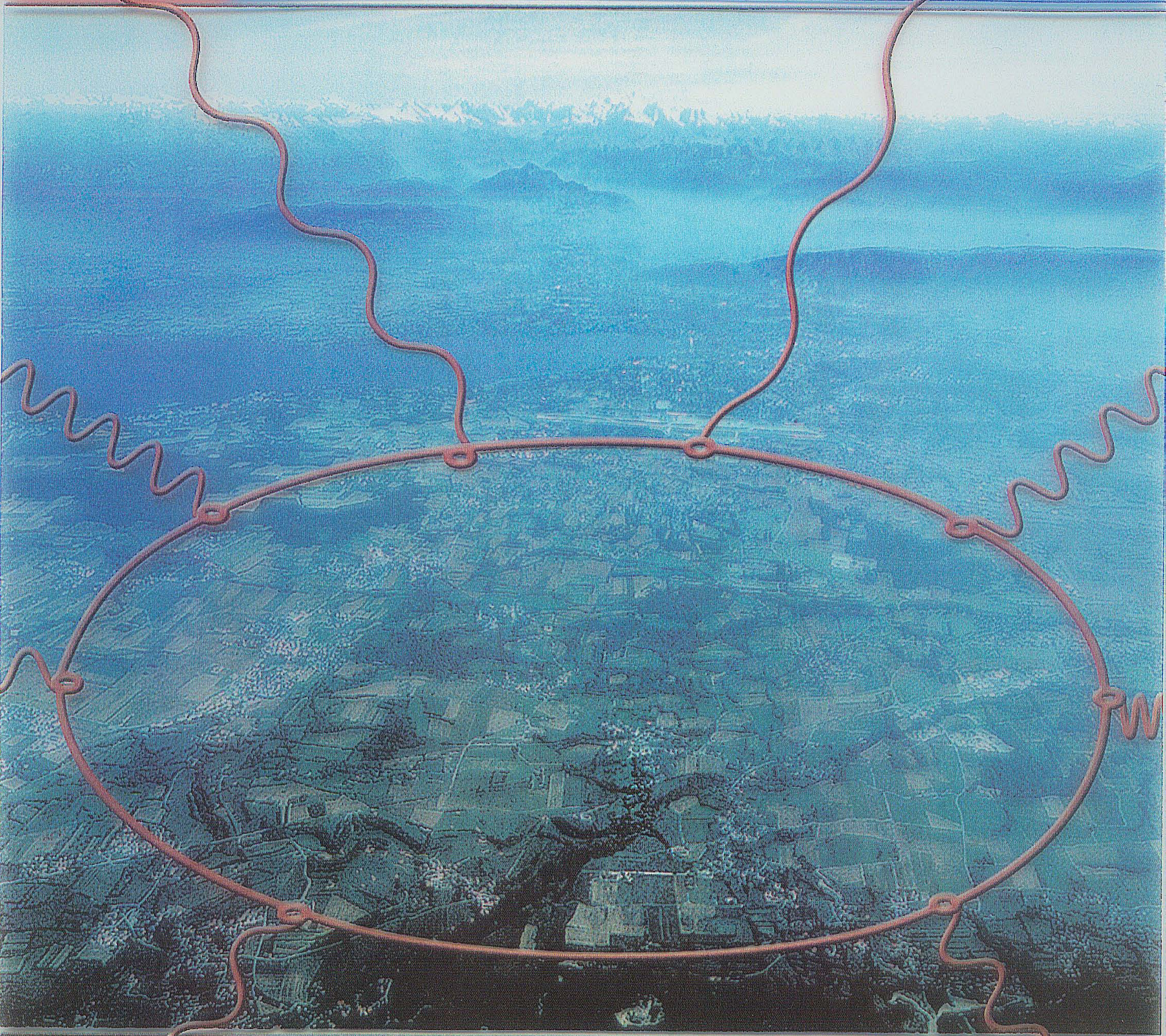
Strings 2008



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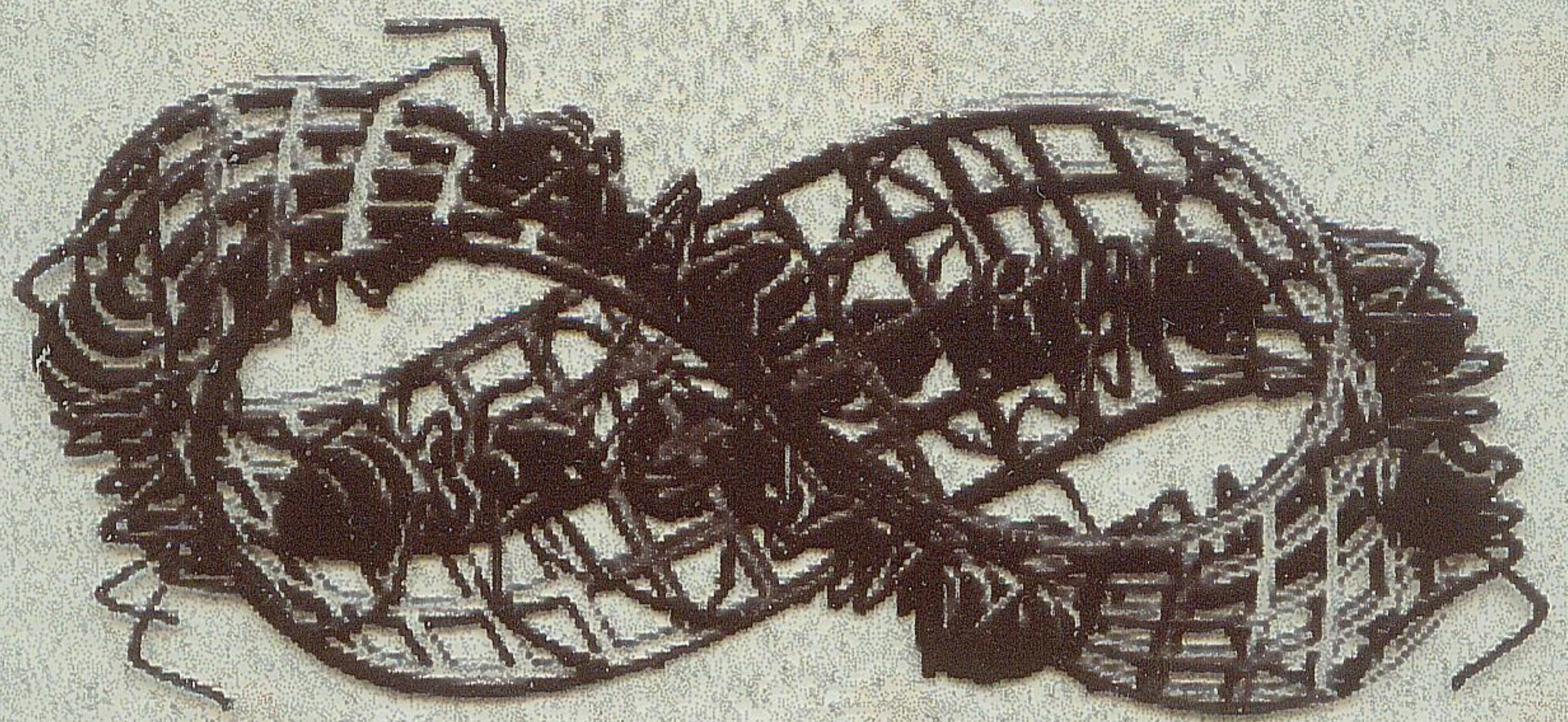
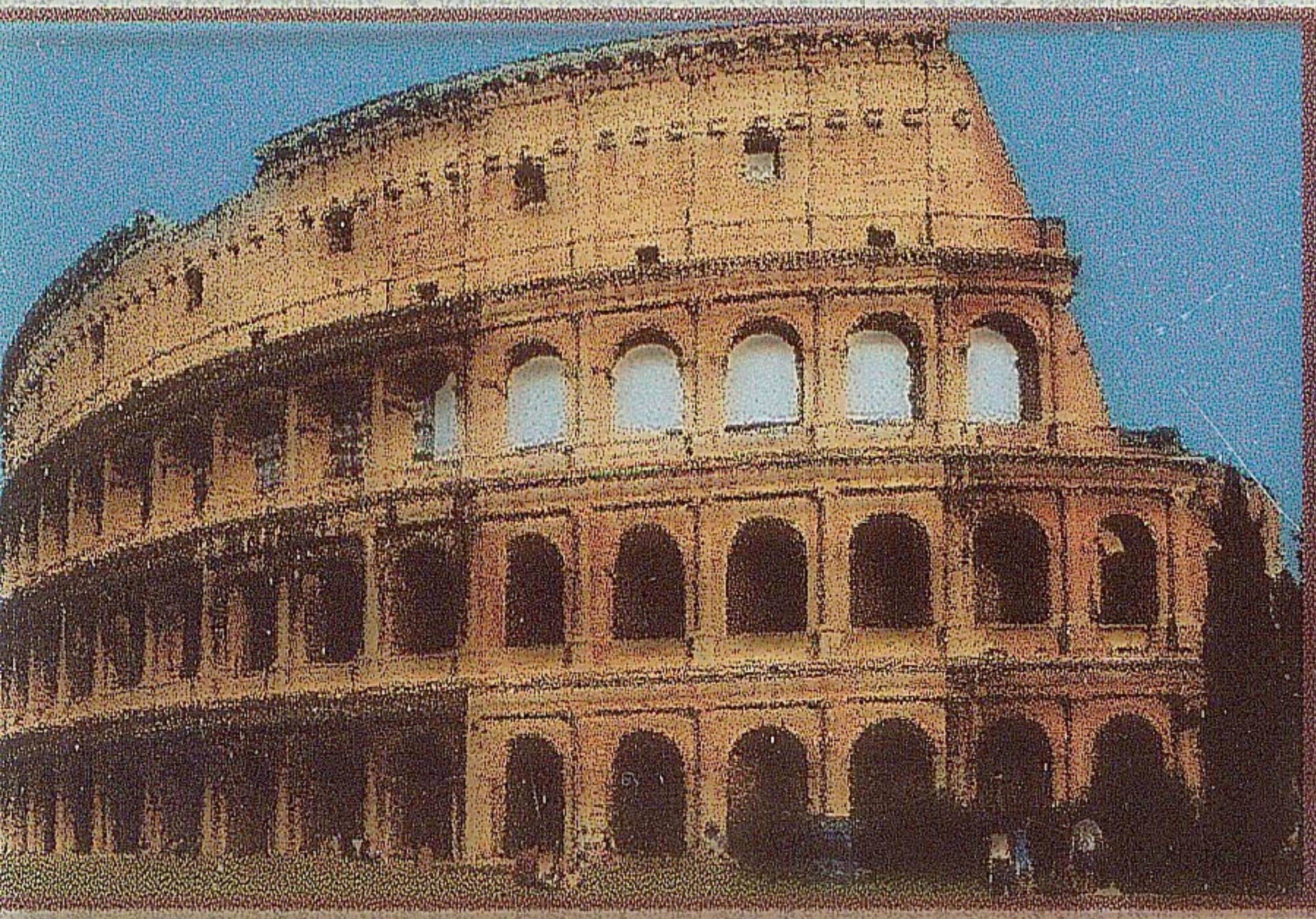
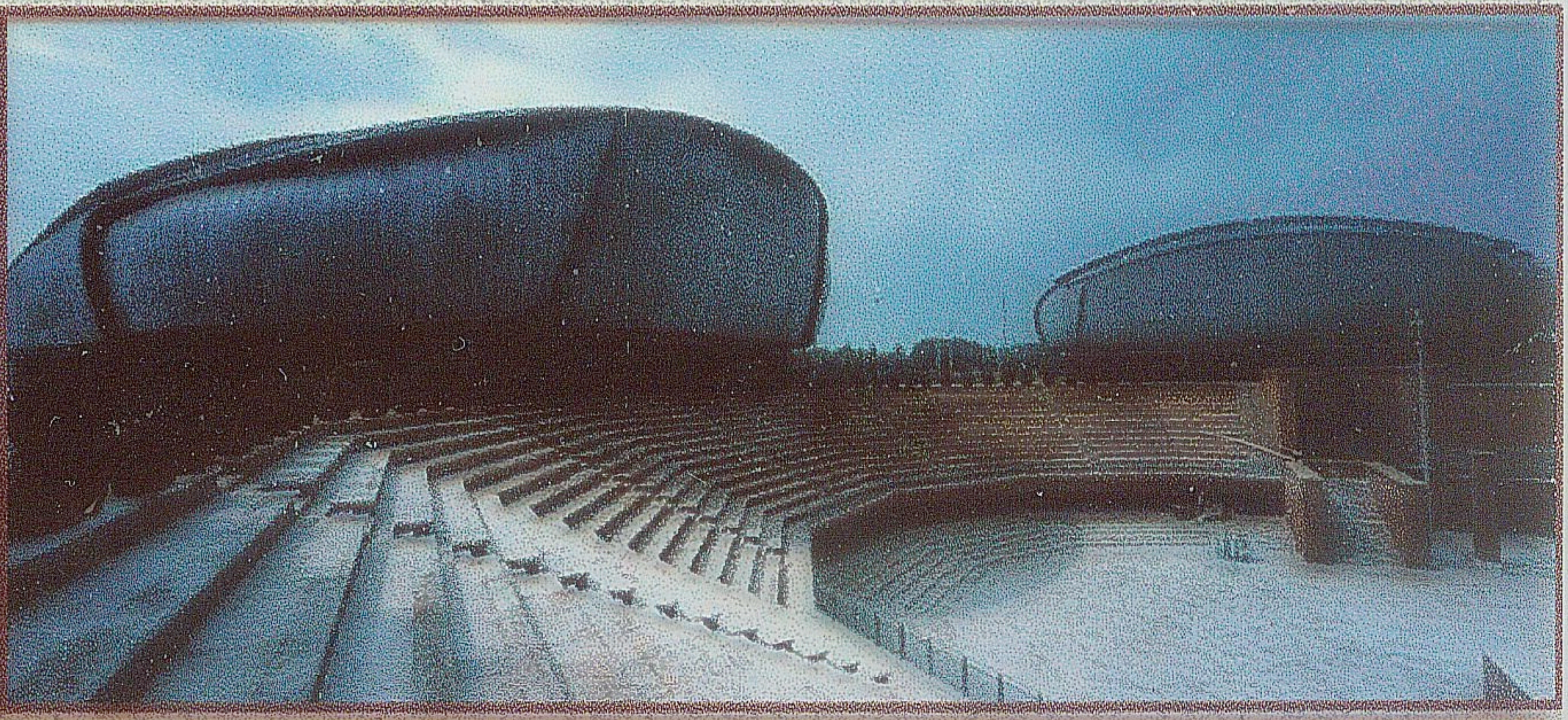


STRINGS '09

ROME, JUNE 2009

University of Rome
"*Tor Vergata*"





(INFN, MIUR, EU)

BPS states and the Enriques Calabi–Yau

Marcos Mariño

CERN, Geneva and IST, Lisbon

Mostly based on A. Klemm and M.M., hep–th/0512227

Motivation

Topological string theory is a toy model for string theory, and it seems a realistic goal to try to solve it on Calabi–Yau backgrounds to all genus i.e. find all F_g couplings. This has *many* applications, including BPS state counting.

We now know how to do it in the *toric, noncompact* case (topological vertex), but the compact case is much harder.

Our goal here is to identify, in a sense, the simplest , nontrivial, compact CY manifold and see how far we can get. This example turns out to be the *Enriques Calabi–Yau*, a very special example of K3 fibration introduced in the FHSV model. So this talk we will be mostly a case study, but I will also try to extract general lessons.

Topological string amplitudes

Consider type IIA on a CY X with Kähler moduli t_i. In the large radius limit $t_i \to \infty$, the $F_g(t)$, can be computed in terms of *worldsheet instantons* i.e. holomorphic maps from the Riemann surface Σ_g to X

$$F_g(t) = \sum_{Q \in H_2(X)} N_{g,Q} e^{-Q \cdot t}$$

where $Q \cdot t = \sum_{i=1}^{b_2(X)(X)} Q_i t_i$.

The numbers $N_{g,Q}$ are *Gromov–Witten invariants* of X. Therefore, $F_g(t_i)$ contain enumerative information about the CY.

BPS counting

As shown by [Gopakumar–Vafa], these amplitudes contain information about the degeneracy of BPS states. In terms of the total partition function

$$Z = \exp\Big\{ \sum_{g=0}^{\infty} g_s^{2g-2} F_g(t) \Big\}$$

we have

$$Z = \prod_{Q, m \geq 1} (1 - \mathrm{e}^{\mathrm{i} m g_s - Q \cdot t})^{\Omega(Q,m)},$$

where $\Omega(Q, m)$ is the number (index) of BPS states with spin m in 4d coming from D2/M2 branes wrapping a 2–cycle with the class $Q \in H_2(X, \mathbb{Z})$. This also looks like a denominator formula for a generalized algebra, where $\Omega(Q, m)$ are root multiplicities [Harvey–Moore].

Computation of topological string amplitudes

Most effective way: use string dualities.

- *Mirror symmetry*. The computation of $F_g(t)$ in type IIB is a problem in deformation of complex structures. For $g = 0$ there is a complete solution in terms of Picard–Fuchs equations. For $g \geq 1$ one can use the *holomorphic anomaly equations* of BCOV.

  Problem: For $g \geq 1$, one needs "initial conditions" to have a complete solution – this is the *holomorphic ambiguity*

- *Large N dualities*. For *toric* (hence noncompact) CY manifolds, a complete solution exists which expresses $F_g(t)$ in terms of Chern–Simons correlation functions (the topological vertex).

  Problem: So far, very hard to generalize to the compact case

- *Heterotic duals*. When X is a *K3 fibration* over $\mathbb{P}^1$ there is often a heterotic dual [Kachru–Vafa]. Let us denote the Kähler moduli as y_i for the fiber and S for the base. Then one can compute

  $$F_g(y_i, S \to \infty)$$

  in closed form by using the heterotic dual on K3 $\times \mathbb{T}^2$ at one loop [AGNT, M.M.–Moore]

  Problem: effectively, a non compact case! No information about the S dependence.

A very special valley in the landscape

Given the difficulties in the *compact case*, look for a *simple model*.

What about $\mathsf{K3} \times \mathbb{T}^2$?

$$F_g = 0, \quad g \neq 1, \qquad F_1(t) = -24 \log \eta(q_S),$$

where $q_S = \mathrm{e}^{-S}$ and S is the Kähler parameter of the $\mathbb{T}^2$.

So *too* simple! (holonomy $SU(2)$ too small and too much supersymmetry)

By doing a $\mathbb{Z}_2$ quotient, we obtain a bigger holonomy: this is the *FHSV model*

The Enriques involution

There is a free $\mathbb{Z}_2$ action on K3 which leads to a smooth quotient

$$\mathrm{K3}/\mathbb{Z}_2 = E$$

the *Enriques surface*. From the $\mathbb{Z}_2$ action on the cohomology one finds

$$H^*(E) = \Gamma_2^{1,9} \oplus \Gamma_{0-4}^{1,1}$$

The type IIA model is obtained as a smooth orbifold

$$X = (\mathsf{K3} \times \mathbb{T}^2)/\mathbb{Z}_2$$

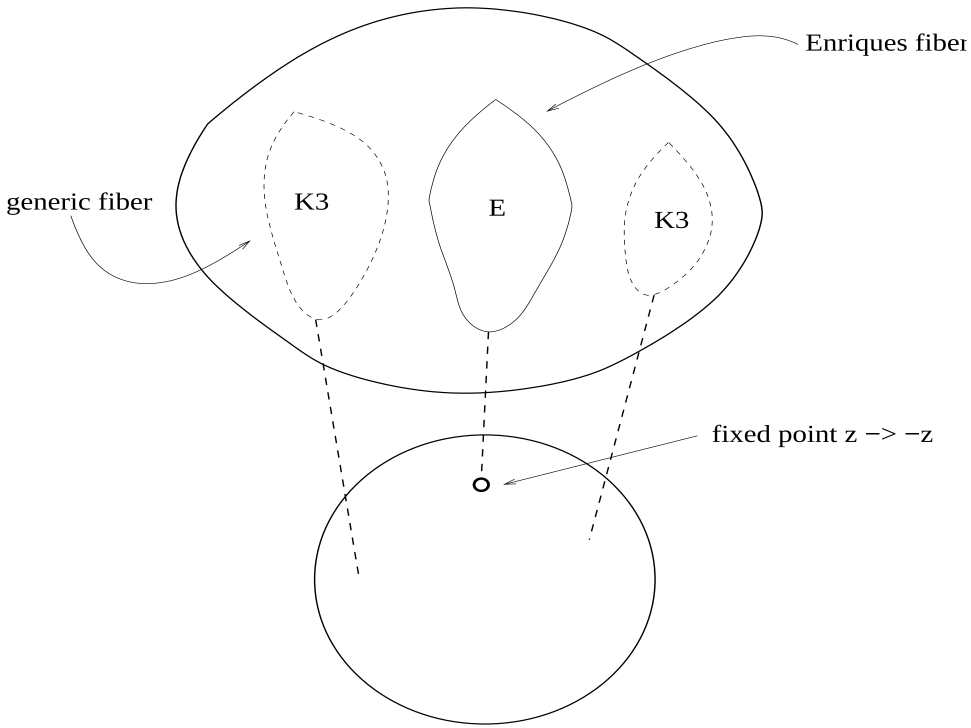
where the $\mathbb{Z}_2$ action on $\mathbb{T}^2$ is $z \mapsto -z$. The resulting CY has

$$b_2(X) = 11, \quad \chi(X) = 0, \quad \text{holonomy}\, SU(2) \times \mathbb{Z}_2$$

Moduli space of VM is *exactly* given by

$$\frac{\mathsf{Sl}(2,\mathbb{R})}{\mathsf{SO}(2)} \times \frac{\mathsf{O}(10,2)}{\mathsf{O}(10) \times \mathsf{O}(2)}$$

X is a *K3 fibration* over $\mathbb{P}^1$ with (double) Enriques fibers over the four fixed points of $z \to -z$



The heterotic FHSV model

This type IIA model is dual to an asymmetric orbifold of heterotic theory on $\mathbb{T}^6$. The action on the lattice $\Gamma^{6,22}$ mimicks the Enriques involution, and the Narain lattices for the orbifold blocks are

$$\Gamma_J = \Gamma^{1,1}_{0-4} \oplus \Gamma^{1,9}_{2}(\zeta_J), \quad J = 1, 2, 3, \quad \zeta_1 = 2,\ \zeta_{2,3} = 1/2$$

The $F_g(t)$ couplings *along the fiber* are one–loop amplitudes [AGNT]

$$F_g = \int_{\mathcal{F}} d^2\tau \sum_J \mathcal{I}_J, \quad \mathcal{I}_J \sim \Theta^g_J(\tau)$$

where Θ^g_J is a Narain–Siegel theta function with momentum insertions

Computing $F_g(t)$ in the heterotic string

General principle: computation of topological string amplitudes needs choice of *basepoint* and of *duality frame* (recall Seiberg–Witten theory).

The one–lop integral is computed with the technique of *lattice reduction* [Harvey–Moore, Borcherds] where one "integrates out" a $\Gamma^{1,1}$ sublattice. This induces

(1) a parametrization of the VM coset $\rightarrow$ basepoint

(2) the result depends only on a sublattice of Γ $\rightarrow$ duality frame

A priori, various choices are possible.

First choice: we integrate out $\Gamma^{1,1}_{0-4}$: *geometric reduction*, $H^2(E)$ left. We obtain

$$Z_E(g_s,t) = \prod_{Q\in H_2(E),m} \left(\frac{1-\mathrm{e}^{\mathrm{i}mg_s/2-Q\cdot t}}{1+\mathrm{e}^{\mathrm{i}mg_s/2-Q\cdot t}}\right)^{\Omega_E(Q,m)}$$

where

$$\sum_{Q,m} \Omega_E(Q,m) q^{Q^2} \mathrm{e}^{\mathrm{i}mg_s} = \frac{64}{\eta^6 \vartheta_2^6} (\xi^2(q,g_s) - \xi^2(-q,g_s)),$$

and

$$\xi(q,g_s) = \frac{1}{2\sin\frac{g_s}{2}} \prod_{n\geq 1} \frac{(1-q^n)^2}{1-2q^n\cos g_s + q^{2n}}$$

This corresponds to *standard Gromov–Witten theory* and counts D2 branes. Notice the "supersymmetric" structure of Z_E.

Second choice: integrate out $\Gamma^{1,1} \subset \Gamma_2^{1,9}$: *Borcherds–Harvey–Moore reduction*. Q lives now in $\Gamma_{0-4}^{1,1} \oplus E_8$ i.e. $H^0 \oplus H^4$ plus $E_8 \subset H^2$, and

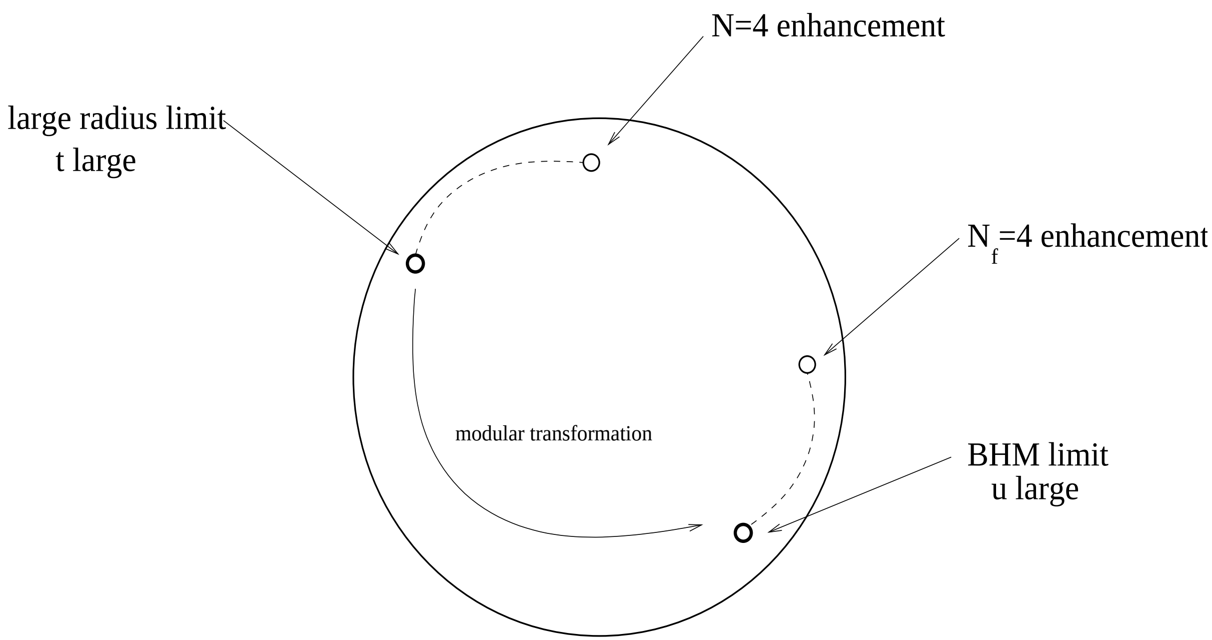
$$Z(u, g_s) = \prod_{Q,m} (1 - \mathrm{e}^{\frac{\mathrm{i}mg_s}{\sqrt{2}} - Q\cdot u})^{\Omega_{\mathrm{u}}(Q,m)} (1 - \mathrm{e}^{\mathrm{i}m\sqrt{2}g_s - Q\cdot u})^{4\Omega_{\mathrm{t}}(Q,m)}$$

$$\sum_{Q,m} \Omega_{\mathrm{u}}(Q,m) q^{Q^2/2} \mathrm{e}^{\mathrm{i}mg_s} = \frac{64}{\eta^6 \vartheta_2^6} \xi^2(q, g_s)$$

$$\sum_{Q,m} \Omega_{\mathrm{t}}(Q,m) q^{Q^2/2} \mathrm{e}^{\mathrm{i}mg_s} = \frac{64}{\eta^6 \vartheta_2^6} \xi^2(q^4, g_s)$$

For $g = 1$ one recovers Borcherds' modular form and Harvey–Moore expression for the FHSV model. This does not display standard GV integrality, and should be counting *D0–D2–D4 bound states* (cf. Xi Yin's talk)

The global picture



The F_g amplitudes at the large radius and BHM points are related by a modular transformation (cf. Mina Aganagic's talk).

Mirror symmetry I

We also analyzed the FHSV model by using mirror symmetry and the B–model. This allows us to

(1) verify that the geometric reduction corresponds to Gromov–Witten invariants

(2) incorporate the Kähler parameter of the base S (unavailable in the heterotic computation)

The complexity of a mirror–symmetric computation increases with the number of Kähler parameters $\Rightarrow$ we use a *reduced model* where we blow down the E_8 two–classes, and we are left with three Kähler parameters $t = \{t_1, t_2\}, S$.

Mirror symmetry II

We solved the holomorphic anomaly equations of [BCOV] up to $g = 4$ and fix the holomorphic ambiguity with various inputs.

$$\boxed{\begin{aligned} &F_1(t,S) = F_1^E(t) - 12\log\eta(q_S),\\ &F_2(t,S) = E_2(q_S)F_2^E(t),\\ &F_3(t,S) = E_2^2(q_S)F_3^E(t) + (E_2^2(q_S) - E_4(q_S))H(t) \end{aligned}}$$

with $q_S = \mathrm{e}^{-S}$, $E_{2n}(q)$ are Eisenstein series, and $H(t)$ is a known function

The mirror description is *arithmetic* i.e. $q_i = \mathrm{e}^{-t_i}, q_S = \mathrm{e}^{-S}$ are modular parameters and all relevant quantities are modular forms $\Rightarrow F_g(t,S)$ is a quasi–modular form with modular weight $2g-2$ for each variable.

Application I: Gromov–Witten invariants

From the GW point of view, the Enriques CY is probably the *simplest compact example*. This allows explicit calculations by deformation+ localization techniques [Maulik–Pandharipande].

For example, $F_g^E(t)$ compute Hodge integrals in the GW theory of the Enriques surface:

$$N_{g,Q}^E = \int_{\overline{\mathcal{M}}_g(E,Q)} \lambda_{g-1}$$

The fiber + base results for F_1, F_2 have been proved in GW theory. The Enriques CY is then the only *compact* CY for which topological string amplitudes have been successfully compared between algebraic geometry and string theory up to genus 2.

Application II: small Enriques black holes

Consider M–theory on the Enriques CY. By wrapping M2 branes around cycles Q in the Enriques fiber we produce a *small* 5d BH with entropy

$$S = 4\pi\sqrt{Q^2/2} + \cdots$$

This maps [Gaiotto–Strominger–Yin] to a 4d BH in type IIA with D6–D2 charge $p_0 = 1, Q_A$. The microscopic degeneracies of this BH can be extracted from the BPS degeneracies [Katz–Klemm–Vafa]

$$\Omega_E(Q, 0)$$

Conclusions and open problems

- An exactly solvable model for topological strings on a compact CY? Indications, but further work is needed

- First compact check of topological string theory≡GW theory for higher genus.

- Choice of basepoint/frame and enumerative meaning of $F_g(t)$: a particularly clear example based on heterotic one–loop integrals, but we expect the phenomenon to be more general

- Applications to BH counting: in general, one can compute $\Omega(Q,m)$ for 5d BHs in fibers of K3 fibrations.