NEW ASPECTS OF GENERALIZED GEOMETRY

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STRINGS 2006 Beijing

- unifies different structures
- incorporates B-fields and H-fluxes
- economical repackaging

GENERALIZED GEOMETRY

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- inner product of signature (n, n): $(X+\xi, X+\xi) = i_X\xi = X^i\xi_i$
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- B-fields $X + \xi \mapsto X + \xi + i_X B = (X^i, \xi_j) \mapsto (X^i, \xi_j + X^i B_{ij})$

A generalized complex structure is:

- $J : T \oplus T^* \to T \oplus T^*, \qquad J^2 = -1$

- $(Ju, v) + (u, Jv) = 0$

- if $Ju = iu, Jv = iv$ then $J[u, v] = i[u, v]$ (*Courant bracket*)

EXAMPLES

- complex manifold $J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$

$$J = i : \left[\ldots \frac{\partial}{\partial z_i} \ldots, \ldots d\bar{z}_i \ldots \right]$$

- symplectic manifold $J = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$

$$J = i : \left[\ldots, \frac{\partial}{\partial x_j} + i\omega_{jk} dx_k, \ldots \right]$$

EXAMPLES

- symplectic manifold + B-field

$$J = i : \left[\ldots, \frac{\partial}{\partial x_j} + (B + i\omega)_{jk} dx_k, \ldots \right]$$

POISSON BRACKETS

- $Jdf = X_f + \xi_f$

- $\{f, g\} = X_f(g)$

A generalized Kähler structure consists of:

- generalized complex structures J_1, J_2 on $T \oplus T^*$
- such that $J_1 J_2 = J_2 J_1$
- and $(J_1 J_2 u, u)$ is positive definite.

Theorem (Gualtieri) Up to the action of a B-field, a generalized Kähler structure consists of:

- complex structures I^+, I^-
- Hermitian metric g
- metric connections ∇^+ and ∇^- with skew torsion $\pm H$
- ... such that $\nabla^+ I^+ = 0 = \nabla^- I^-$

- S J Gates, C M Hull, M Roček, *Twisted multiplets and new supersymmetric nonlinear σ-models*. Nuclear Phys. B **248** (1984), 157–186.

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- V Apostolov, P Gauduchon, G Grantcharov, *Bihermitian structures on complex surfaces*, Proc. London Math. Soc. **79** (1999), 414–428

WHICH 4-MANIFOLDS ARE GENERALIZED COMPLEX?

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GENERALIZED COMPLEX MANIFOLDS

- NJH: *Generalized Calabi-Yau manifolds*

  math.DG/0209099 (QJM **54** (2003) 281–308)

- Marco Gualtieri:*Generalized complex geometry*

  math.DG/0401221

COMPLEX POISSON MANIFOLDS

$$\sigma = \sigma^{ij} \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j} \qquad (\sigma^{ij} \text{ holomorphic})$$

$$[\sigma, \sigma] = 0 \ \text{(Schouten bracket)}$$

$$J = i : \left[\ldots, \frac{\partial}{\partial z_j}, \ldots, d\bar{z}_k + \sum_\ell \bar{\sigma}^{k\ell} \frac{\partial}{\partial \bar{z}_\ell}, \ldots \right]$$

COMPLEX POISSON SURFACES

$$\sigma = s \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2} \qquad (s \text{ holomorphic})$$

- $[\sigma, \sigma] = 0$ automatically
- $s^{-1} dz_1 \wedge dz_2$ meromorphic 2-form
- $s^{-1} dz_1 \wedge dz_2 = B + i\omega$

SURGERY

G. Cavalcanti and M. Gualtieri,
A surgery for generalized complex structures on 4-manifolds
math.DG/0602333

$\mathbf{R}^2 \times T^2 = \mathbf{C} \times \mathbf{C}/\Gamma$ AS COMPLEX POISSON

- $B + i\omega = d \log z_1 \wedge dz_2$

- $\omega = d \log r \wedge d\theta_3 + d\theta_1 \wedge d\theta_2$

- $B = d \log r \wedge d\theta_2 - d\theta_1 \wedge d\theta_3$

$\mathbf{R}^2 \times T^2 = \mathbf{C} \times \mathbf{C}/\Gamma$ AS SYMPLECTIC

- $dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2$

- $\omega' = r dr \wedge d\theta_1 + d\theta_2 \wedge d\theta_3$

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- $\omega' = r dr \wedge d\theta_1 + d\theta_2 \wedge d\theta_3$

- $f : T^3 \to T^3$

  $f(\theta_1, \theta_2, \theta_3) = (\theta_3, \theta_2, -\theta_1)$

- $\partial(D^2 \times T^2) = T^3$

- symplectic manifold M, symplectic $T^2 \subset M$ trivial normal bundle

- do surgery, (Gompf-Mrowka C^∞ logarithmic transform) get generalized complex $\tilde{M}$

- if $M =$ elliptic K3, $\tilde{M} = 3\mathbf{C}P^2 \# 19\mathbf{C}\bar{P}^2$

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- **neither complex nor symplectic (Seiberg-Witten)**

GENERALIZED KÄHLER MANIFOLDS

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$$\sigma = h^{i\bar{k}} h^{j\bar{\ell}} a_{\bar{k}\bar{\ell}} \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$$

σ is a holomorphic Poisson structure

C Bartocci and E Macrì, *Classification of Poisson surfaces* math.AG/0402338

- Poisson structure = anticanonical section
- K3, abelian surface – holomorphic symplectic
- blow up points on anticanonical divisor
- minimal ruled surface – some restrictions

MANIFOLDS ($\pi_1 = 0$)

- K3

- $S^2 \times S^2$

- $\mathbf{C}P^2 \# k \mathbf{C}\overline{P}^2$

SOLVING THE EQUATIONS

U Lindström, M Roček, R von Unge & M. Zabzine *Generalized Kähler manifolds and off-shell supersymmetry* hep-th/0512164

Dominic Joyce:

V Apostolov, P Gauduchon, G Grantcharov, *Bihermitian structures on complex surfaces*, Proc. London Math. Soc. **79** (1999), 414–428

- Poisson structures = meromorphic 2-forms

- $\omega + i\omega'$ for I^+

- $\omega + i\omega''$ for I^-

- Hermitian form $\omega_+ = (\omega'')^{(1,1)}$

LOCALLY...

- $f(z_1, z_2) dz_1 \wedge dz_2 = \omega + i\omega' \qquad g(w_1, w_2) dw_1 \wedge dw_2 = \omega + i\omega''$,

- holomorphic Darboux $\Rightarrow$ local diffeomorphism φ such that
$$\varphi^*(\omega + i\omega') = \omega + i\omega''$$

- $\varphi^*\omega = \omega \qquad \varphi^*\omega' = \omega''$

- symplectic diffeomorphism

- real function f
- Hamiltonian (w.r.t. ω) vector field X
- one parameter group φ_t of symplectic diffeomorphisms
- define $\omega'' = \varphi_t^* \omega'$

POSITIVITY OF $\omega''^{(1,1)}$?

$$\frac{\partial}{\partial t}\varphi_t^*\omega'|_{t=0} = \mathcal{L}_X\omega' = d(i_X\omega').$$

- But $(\omega + i\omega')(X,Y) = i(\omega + i\omega')(X, I^+Y)$ and $i_X\omega = df$ ...
- ... so that $i_X\omega' = I^+df$ and hence

$$\frac{\partial}{\partial t}\varphi_t^*\omega'|_{t=0} = dI^+df = dd^cf.$$

- $dd^c f > 0$ = Kähler form
- $\Rightarrow \varphi_t^* \omega'^{(1,1)}$ positive (small t)
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- $\varphi_2 \circ \varphi_1^{-1}$ holomorphic $\Rightarrow$ same bihermitian metric

DEL PEZZO SURFACES

- $\mathbf{C}P^1 \times \mathbf{C}P^1$

- $\mathbf{C}P^2$ blown up at $k \leq 8$ points ..

- ... such that no three are collinear and ..

- ... no six lie on a conic.

- Del Pezzo surface: K^* is ample $=$

- there is a Hermitian metric on K^* whose curvature $F > 0$ (i.e. a Kähler form).

- Take a holomorphic section σ of K^*
- $=$ Poisson structure for I^+
- σ vanishes on an elliptic curve C
- Define $f : S \setminus C \to \mathbf{R}$ by

$$f = \log \|\sigma\|^2$$

- $dd^c \log \|\sigma\|^2 = F$

NEAR C...

- $\omega + i\omega' = dz_1 \wedge dz_2 / z_1$

- $f = \log(h|z_1|^2)$

Hamiltonian vector field =

$$\mathrm{Re}\left(\frac{z_1}{h}\frac{\partial h}{\partial z_2}\frac{\partial}{\partial z_1} - \left(1 + \frac{z_1}{h}\frac{\partial h}{\partial z_1}\right)\frac{\partial}{\partial z_2}\right)$$

THE HERMITIAN FORM

- $\omega_+ = (\varphi_t^* \omega')^{(1,1)}$

$$\frac{\partial \omega_+}{\partial t} = (\varphi_t^* \mathcal{L}_X \omega')^{(1,1)} = (\varphi_t^* dd^c f)^{(1,1)} = (\varphi_t^* F)^{(1,1)}$$

- at $t = 0$, $\omega_+^{(1,1)} = \omega'^{(1,1)} = 0$

- smooth solution $F > 0 \Rightarrow \omega_+$ positive for small t

OTHER CONSTRUCTIONS

- NJH: $SU(2)$-invariant

  $\mathbf{C}P^2$, $C =$ triple line, $\mathbf{C}P^1 \times \mathbf{C}P^1$, $C =$ double diagonal

- H. Bursztyn, M. Gualtieri & G. Cavalcanti:

  $\mathbf{C}P^2$, $C =$ three lines

- Y. Lin & S. Tolman:

  Hirzebruch surfaces, certain blow-ups of $\mathbf{C}P^2$

- NJH, *Instantons, Poisson structures and generalized Kähler geometry*, math.DG/0503432

- H. Bursztyn, M. Gualtieri & G. Cavalcanti, *Reduction of Courant algebroids and generalized complex structures*,

  math.DG/0509640

- Y. Lin & S. Tolman, *Symmetries in generalized Kähler geometry*, math.DG/0509069

- Del Pezzo examples: $\varphi_t^* I^+ = I^-$

- ... not true in general...

- Suppose D is an I^+-holomorphic curve with $C \cap D = \emptyset$

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$$\int_D (\omega + i\omega') = 0$$

$$\int_{\varphi_t(D)} (\omega + i\omega') = 0 = \int_D \varphi_t^*(\omega + i\omega') = \int_D (\omega + i\varphi_t^*\omega')$$

- ω type $(2,0)+(0,2)$

- $(\varphi_t^* \omega')^{1,1} = \omega_+$

$$\int_D \varphi_t^*(\omega + i\omega') = i \int_D \omega_+ \neq 0$$

NONCOMMUTATIVE GEOMETRY?

Hamiltonian vector field on C...

$$\mathrm{Re}\left(\frac{z_1}{h}\frac{\partial h}{\partial z_2}\frac{\partial}{\partial z_1} - \left(1 + \frac{z_1}{h}\frac{\partial h}{\partial z_1}\right)\frac{\partial}{\partial z_2}\right)$$

$$z_1 = 0 \Rightarrow \mathrm{Re}\left(-\frac{\partial}{\partial z_2}\right)$$

non-vanishing holomorphic vector field on elliptic curve C
$\Rightarrow \varphi_t : C \to C$ is a translation

EXAMPLE: $\mathbf{C}P^2$

- C = cubic curve

- $\varphi_t : (\mathbf{C}P^2, I^+) \to (\mathbf{C}P^2, I^-)$

- $\mathbf{C}P^1 \cap C = x_1 + x_2 + x_3$

- $\varphi_t(x_i) = x_i + tc$

NONCOMMUTATIVE GEOMETRY

- Algebraic (Artin, van den Bergh, Stafford)
- Non-commmutative $\mathbf{C}P^2$: noncommutative deformation of

$$\bigoplus_{0}^{\infty} H^0(\mathbf{C}P^2, \mathcal{O}(n))$$

- cubic curve C + automorphism σ

- $L = \mathcal{O}(1)$

- $L_n = L \otimes L^\sigma \otimes \cdots \otimes L^{\sigma^{n-1}}$

- $B_n = H^0(C, L_n)$

- twisted coordinate ring $B = \mathbf{C} \oplus \bigoplus_{n \geq 1} B_n$

- $x \in H^0(C, L_n), y \in H^0(C, L_m), \quad x \cdot y = x\sigma^n y \in H^0(C, L_{n+m})$

- $t \to 0$ noncommutative $\to$ commutative

- $I^{-}(t) \to I^{+}$

- $g(t) \to 0$

- $g(t)/t \to$ Kähler metric F.