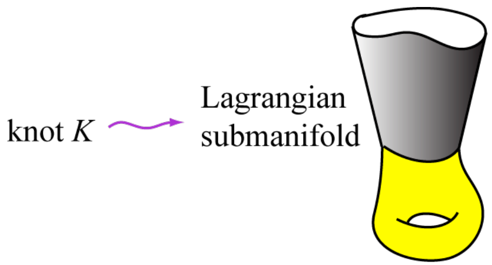
$\mathcal{H} = \mathcal{H}_{BPS}$ space of BPS states [S.G., A.Schwarz, C.Vafa]

M-theory on $\mathbb{R}^5 \times$ (conifold)

$\cup \qquad \cup$

M5-brane on $\mathbb{R}^3 \times$ Lagrangian

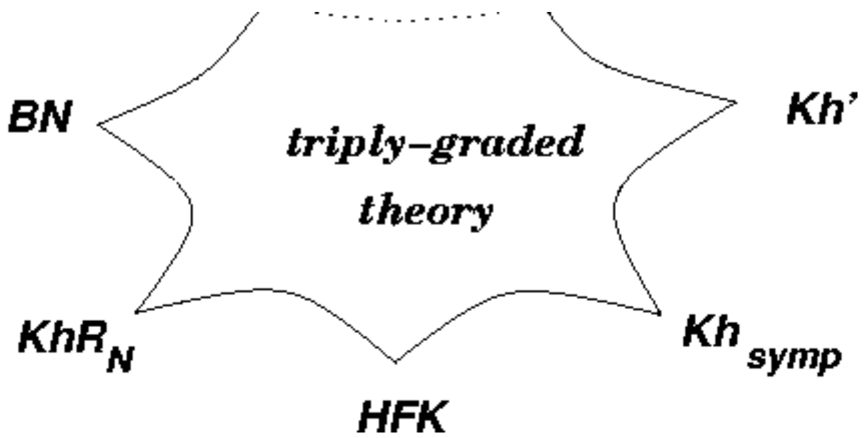
Earlier work:
[H.Ooguri, C.Vafa]
[J.Labastida, M.Marino, C.Vafa]



BPS state:
membrane ending on the Lagrangian five-brane

- Surprisingly, this physical interpretation leads to a rich theory, which unifies all the existing knot homologies

[N.Dunfield, S.G., J.Rasmussen]



$\mathcal{H}_{BPS}$ graded by J_L, J_R, and membrane charge Q

What's Next?

- Generalization to other groups and representations
- The role of matrix factorizations [S.G., J.Walcher]
- Finite N (stringy exclusion principle)
- Realization in topological gauge theory
    - Boundaries, corners, …
    - Surface operators
    - Braid group actions on D-branes

Gauge Theory and Categorification

gauge theory on a 4-manifold X	$\rightsquigarrow$	number $Z(X)$ (partition function)
gauge theory on $\mathbb{R} \times Y$, Y = 3-manifold	$\rightsquigarrow$	vector space $\mathcal{H}_Y$ (Hilbert space)
gauge theory on $\mathbb{R}^2 \times \Sigma$, Σ = surface	$\rightsquigarrow$	category of branes $\mathcal{F}_\Sigma$ (boundary conditions)

gauge theory on X
self-duality equations:

$$F_A^+ + \dots = 0$$

$\rightsquigarrow$ $Z(X)$ counts solutions

gauge theory on $\mathbb{R} \times Y$
monopole equations:

$$F_A + \bar{M}M + \dots = 0$$

$\rightsquigarrow$ $\mathcal{H}_Y = H^*(\mathcal{M}_Y)$

$\mathcal{M}_Y$ = moduli space

gauge theory on $\mathbb{R}^2 \times \Sigma$
vortex equations:

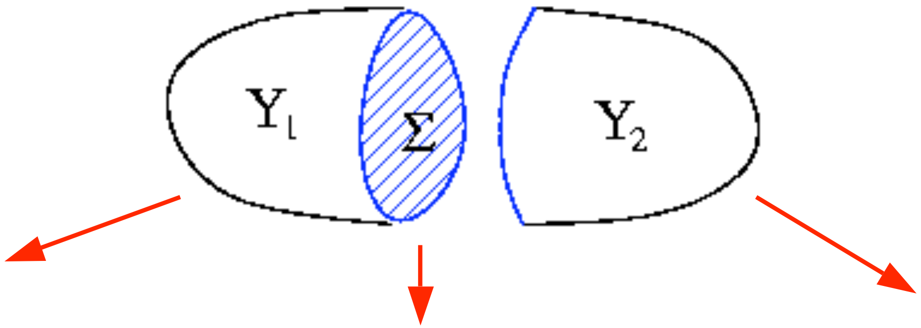
$$*F + |\phi|^2 + \dots = 0$$

$\rightsquigarrow$ topological A-model/B-model

$$\mathcal{F}_\Sigma = \begin{cases} \mathrm{Fuk}(\mathcal{M}_\Sigma) \\ D^b(\mathcal{M}_\Sigma) \end{cases}$$

Gauge Theory with Boundaries

In three-dimensional topological gauge theory:



vector

$Z_{Y_1} \in \mathcal{H}_\Sigma$

vector space

$\mathcal{H}_\Sigma$

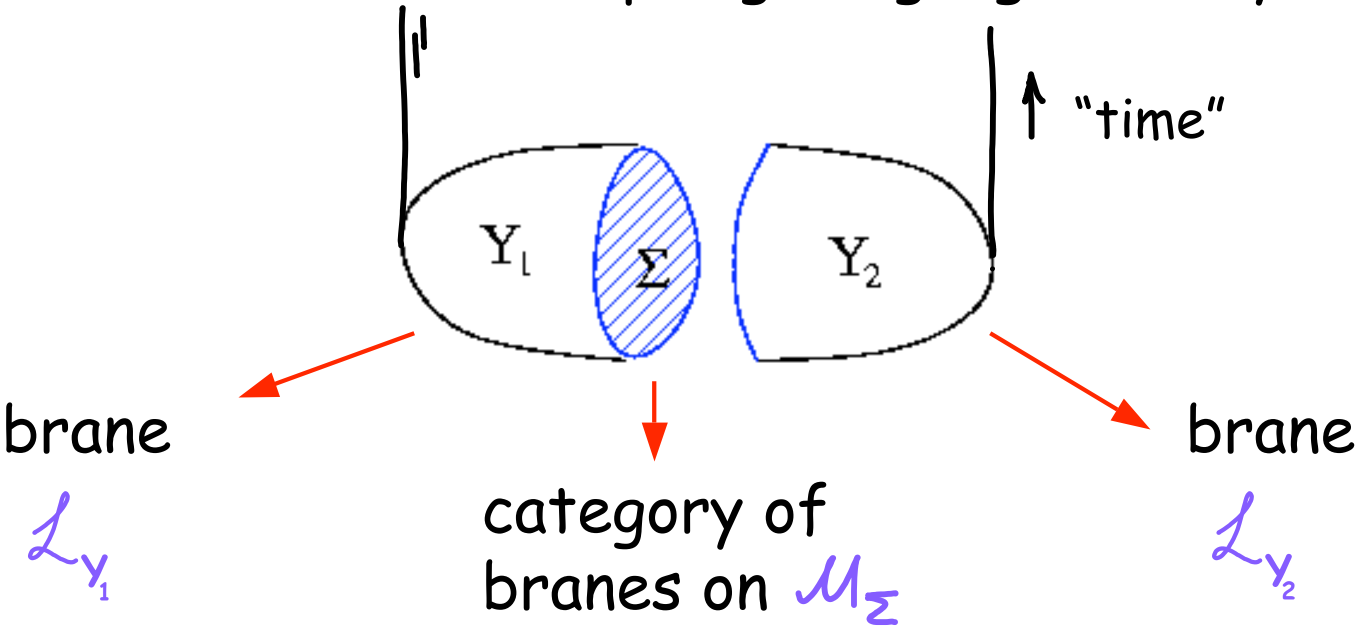
vector

$Z_{Y_2} \in \mathcal{H}_\Sigma$

$Y = Y_1 \cup Y_2 \quad \Rightarrow \quad Z_Y = \langle Z_{Y_1} | Z_{Y_2} \rangle$

Gauge Theory with Corners

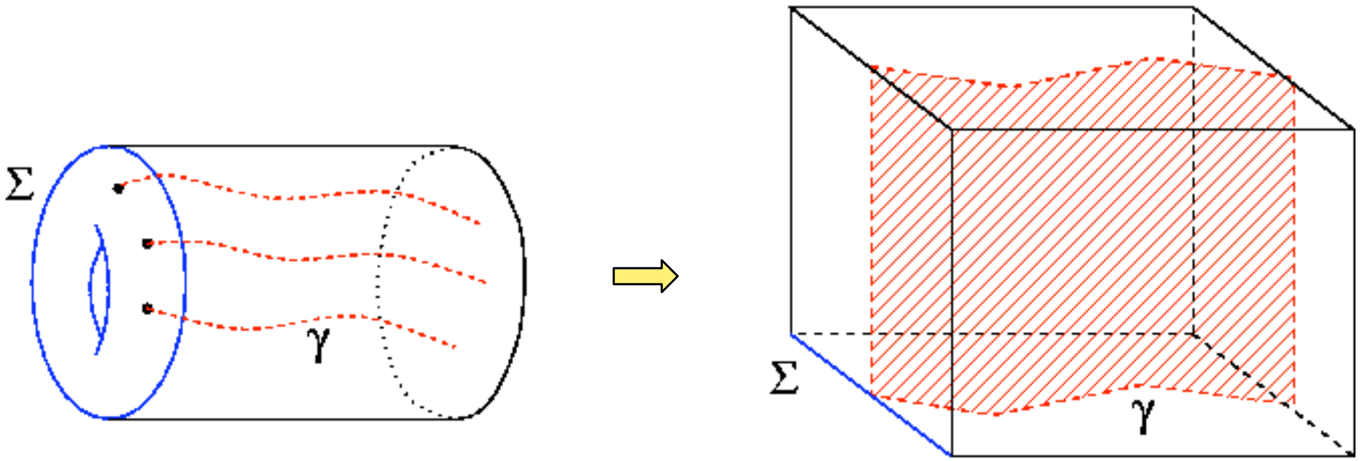
In four-dimensional topological gauge theory:



⇨ A-model: $\mathcal{H}_Y = HF_*^{symp}(\mathcal{M}_\Sigma; \mathcal{L}_{Y_1}, \mathcal{L}_{Y_2})$

("Atiyah-Floer conjecture")

From Lines to Surfaces

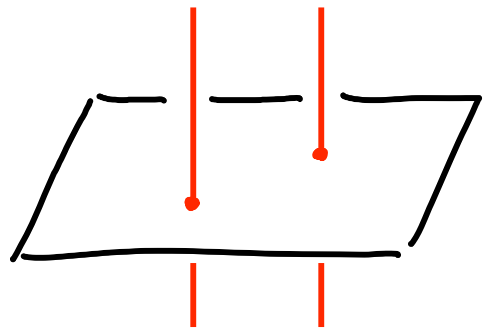


- A line operator lifts to an operator in 4D gauge theory localized on the surface $S = \mathbb{R} \times \gamma$ where the gauge field A has a prescribed singularity

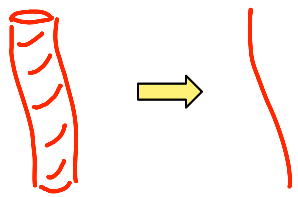
$$\mathrm{Hol}\,(A) \in C \qquad \text{fixed conjugacy class in } G$$

Sometimes a surface operator (a.k.a. “impurity” or “singular vortex”) can be also described using:

- σ-model on $S \subset X$ coupled to gauge theory on X [E.Witten]
- Intersecting D-branes in string theory [A.Kapustin, S.Sethi] …



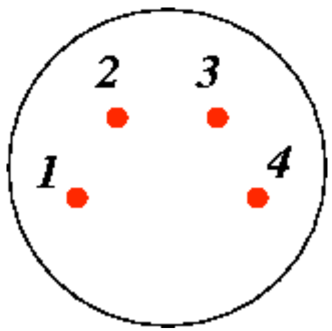
- Singular limits of smooth vortex solutions in four-dimensional gauge theory



Braid Group Actions on D-branes

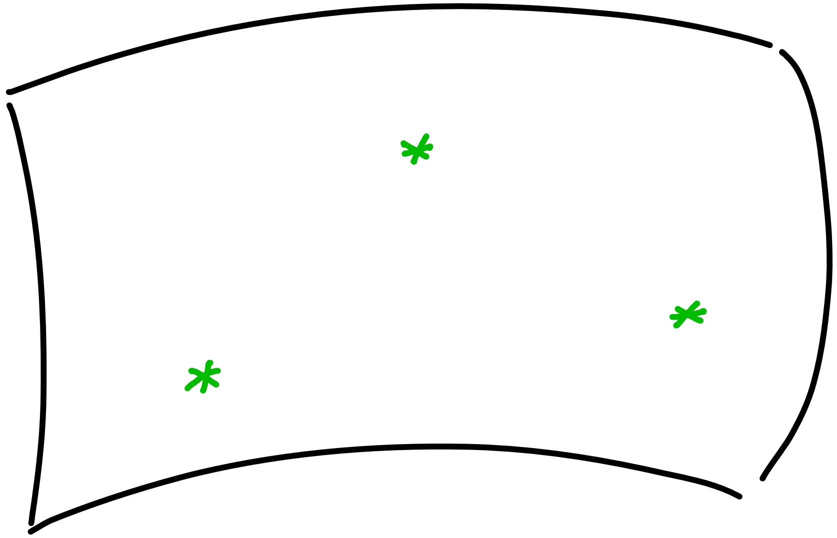
- Any four-dimensional topological gauge theory which admits supersymmetric surface operators provides (new) examples of braid group actions on D-branes

- Example: topological twist (GL twist) of N=4 super-Yang-Mills on $\mathbb{R}^2 \times S^2$ with 4 surface operators

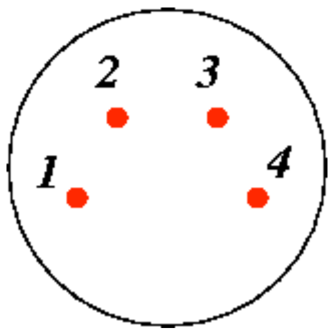
$\Sigma =$ 

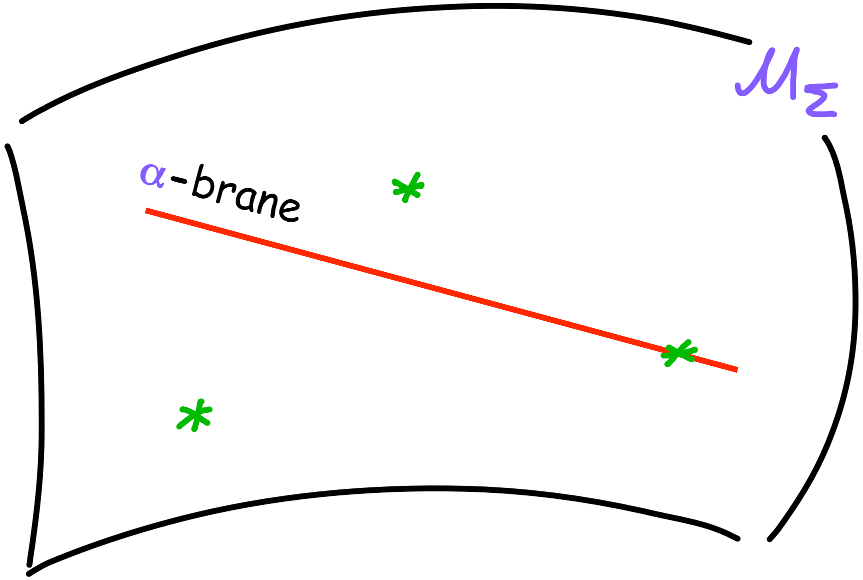
Moduli space:

$\mathcal{M}_{\Sigma}$ = complex surface with three singularities

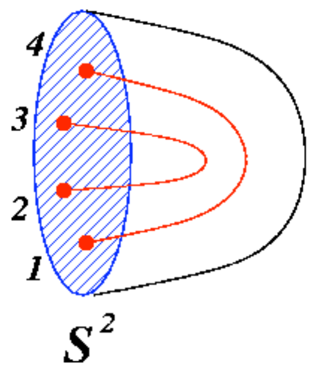


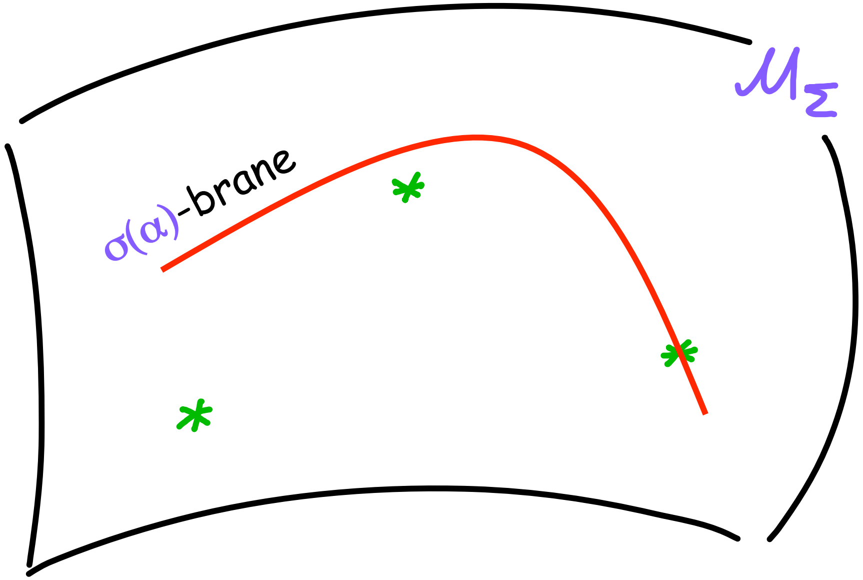
Σ =



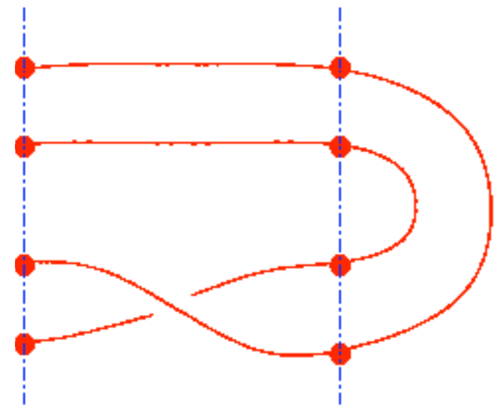


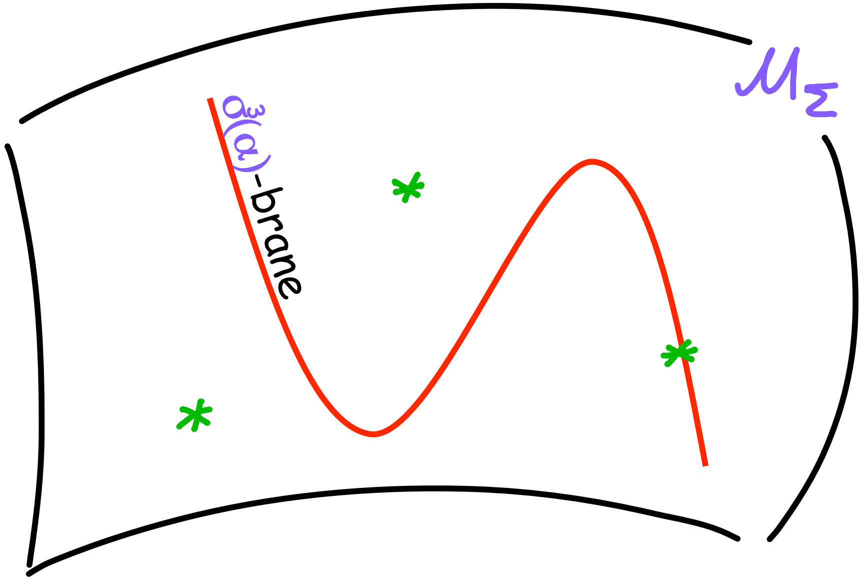
α-brane corresponds to the static configuration of surface operators below ("time" direction not shown)



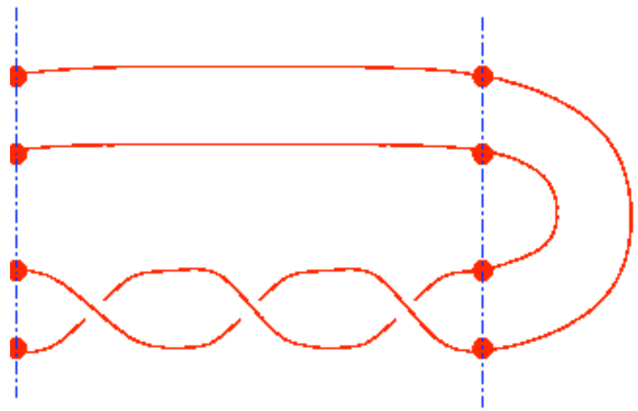


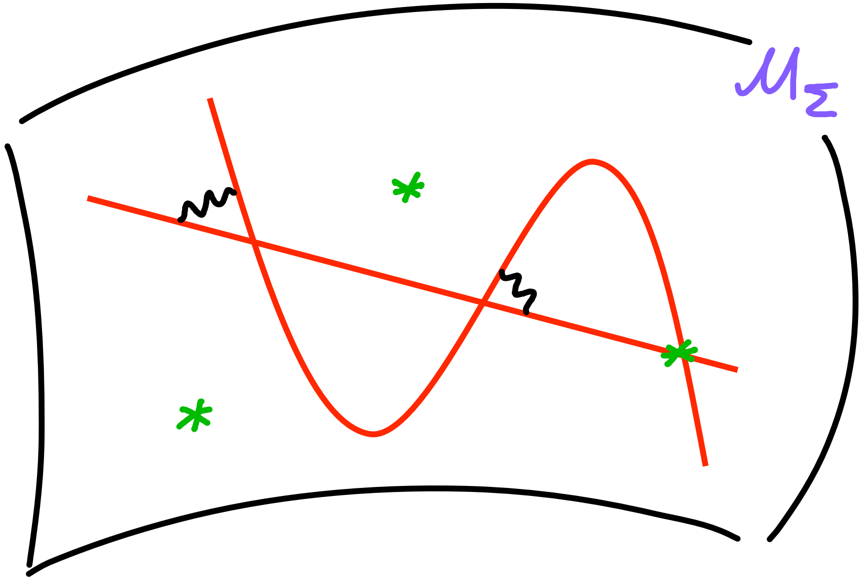
σ(α)-brane corresponds to the static configuration of surface operators with a half-twist

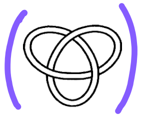




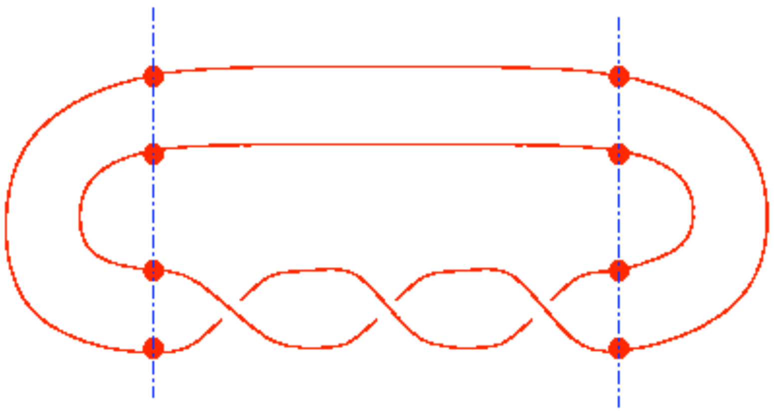
$\sigma^3(\alpha)$-brane corresponds to the static configuration of surface operators with three half-twists





Closing the braid gives $\mathcal{H}($$)$ = space of α - $\sigma^3(\alpha)$ strings

$\chi(\mathcal{H})$ = Casson-like invariant for knots



Topological Twists of SUSY Gauge Theory

- N=2 twisted gauge theory:

  $\chi(\mathcal{H}) = \Delta(q)$ Alexander polynomial

- N=4 twisted SYM (adjoint non-Abelian monopoles):

  $\mathcal{H}^{i,j}$ doubly-graded knot homology

- Partial twist of 5D super-Yang-Mills:

  $\chi(\mathcal{H}) = Z_{\text{Vafa-Witten}}$ $\mathcal{H} = H^*(\mathcal{M}_{\text{instanton}})$

String Theory

Gauge Theory

Topological Strings

Categories

D-branes