

# Localization and Conjectures from String Duality

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String Duality is to identify different theories in string theory. Such identifications have produced many surprisingly beautiful mathematical formulas.

The mathematical proofs of many of such formulas depend on *Localization Techniques* on various finite dimensional moduli spaces. More precisely integrals of Chern classes on moduli spaces:

Combined with various mathematics: Chern-Simons knot invariants, Kac-Moody algebras' representations, Calabi-Yau, geometry and topology of moduli space of stable maps, moduli of stable bundles....

A simple technique we use: *Functorial Localization* transfers computations on complicated spaces to simple spaces: Connects computations of mathematicians and physicists.

## Functorial Localization Formula (LLY):

$f : X \rightarrow Y$  equivariant map.  $F \subset Y$  a fixed component,  $E \subset f^{-1}(F)$  fixed components in  $f^{-1}(F)$ . Let  $f_0 = f|_E$ , then for  $\omega \in H_T^*(X)$  an equivariant cohomology class, we have identity on  $F$ :

$$f_{0*} \left[ \frac{i_E^* \omega}{e_T(E/X)} \right] = \frac{i_F^*(f_* \omega)}{e_T(F/Y)}.$$

This is generalization of Atiyah-Bott localization, compared with Grothendieck Riemann-Roch.

This formula is used to push computations on complicated moduli space to simple moduli space. In most cases,

*Complicated moduli:* moduli of stable maps.

*Simple moduli:* projective spaces.

I will discuss the following topics:

- (1) The mirror principle; The discussion will be very brief. (Joint with Lian, Yau).
- (2) A proof of the Witten conjecture through localization. (Joint with Yon Seo Kim).
- (3) The proof of the Mariño-Vafa formula. (Joint with M. Liu, Zhou).
- 4) Work in progress: an effective method to derive recursion formulas from localization. (Joint with Yon Seo Kim).
- (5) Mathematical theory of topological vertex;
- (6) Applications of topological vertex. (Joint with J. Li, M. Liu, Zhou; others).

The proofs of Hori-Vafa formula and ELSV formula use the same method, will not be discussed.

## (1). Mirror Principle:

*Mathematical moduli:*  $M_d^g(X)$  = stable map moduli of degree  $(1, d)$  into  $\mathbf{P}^1 \times X = \{(f, C) : f : C \rightarrow \mathbf{P}^1 \times X\}$  with  $C$  a genus  $g$  (nodal) curve, modulo obvious equivalence.

*Physical moduli:*  $W_d$  for toric  $X$ . (Witten, Aspinwall-Morrison):

**Example:**  $\mathbf{P}^n$  with homogeneous coordinate  $[z_0, \dots, z_n]$ . Then

$$W_d = [f_0(w_0, w_1), \dots, f_n(w_0, w_1)]$$

$f_j(w_0, w_1)$ : homogeneous polynomials of degree  $d$ . A large projective space.

There exists an explicit equivariant collapsing map(LLY+Li; Givental for  $g = 0$ ):

$$\varphi : M_d^g(\mathbf{P}^n) \longrightarrow W_d.$$

$M_d^g(X)$ , embedded into  $M_d^g(\mathbf{P}^n)$ , is very "singular" and complicated. But  $W_d$  smooth and simple. The embedding induces a map of  $M_d^g(X)$  to  $W_d$ .

Functorial localization formula pushes all computations on the stable map moduli to the simple moduli  $W_d$ .

Mirror formulas are to compute certain Chern numbers on the moduli spaces of stable maps in terms of hypergeometric type functions from mirror manifolds. In fact we can say:

*Mirror formulas = Comparison of computations on the two moduli spaces  $M_d^g$  and  $W_d$ !*

Hypergeometric functions arise naturally on  $W_d$  through localization formula.

In the following discussions, we will study Hodge integrals (i.e. intersection numbers of  $\lambda$  classes and  $\psi$  classes) on the Deligne-Mumford moduli space of stable curves  $\overline{\mathcal{M}}_{g,h}$ .

A point in  $\overline{\mathcal{M}}_{g,h}$  consists of  $(C, x_1, \dots, x_h)$ , a (nodal) curve and  $h$  smooth points on  $C$ .

The Hodge bundle  $\mathbb{E}$  is a rank  $g$  vector bundle over  $\overline{\mathcal{M}}_{g,h}$  whose fiber over  $[(C, x_1, \dots, x_h)]$  is  $H^0(C, \omega_C)$ . The  $\lambda$  classes are Chern Classes:

$$\lambda_i = c_i(\mathbb{E}) \in H^{2i}(\overline{\mathcal{M}}_{g,h}; \mathbb{Q}).$$

Let

$$\Lambda_g^\vee(u) = u^g - \lambda_1 u^{g-1} + \dots + (-1)^g \lambda_g.$$

denote the total Chern class.

The cotangent line  $T_{x_i}^*C$  of  $C$  at the  $i$ -th marked point  $x_i$  gives a line bundle  $\mathbb{L}_i$  over  $\overline{\mathcal{M}}_{g,h}$ . The  $\psi$  classes are also Chern classes:

$$\psi_i = c_1(\mathbb{L}_i) \in H^2(\overline{\mathcal{M}}_{g,h}; \mathbb{Q}).$$

## (2). Localization Proof of the Witten Conjecture

The Witten conjecture for moduli spaces states that the generating series  $F$  of the integrals of the  $\psi$  classes for all genera and any number of marked points satisfies the KdV hierarchy and the Virasoro constraint. For example the Virasoro constraint states that  $F$  satisfies

$$L_n \cdot F = 0, \quad n \geq -1$$

where  $L_n$  denote certain Virasoro operators.

Witten conjecture was first proved by Kontsevich using combinatorial model of the moduli space and matrix model, with later approaches by Okounkov-Pandhripande using ELSV formula and combinatorics, by Mirzakhani using Weil-Petersson volumes on moduli spaces of bordered Riemann surfaces.



I will present a much simpler proof, with Y.-S. Kim. It is a direct application of the functorial localization formula and an asymptotic expansion.

The method is being developed into an effective technique in deriving more general recursion formulas in computing Hodge integrals and GW invariants.

The basic idea is to directly prove the following recursion formula which, as derived in physics by Dijkgraaf, Verlinde and Verlinde using quantum field theory, it implies the Virasoro constraints and the KdV hierarchy for the generating series  $F$  of the integrals of the  $\psi$  classes.

**Theorem:** *We have the identity of DVV:*

$$\begin{aligned}
\langle \tilde{\sigma}_n \prod_{k \in S} \tilde{\sigma}_k \rangle_g &= \sum_{k \in S} (2k + 1) \langle \tilde{\sigma}_{n+k-1} \prod_{l \neq k} \tilde{\sigma}_l \rangle_g + \\
&\frac{1}{2} \sum_{a+b=n-2} \langle \tilde{\sigma}_a \tilde{\sigma}_b \prod_{l \neq a,b} \tilde{\sigma}_l \rangle_{g-1} + \\
&\frac{1}{2} \sum_{\substack{S=X \cup Y, \\ a+b=n-2, \\ g_1+g_2=g}} \langle \tilde{\sigma}_a \prod_{k \in X} \tilde{\sigma}_k \rangle_{g_1} \langle \tilde{\sigma}_b \prod_{l \in Y} \tilde{\sigma}_l \rangle_{g_2}.
\end{aligned}$$

Here  $\tilde{\sigma}_n = (2n + 1)!! \psi^n$  and

$$\langle \prod_{j=1}^n \tilde{\sigma}_{k_j} \rangle_g = \int_{\overline{\mathcal{M}}_{g,n}} \prod_{j=1}^n \tilde{\sigma}_{k_j}.$$

The notation  $S = \{k_1, \dots, k_n\} = X \cup Y$ .

To prove the above recursion relation, we use functorial localization formula for moduli space of relative stable maps and branching morphism.

Given a partition  $\mu = (\mu_1 \geq \cdots \geq \mu_{l(\mu)} \geq 0)$  of length  $l(\mu)$ , write  $|\mu| = \sum_j \mu_j$ .

$\overline{\mathcal{M}}_g(\mathbf{P}^1, \mu)$ : moduli space of relative stable maps from a genus  $g$  (nodal) curve to  $\mathbf{P}^1$  with fixed ramification type  $\mu$  at  $\infty$ . (J. Li).

The functorial localization applied to the natural branching map (to ramification points):

$$\text{Br} : \overline{\mathcal{M}}_g(\mathbf{P}^1, \mu) \rightarrow \mathbf{P}^r.$$

Label the isolated fixed points  $\{p_0, \cdots, p_r\}$  of  $\mathbf{P}^r$  and denote by

$I_{g,\mu}^0$  : the fixed points contribution in  $\text{Br}^{-1}(p_0)$ .

$I_{g,\mu}^1$  : the fixed points contribution in  $\text{Br}^{-1}(p_1)$ .

Denote the one Hodge integral by

$$I_{g,\mu} = \frac{1}{|\text{Aut } \mu|} \prod_{i=1}^n \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\mathcal{M}_{g,n}} \frac{\Lambda_g^\vee(1)}{\prod (1 - \mu_i \psi_i)}.$$

Then we have

$$I_{g,\mu}^0 = I_{g,\mu}$$

and

$$\begin{aligned} I_{g,\mu}^1 = & \sum_{\nu \in J(\mu)} I_1(\nu) I_{g,\nu} + \sum_{\nu \in C(\mu)} I_2(\nu) I_{g-1,\nu} + \\ & \sum_{g_1+g_2=g} \sum_{\nu^1 \cup \nu^2 \in C(\mu)} I_3(\nu^1, \nu^2) I_{g_1,\nu^1} I_{g_2,\nu^2}, \end{aligned}$$

where  $I_1$ ,  $I_2$ ,  $I_3$  are some explicit combinatorial coefficients.

Here  $J(\mu)$ ,  $C(\mu)$  are the join and cut operations of the partition  $\mu$ . Will be explained more later.

By comparing the contributions in  $\text{Br}^{-1}(p_0)$  and  $\text{Br}^{-1}(p_1)$  through functorial localization formula, we easily get the identity:

$$I_{g,\mu}^0 = I_{g,\mu}^1.$$

This is just the cut-and-join equation for one Hodge integral:

$$\begin{aligned} I_{g,\mu} = & \sum_{\nu \in J(\mu)} I_1(\nu) I_{g,\nu} + \sum_{\nu \in C(\mu)} I_2(\nu) I_{g-1,\nu} + \\ & \sum_{g_1+g_2=g} \sum_{\nu^1 \cup \nu^2 \in C(\mu)} I_3(\nu^1, \nu^2) I_{g_1,\nu^1} I_{g_2,\nu^2}. \end{aligned}$$

This equation already implies the ELSV formula to relate the Hurwitz numbers to  $I_{g,\mu}$ .

We then do asymptotic expansion. Write  $\mu_i = Nx_i$  and let  $N$  go to infinity and expand in  $N$  and  $x_i$ , and take the coefficient of  $N^{m+\frac{1}{2}}$  with  $m = 3g - 3 + \frac{n}{2}$ .

We obtained the identity:

$$\begin{aligned}
& \sum_{i=1}^n \left[ \frac{(2k_i + 1)!!}{2^{k_i+1} k_i!} x_i^{k_i} \prod_{j \neq i} \frac{x_j^{k_j - \frac{1}{2}}}{\sqrt{2\pi}} \int_{\overline{\mathcal{M}}_{g,n}} \prod \psi_j^{k_j} - \right. \\
& \sum_{j \neq i} \frac{(x_i + x_j)^{k_i + k_j - \frac{1}{2}}}{\sqrt{2\pi}} \prod_{l \neq i,j} \frac{x_l^{k_l - \frac{1}{2}}}{\sqrt{2\pi}} \int_{\overline{\mathcal{M}}_{g,n-1}} \psi^{k_i + k_j - 1} \prod \psi_l^{k_l} \\
& - \frac{1}{2} \sum_{k+l=k_i-2} \frac{(2k+1)!!(2l+1)!!}{2^{k_i} k_i!} x_i^{k_i} \prod_{j \neq i} \frac{x_j^{k_j - \frac{1}{2}}}{\sqrt{2\pi}} \\
& \left[ \int_{\overline{\mathcal{M}}_{g-1,n+1}} \psi_1^k \psi_2^l \prod \psi_j^{k_j} + \right. \\
& \left. \sum_{\substack{g_1+g_2=g, \\ \nu_1 \cup \nu_2 = \nu}} \int_{\overline{\mathcal{M}}_{g_1,n_1}} \psi_1^k \prod \psi_j^{k_j} \int_{\overline{\mathcal{M}}_{g_2,n_2}} \psi_1^l \prod \psi_j^{k_j} \right] = 0
\end{aligned}$$

Performing Laplace transforms on the  $x_i$ 's, we get the recursion formula which implies both the KdV hierarchy and the Virasoro constraints.

### (3). Proof of the Mariño-Vafa Formula

Recall the total Chern classes of the Hodge bundle on the moduli space of curves:

$$\Lambda_g^\vee(u) = u^g - \lambda_1 u^{g-1} + \cdots + (-1)^g \lambda_g.$$

Mariño-Vafa formula: the generating series over  $g$  of triple Hodge integrals

$$\int_{\overline{\mathcal{M}}_{g,h}} \frac{\Lambda_g^\vee(1) \Lambda_g^\vee(\tau) \Lambda_g^\vee(-\tau - 1)}{\prod_{i=1}^h (1 - \mu_i \psi_i)},$$

can be expressed by close formulas of **finite** expression in terms of representations of symmetric groups, or Chern-Simons knot invariants. Here  $\tau$  is a parameter.

Mariño-Vafa conjectured the formula from large  $N$  duality between Chern-Simons and string theory, following works of Witten, Gopakumar-Vafa, Ooguri-Vafa.

*The Mariño-Vafa Formula:*

**Geometric side:** For every partition  $\mu = (\mu_1 \geq \dots \geq \mu_{l(\mu)} \geq 0)$ , define triple Hodge integral:

$$G_{g,\mu}(\tau) = A(\tau) \cdot \int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{\Lambda_g^\vee(1) \Lambda_g^\vee(-\tau-1) \Lambda_g^\vee(\tau)}{\prod_{i=1}^{l(\mu)} (1 - \mu_i \psi_i)},$$

with

$$A(\tau) = -\frac{\sqrt{-1}^{|\mu|+l(\mu)}}{|\text{Aut}(\mu)|} [\tau(\tau+1)]^{l(\mu)-1} \prod_{i=1}^{l(\mu)} \frac{\prod_{a=1}^{\mu_i-1} (\mu_i \tau + a)}{(\mu_i-1)!}.$$

Introduce generating series

$$G_\mu(\lambda; \tau) = \sum_{g \geq 0} \lambda^{2g-2+l(\mu)} G_{g,\mu}(\tau).$$

Special case when  $g = 0$ :

$$\begin{aligned} & \int_{\overline{\mathcal{M}}_{0,l(\mu)}} \frac{\Lambda_0^\vee(1) \Lambda_0^\vee(-\tau-1) \Lambda_0^\vee(\tau)}{\prod_{i=1}^{l(\mu)} (1 - \mu_i \psi_i)} \\ &= \int_{\overline{\mathcal{M}}_{0,l(\mu)}} \frac{1}{\prod_{i=1}^{l(\mu)} (1 - \mu_i \psi_i)} = |\mu|^{l(\mu)-3} \end{aligned}$$



for  $l(\mu) \geq 3$ , and we use this expression to extend the definition to the case  $l(\mu) < 3$ .

Introduce formal variables  $p = (p_1, p_2, \dots, p_n, \dots)$ , and define

$$p_\mu = p_{\mu_1} \cdots p_{\mu_{l(\mu)}}$$

for any partition  $\mu$ .

Generating series for all genera and all possible marked points:

$$G(\lambda; \tau; p) = \sum_{|\mu| \geq 1} G_\mu(\lambda; \tau) p_\mu.$$

**Representation side:**  $\chi_\mu$ : the character of the irreducible representation of symmetric group  $S_{|\mu|}$  indexed by  $\mu$  with  $|\mu| = \sum_j \mu_j$ ,

$C(\mu)$ : the conjugacy class of  $S_{|\mu|}$  indexed by  $\mu$ .

Introduce:

$$\mathcal{W}_\mu(q) = q^{\kappa_\mu/4} \prod_{1 \leq i < j \leq \ell(\mu)} \frac{[\mu_i - \mu_j + j - i]}{[j - i]}$$

$$\prod_{i=1}^{\ell(\mu)} \frac{1}{\prod_{v=1}^{\mu_i} [v - i + \ell(\mu)]}$$

where

$$\kappa_\mu = |\mu| + \sum_i (\mu_i^2 - 2i\mu_i), \quad [m] = q^{m/2} - q^{-m/2}$$

and  $q = e^{\sqrt{-1}\lambda}$ . The expression  $\mathcal{W}_\mu(q)$  has an interpretation in terms of *quantum dimension* in Chern-Simons knot theory.

Define:

$$R(\lambda; \tau; p) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} \sum_{\mu} \left[ \sum_{\cup_{i=1}^n \mu^i = \mu} \right]$$

$$\prod_{i=1}^n \sum_{|\nu^i| = |\mu^i|} \frac{\chi_{\nu^i}(C(\mu^i))}{z_{\mu^i}} e^{\sqrt{-1}(\tau + \frac{1}{2})\kappa_{\nu^i}\lambda/2} \mathcal{W}_{\nu^i}(\lambda) p_\mu$$

where  $\mu^i$  are sub-partitions of  $\mu$ ,  $z_\mu = \prod_j \mu_j! j^{\mu_j}$  for a partition  $\mu$ : standard for representations of symmetric groups.

**Theorem:** *Mariño-Vafa Conjecture is true:*

$$G(\lambda; \tau; p) = R(\lambda; \tau; p).$$

**Remarks:** (1). Equivalent expression:

$$G(\lambda; \tau; p)^\bullet = \exp [G(\lambda; \tau; p)] = \sum_{\mu} G(\lambda; \tau)^\bullet p_{\mu} =$$

$$\sum_{|\mu| \geq 0} \sum_{|\nu| = |\mu|} \frac{\chi_{\nu}(C(\mu))}{z_{\mu}} e^{\sqrt{-1}(\tau + \frac{1}{2})\kappa_{\nu}\lambda/2} \mathcal{W}_{\nu}(\lambda) p_{\mu}$$

(2). Each  $G_\mu(\lambda, \tau)$  is given by a **finite and closed** expression in terms of representations of symmetric groups:

$$G_\mu(\lambda, \tau) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} \sum_{\cup_{i=1}^n \mu^i = \mu} \prod_{i=1}^n \sum_{|\nu^i|=|\mu^i|} \frac{\chi_{\nu^i}(C(\mu^i))}{z_{\mu^i}} e^{\sqrt{-1}(\tau + \frac{1}{2})\kappa_{\nu^i}\lambda/2} \mathcal{W}_{\nu^i}(\lambda).$$

$G_\mu(\lambda, \tau)$  gives triple Hodge integrals for moduli spaces of curves of all genera with  $l(\mu)$  marked points.

(3). Mariño-Vafa formula gives explicit values of many interesting Hodge integrals up to three Hodge classes:

- Taking limit  $\tau \longrightarrow 0$  we get the  $\lambda_g$  conjecture (FP),

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \lambda_g = \binom{2g+n-3}{k_1, \dots, k_n} \frac{2^{2g-1} - 1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!},$$

for  $k_1 + \cdots + k_n = 2g - 3 + n$ , and the following identity for Hodge integrals:

$$\begin{aligned} \int_{\overline{\mathcal{M}}_g} \lambda_{g-1}^3 &= \int_{\overline{\mathcal{M}}_g} \lambda_{g-2} \lambda_{g-1} \lambda_g \\ &= \frac{1}{2(2g-2)!} \frac{|B_{2g-2}|}{2g-2} \frac{|B_{2g}|}{2g}, \end{aligned}$$

$B_{2g}$  are Bernoulli numbers. And other identities.

- Taking limit  $\tau \longrightarrow \infty$ , we get the ELSV formula, then the Witten conjecture.
- Yi Li recently has derived more Hodge integral identities from the MV formula.

The idea to prove the Mariño-Vafa formula is to prove that both  $G$  and  $R$  satisfy the Cut-and-Join equation:

**Theorem :** *Both  $R$  and  $G$  satisfy the following differential equation:*

$$\begin{aligned} \frac{\partial F}{\partial \tau} = & \frac{1}{2} \sqrt{-1} \lambda \sum_{i,j=1}^{\infty} \left( (i+j) p_i p_j \frac{\partial F}{\partial p_{i+j}} \right. \\ & \left. + i j p_{i+j} \left( \frac{\partial F}{\partial p_i} \frac{\partial F}{\partial p_j} + \frac{\partial^2 F}{\partial p_i \partial p_j} \right) \right) \end{aligned}$$

This is equivalent to linear systems of ODE. They have the same initial value at  $\tau = 0$ :

**The solution is unique!**

$$G(\lambda; \tau; p) = R(\lambda; \tau; p).$$

*Cut-and-Join operator, denoted by  $(CJ)$ , in variables  $p_j$  on the right hand side gives a nice match of Combinatorics and Geometry from collecting the following operations:*

**Combinatorics:** Cut and join of cycles:

*Cut:* a  $k$ -cycle is cut into an  $i$ -cycle and a  $j$ -cycle, denote the set by  $C(\mu)$ :

*Join:* an  $i$ -cycle and a  $j$ -cycle are joined to an  $(i + j)$ -cycle, denote the set by  $J(\mu)$ :

**Geometry:** How curves stably vary,

*Cut:* One curve split into two lower degree or lower genus curves.

*Join:* Two curves joined together to give a higher genus or higher degree curve.

The proof of cut-and-join equation for  $R$  is a direct computation in combinatorics.

The first proof of the cut-and-join equation for  $G$  used functorial localization formula.

$J_{g,\mu}^0$  : the fixed points contribution in  $\text{Br}^{-1}(p_0)$

$J_{g,\mu}^1$  : the fixed points contribution in  $\text{Br}^{-1}(p_1)$ .

Then

$$J_{g,\mu}^0(\tau) = \sqrt{-1}^{|\mu|-l(\mu)} G_{g,\mu}(\tau),$$

$$J_{g,\mu}^1(\tau) = \sqrt{-1}^{|\mu|-l(\mu)-1}.$$

$$\left( \sum_{\nu \in J(\mu)} I_1(\nu) G_{g,\nu}(\tau) + \sum_{\nu \in C(\mu)} I_2(\nu) G_{g-1,\nu}(\tau) \right. \\ \left. + \sum_{g_1+g_2=g, \nu^1 \cup \nu^2 \in C(\mu)} I_3(\nu^1, \nu^2) G_{g_1,\nu^1}(\tau) G_{g_2,\nu^2}(\tau) \right).$$

We proved the following identity by analyzing equivariant cohomology classes on  $\mathbf{P}^r$ :

$$\frac{d}{d\tau} J_{g,\mu}^0(\tau) = -J_{g,\mu}^1(\tau).$$

which is the cut-and-join equation we need.



In fact we have more higher order cut-and-join equations:

$$\frac{(-1)^k}{k!} \frac{d^k}{d\tau^k} J_{g,\mu}^0(\tau) = J_{g,\mu}^k(\tau)$$

from comparing contributions from the first and the  $k$ -th fixed point on  $\mathbf{P}^r$ .

**Remark:** Cut-and-join equation is encoded in the geometry of the moduli spaces of stable maps: convolution formula of the form:

$$G_{\mu}^{\bullet}(\lambda, \tau) = \sum_{|\nu|=|\mu|} \Phi_{\mu,\nu}^{\bullet}(-\sqrt{-1}\tau\lambda) z_{\nu} K_{\nu}^{\bullet}(\lambda)$$

where  $\Phi_{\mu,\nu}^{\bullet}$  is series of double Hurwitz numbers. This gives the explicit solution of the cut-and-join equation, with initial value  $K^{\bullet}(\lambda)$ , the integrals of Euler classes on moduli of relative stable maps.

#### (4). Work in Progress:

We note that *asymptotic cut-and-join equation* from the functorial localization formula and moduli space of relative stable maps always holds. This gives a universal and very effective way to derive recursive type formulas in Hodge integrals and in GW invariants.

*From cut-and-join to recursion through functorial localization and asymptotic analysis.*

Applications may include:

- (1) The recursion formula related to the generalized Witten conjecture for  $W$ -algebra constraints.
- (2) Recursion formula for GW invariants of general projective manifolds related to Virasoro conjecture.
- (3) Recursion formula for other Hodge integrals related to the Faber conjecture.

## **(5). Mathematical Theory of Topological Vertex**

Mirror symmetry used periods and holomorphic anomaly to compute Gromov-Witten series, difficult for higher genera. Topological vertex theory, as developed by Aganagic-Klemm-Marino-Vafa from string duality and geometric engineering, gives complete answers for all genera and all degrees in the toric Calabi-Yau cases in terms of Chern-Simons knot invariants!

We developed the mathematical theory of topological vertex by using localization technique, to first prove a three partition analogue of the Mariño-Vafa formula. This formula gives closed formula for the generating series of the Hodge integrals involving three partitions in terms of Chern-Simons knot invariants of Hopf links.

The corresponding cut-and-join equation has the form:

$$\frac{\partial}{\partial \tau} F^\bullet = (CJ)^1 F^\bullet + \frac{1}{\tau^2} (CJ)^2 F^\bullet + \frac{1}{(\tau + 1)^2} (CJ)^3 F^\bullet$$

where  $(CJ)$  denotes the cut-and-join operator with respect to the three groups of infinite numbers of variables associated to the three partitions.

We derived the convolution formulas both in combinatorics and in geometry. Then we proved the identity of initial values at  $\tau = 1$ .

We introduced the new notion of *formal toric Calabi-Yau manifolds* to work out the gluing of Calabi-Yau and the topological vertices. We then derived all of the basic properties of topological vertex, like the fundamental gluing formula.

By using gluing formula of the topological vertex, we can derive closed formulas for generating series of GW invariants, all genera and all degrees, open or closed, for all formal toric Calabi-Yau, in terms Chern-Simons invariants, by simply looking at

*The moment map graph of the toric Calabi-Yau.*

Each **vertex** of the moment map graph contributes a closed expression to the generating series of the GW invariants in terms of explicit combinatorial Chern-Simons knot invariants.

Let us look at an example to see the computational power of topological vertex.

Let  $N_{g,d}$  denote the GW invariants of a toric Calabi-Yau, total space of canonical bundle on a toric surface  $S$ .

It is the Euler number of the obstruction bundle on the moduli space  $\overline{\mathcal{M}}_g(S, d)$  of stable maps of degree  $d \in H_2(S, \mathbb{Z})$  from genus  $g$  curve into the surface  $S$ :

$$N_{g,d} = \int_{[\overline{\mathcal{M}}_g(S,d)]^v} e(V_{g,d})$$

with  $V_{g,d}$  a vector bundle induced by the canonical bundle  $K_S$ . At point  $(\Sigma; f) \in \overline{\mathcal{M}}_g(S, d)$ , its fiber is  $H^1(\Sigma, f^*K_S)$ .

Write

$$F_g(t) = \sum_d N_{g,d} e^{-d \cdot t}.$$

**Example:** Topological vertex formula of GW generating series in terms of Chern-Simons invariants. For the total space of canonical bundle  $\mathcal{O}(-3)$  on  $\mathbf{P}^2$ :

$$\exp\left(\sum_{g=0}^{\infty} \lambda^{2g-2} F_g(t)\right) = \sum_{\nu_1, \nu_2, \nu_3} \mathcal{W}_{\nu_1, \nu_2} \mathcal{W}_{\nu_2, \nu_3} \mathcal{W}_{\nu_3, \nu_1} \cdot (-1)^{|\nu_1|+|\nu_2|+|\nu_3|} q^{\frac{1}{2} \sum_{i=1}^3 \kappa_{\nu_i}} e^{t(|\nu_1|+|\nu_2|+|\nu_3|)}.$$

Here  $q = e^{\sqrt{-1}\lambda}$ , and  $\mathcal{W}_{\mu, \nu}$  are explicit from the Chern-Simons knot invariants of Hopf link. Sum over three partitions  $\nu_1, \nu_2, \nu_3$ .

Three vertices of moment map graph of  $\mathbf{P}^2 \leftrightarrow$  three  $\mathcal{W}_{\mu, \nu}$ 's, explicit in Schur functions.

For general (formal) toric Calabi-Yau, the expressions are just similar: **closed formulas**.

## 6. Applications of Topological Vertex.

Recall:

**Gopakumar-Vafa conjecture:** *There exists expression:*

$$\sum_{g=0}^{\infty} \lambda^{2g-2} F_g(t) = \sum_{k=1}^{\infty} \sum_{g,d} n_d^g \frac{1}{d} (2 \sin \frac{d\lambda}{2})^{2g-2} e^{-kd \cdot t},$$

*such that  $n_d^g$  are integers, called instanton numbers.*

By using the explicit knot invariant expressions from topological vertex in terms of the Schur functions, we have the following applications:

(1). First motivated by the Nekrasov's work, by comparing with Atiyah-Bott fixed point formulas on instanton moduli we have proved:



**Theorem:** *For conifold and the toric Calabi-Yau from the canonical line bundle of the Hirzebruch surfaces, we can identify the  $n_d^g$  as equivariant indices of twisted Dirac operators on moduli spaces of anti-self-dual connections on  $\mathbb{C}^2$ .*

A complicated change of variables like mirror transformation is performed.

(2). The following theorem was first proved by Pan Peng:

**Theorem:** *The Gopakumar-Vafa conjecture is true for all (formal) local toric Calabi-Yau for all degree and all genera.*

(3). The proof of the flop invariance of the GW invariants of (toric) Calabi-Yau by Konishi and Minabe, (with previous works of Li-Ruan and Chien-Hao Liu-Yau.)

The above applications have shown the power of topological vertex. More applications are expected from the computational power of the topological vertex.

We have seen close connection between knot invariants and Gromov-Witten invariants. There may be a more interesting and grand duality picture between Chern-Simons invariants for real three dimensional manifolds and Gromov-Witten invariants for complex three dimensional toric Calabi-Yau.

General correspondence between the geometry of real dimension 3 and complex dimension 3?!.

In any case *String Duality* has already inspired exciting duality and unification among various mathematical subjects.

**Thank You Very Much!**