Topological reduction of supersymmetric gauge theories and S-duality

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Outline

- Twisting supersymmetric gauge theories
- Topological reduction of $N = 4$ gauge theory
- S-duality and mirror symmetry
- Semi-topological reduction of finite $N = 2$ field theories

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Motivation

An outstanding problem is understanding the nonperturbative dynamics of gauge theories in 4d.

Qualitative behavior can be very rich: spontaneous breaking of symmetries, nontrivial IR fixed points, confinement of electric or magnetic charges, nonperturbative dualities, etc.

Quantitative understanding of nonperturbative behavior is more feasible in the supersymmetric case.

Compactification on a Riemann surface relates 4d gauge theories and 2d theories (typically, sigma-models).

Supersymmetric gauge theories

Extended supersymmetry algebra ($i = 1, \ldots, N$):

$$\{Q^i_\alpha, \bar{Q}_{j\dot{\alpha}}\} = i\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \delta^i_j,$$

$$\{Q^i_\alpha, Q^j_\beta\} = \epsilon_{\alpha\beta} Z^{ij}$$

Admits $U(N)$ automorphism group, known as the R-symmetry group. Superpotential often breaks $U(N)$ completely, or down to a subgroup.

In gauge theories, $U(1)$ subgroup of $U(N)$ often suffers from anomaly.

Twisting SUSY gauge theories

In 1987, E. Witten introduced a procedure of “twisting” a supersymmetric gauge theory into a topological field theory.

Main idea: modify the spin assignments of fields so that at least one of the supercharges become a scalar. Then the corresponding current is conserved even on a curved manifold.

Equivalent description: embed the space-time holonomy group into the R-symmetry group; use the embedding to couple the spin-connection to R-currents so that one of the SUSY parameters becomes a scalar.

Twisting SUSY gauge theories (cont.)

The conserved fermionic symmetry Q obeys $Q^2 = 0$ and is taken as the BRST charge (i.e. observables are elements of Q-cohomology).

The stress-energy tensor of the twisted theory is exact:

$$T_{\mu\nu} = \{Q, G_{\mu\nu}\}$$

Usually, this comes about because the action can be written as

$$S = \frac{i\Psi}{4\pi} \int_M \mathrm{Tr}\, F \wedge F + \{Q, V\}$$

This implies that the twisted theory does not depend on the metric of the space-time manifold M.

A generalization

It is interesting to consider the case when the metric has restricted holonomy.

For example, if M is a product of two Riemann surfaces $M = \Sigma \times C$, then the holonomy is $U(1) \times U(1)$. More generally, if M is a Kähler manifold, the holonomy group is $U(2)$.

Applying the twisting procedure in this case leads, in general, to a "semi-topological theory" (i.e. only some components of $T_{\mu\nu}$ are BRST-exact).

For example, if an $N = 1$ gauge theory has nonanomalous $U(1)$ symmetry, it admits a semi-topological twist on any Kähler 4-manifold (A. Johansen, 1994).

Twists of $N = 4$ gauge theory

$N = 4$ SYM has R-symmetry group $SU(4) \cong Spin(6)$. Three different topological twists are possible.

- Vafa-Witten twist (1994): chiral, leaves $SU(2)$ subgroup of R-symmetry unbroken, two scalar BRST charges which transform as a doublet. Partition function computes the Euler characteristic of the moduli space of instantons.
- GL twist (N. Marcus (1995), M. Blau and G. Thomspon (1996), J. M. F. Labastida and C. Lozano (1997), E. Witten and A.K. (2006)): nonchiral, preserves $U(1)$ symmetry, two scalar BRST charges which have the same $U(1)$ charge.
- Donaldson-Witten twist

GL twist

The most general BRST charge in the GL theory is

$$Q = uQ_\ell + vQ_r$$

The twisted theory has an extra parameter $t = v/u$ taking values in $\mathbb{CP}^1$.

Other features: the six adjoint scalars of $N = 4$ SYM become a bosonic 1-form ϕ with values in the adjoint representation of G and a pair of bosonic 0-forms ϕ_5, ϕ_6.

There are also fermionic 0-forms, 1-forms, and 2-forms.

The theory depends on t, e^2, θ in the combination

$$\Psi = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}\frac{t^2-1}{t^2+1}$$

BPS equations

The path-integral localizes on solutions of BPS equations which are t-dependent:

$$(F - \phi \wedge \phi + tD\phi)^+ = 0, \tag{1}$$

$$(F - \phi \wedge \phi - t^{-1}D\phi)^- = 0, \tag{2}$$

$$D_\mu \phi^\mu = 0. \tag{3}$$

Special case: $t = i$. Let $\mathcal{A} = A + i\phi$ be a new complex connection, $\mathcal{F}$ be its curvature. Then the first two equations combine into

$$\mathcal{F} = 0$$

Topological reduction to 2d

Let $M = \Sigma \times C$, where Σ, C are Riemann surfaces.

If C has very small size, a topological gauge theory on M must reduce to a topological field theory on Σ (M. Bershadsky, A. Johansen, V. Sadov, C. Vafa, 1995; related work by J. Harvey, A. Strominger, G. Moore, 1995).

For twisted $N = 4$ YM theory, this 2d TFT is a topological sigma-model whose target is the Hitchin moduli space.

Since the 4d theory is topological, the reduction to 2d TFT is in fact an exact statement, no matter how big C is.

Statements about 4d TFTs, such as S-duality, imply duality relations between topological sigma-models.

The Hitchin moduli space

The Hitchin moduli space is the moduli space of solutions of the following PDEs on C:

$$F_{z\bar{z}} - [\phi_z, \phi_{\bar{z}}] = 0, \quad D_{\bar{z}}\phi_z = D_z\phi_{\bar{z}} = 0.$$

Here $\phi_z, \phi_{\bar{z}}$ are components of the adjoint 1-form ϕ along C.

The Hitchin moduli space $\mathcal{M}_H(G, C)$ depends on the gauge group G and has disconnected components labelled by $H^2(C, \pi_1(G))$.

The components of $\mathcal{M}_H(G, C)$ are hyper-Kähler manifolds (in general, with singularities).

Topological sigma-models

If X is a Kähler manifold, there are two kinds of topological sigma-models with target X: A-model and B-model (E. Witten).

- A-model counts holomorphic maps from Σ to X and depends only on the symplectic structure of X and the B-field (a closed 2-form on X).
- B-model depends only on the complex structure of X and the $(0, 2)$ part of the B-field. It is related to the theory of deformations of the complex manifold X.

But if X is hyper-Kähler, then one can choose different complex structures for left and right movers. This leads to a more general class of topological sigma models related to Generalized Complex Geometry.

Generalized complex sigma-models

Generalized Complex Geometry was introduced by N. Hitchin (2002) and studied in detail by M. Gualtieri (2004). It is very useful for understanding topological sigma-models with different complex structures for left-moving and right-moving fields.

I will avoid using GC geometry in this talk and will state the results in terms of a pair of ordinary complex structures on $\mathcal{M}_H(G,C)$, which will be denoted I_+, I_-.

The sigma-model with target $\mathcal{M}_H(G,C)$ has $(4,4)$ supersymmetry, and choosing $I_\pm$ tells us which particular $(2,2)$ subalgebra is used for twisting.

If $I_+ = I_-$, one gets a B-model, if $I_+ = -I_-$, one gets an A-model (with the symplectic form $\omega = gI_+$).

Geometry of the Hitchin moduli space

The most general complex structure on $\mathcal{M}_H(G,C)$ has the form $aI+bJ+cK$, where $a^2+b^2+c^2=1$, and I,J,K are three special complex structures such that $IJ=K$.

Alternatively, choose $A_{\bar{z}}-w\phi_{\bar{z}}$ and $A_z+w^{-1}\phi_z$ as holomorphic coordinates on $\mathcal{M}_H(G,C)$. The complex number w parametrizes the sphere of complex structures.

Complex structure I ($w=0$): $\mathcal{M}_H(G,C)$ can be identified with the moduli space of (stable) pairs (E,φ), where E is a holomorphic G-bundle and φ is a holomorphic adjoint Higgs field.

Complex structure J ($w=-i$): $\mathcal{M}_H(G,C)$ can be identified with the moduli space of (stable) flat $G_{\mathbb{C}}$-valued connections on C.

Reduction of GL theory

The gauge coupling of the GL theory determines the overall scale of the metric on $\mathcal{M}_H(G, C)$.

The theta-angle determines the B-field on $\mathcal{M}_H(G, C)$:

$$B = -\frac{\theta}{2\pi}\omega_I.$$

The parameter t determines I_+ and I_-, or equivalently w_+ and w_-. Note that in this way we do not obtain the most general topological sigma-model with target $\mathcal{M}_H(G, C)$.

Reduction of GL theory (cont.)

By analyzing the BPS equations in the 2d limit, one finds:

$$w_+ = -t, \quad w_- = t^{-1}$$

When does one get an A-model? For this one must have $w_+ = -1/\bar{w}_-$, i.e. $t \in \mathbb{R}$. Thus for real t we get a family of A-models with symplectic structure $\omega = a\omega_I + b\omega_K$.

When does one get a B-model? For $t = \pm i$. The corresponding complex structure is $\pm J$.

For all other values of t one gets a Generalized Complex sigma-model.

S-duality in 4d gauge theory

The S-duality conjecture (C. Montonen and D. Olive, 1977, D. Olive and E. Witten, 1978, H. Osborn, 1979) is a highly nontrivial generalization of the duality symmetry of free Maxwell theory under the exchange of electric and magnetic fields.

According to the S-duality conjecture, $N = 4$ SYM with gauge group G and coupling $\tau = \theta/2\pi + 4\pi i/e^2$ is equivalent to the $N = 4$ SYM with gauge group ${}^LG$ and coupling ${}^L\tau = -1/q\tau$ ($q = 1$ for simply-laced groups, $q = 3$ for $G = G_2$, for all other groups $q = 2$).

For example, if $G = SU(N)$, then ${}^LG = SU(N)/\mathbb{Z}_N$.

S-duality (cont.)

Under S-duality left and right supercharges are multiplied by nontrivial (and opposite) phases.

This can be seen by inspecting the formula $\{Q^i_\alpha, Q^j_\beta\} = \epsilon_{\alpha\beta} Z^{ij}$ and noting that the central charge Z is multiplied by a phase under S-duality.

Consequently, t is also multiplied by a phase:

$$t \mapsto -t \frac{\tau}{|\tau|}$$

Thus S-duality does not map the GL theory at fixed t to itself, but to a GL theory with another value of t. The only exceptions are $t = 0$ and $t = \infty$ (these points are self-dual).

S-duality as mirror symmetry

Now we are ready to see that S-duality becomes a statement about mirror symmetry for $\mathcal{M}_H(G, C)$ and $\mathcal{M}_H({}^LG, C)$.

Let $\theta = 0$. Then S-duality maps $t \mapsto -it$. In particular, it maps $t = i$ to $t = 1$.

But $t = i$ corresponds to B-model in complex structure J, while $t = 1$ corresponds to A-model in complex structure K. Hence S-duality implies that $\mathcal{M}_H(G, C)$ with symplectic structure ω_K is mirror to $\mathcal{M}_H({}^LG, C)$ with complex structure J.

This mirror relation provides a starting point for a physical explanation of the Geometric Langlands Program.

Finite $N = 2$ gauge theories

Consider now an $N = 2$ $d = 4$ gauge theory with vanishing β-function. That is, consider $N = 2$ SYM coupled to hypermultiplets in representation R_i such that

$$C_2(G) - \sum_i C(R_i) = 0.$$

For example, one can take $G = SU(N_c)$ and $2N_c$ hypermultiplets in the fundamental representation.

If hypermultiplet masses vanish, the theory has $U(1) \times SU(2)$ R-symmetry group. For $N_c = 2$, the theory is believed to enjoy S-duality which maps τ to $-1/4\tau$ and leaves the theory otherwise unchanged (N. Seiberg and E. Witten, 1994).

Semi-topological reduction

If $M = \Sigma \times C$, then the holonomy of the spin connection is $U(1)_\Sigma \times U(1)_C$.

- Step 1: identify $U(1)_C$ with the $U(1)$ factor in the R-symmetry group $U(1) \times SU(2)$. This is a twist along C. The resulting theory is independent of the size of C and must be equivalent to a field theory on Σ.

One can show that this is a $(4,0)$ sigma-model whose target is $\mathcal{M}_H(G,C)$.

The left-moving fermions take values in a certain vector bundle V on $\mathcal{M}_H(G,C)$. It is associated (via R_i) to the push-forward of the universal G-bundle on $\mathcal{M}_H(G,C) \times C$ to $\mathcal{M}_H(G,C)$.

Semi-topological reduction (cont.)

Note that $(4,0)$ model with target X and vector bundle V is anomalous unless $p_2(X) = p_2(V)$. In the present case, this is ensured by the condition $C_2(G) = \sum_i C(R_i)$ in the gauge theory.

- Step 2: identify $U(1)_\Sigma$ with a maximal torus of the $SU(2)$ R-symmetry. This yields a half-twisted sigma-model with target $\mathcal{M}_H(G, C)$ and fermions taking values in a holomorphic vector bundle.

The half-twisted model is a holomorphic CFT. From the mathematical viewpoint, its large-volume limit is related to the sheaf of Chiral Differential Operators on $\mathcal{M}_H(G, C)$ with values in V (Witten, 2005; related results by A.K. 2005).

The half-twisted model

The large-volume limit is actually misleading in this case: there are many more operators in the half-twisted model than seen classically.

One can obtain local operators in the half-twisted model from Wilson or 't Hooft loops wrapped on C (and localized at a point on Σ). In the large-volume limit, only Wilson loops are visible.

If the gauge theory enjoys S-duality, this implies that certain half-twisted models with target $\mathcal{M}_H(G, C)$ are equivalent.

In the abelian case, this equivalence is ordinary T-duality. It would be interesting to study the nonabelian case in more detail.