Supersymmetric States in $\mathcal{N}$=4 Yang Mills

Shiraz Minwalla

Tata Institute of Fundamental Research

Based on

hep-th 0510251 (J. Kinney, J. Maldacena, S.M, S. Raju)

hep-th 0606087 (I. Biswas, D. Gaiotto, S. Lahiri, S.M)

WIP (above + C. Beasley, L. Grant, K. Narayan)

Closely Related Work

hep-th 0010206 (A. Mikhailov)

hep-th 0207125 (C. Beasley)

hep-th 0606088 (G. Mandal, N. Suryanarayana)

hep-th 0604023 (Berkooz, Reichmann, Simon)

Motivation

$\mathcal{N}$ =4 Yang Mills on S^3 characterized by

$$Z(\lambda, N) = \text{Tr} e^{-\beta_i Q_i}$$

Can we hope to compute this object?

Perhaps Integrability will yield $Z(\lambda, \infty)$. However $Z(\lambda, N)$ at finite N, at the moment, seems out of reach.

Pity as several interesting bulk dual phenonmena are invisible at $N = \infty$ (examples ahead)

However

$$Z_{\mathsf{SUSY}}(\lambda, N) = \mathsf{Tr}_{\mathsf{BPS}} e^{-\beta_i Q_i}$$

may be computable at all λ, N.

If so, we would be able to use this quantity to extract finite N (hence bulk interaction) information, though only for supersymmetric states. See examples ahead

In this talk I present a conjecture about - and report some progress on - the characterization of Z_{SUSY}.

Supersymmetric States and Q Cohomology

Recall that $\mathcal{N} = 4$ Yang Mills has $8Qs, 8\bar{Q}s +$ Hermitian Conjugates $S, \bar{S}$.

Qs have $j_1 = \pm 1/2$ and are chiral spinors under $SO(6)$. The $\bar{Q}s$ have $j_2 = \pm 1/2$ and are antichiral spinors under $SO(6)$.

By definition, supersymmetric states are part of short multiplets of the $PSU(4/4)$ superconformal algebra. All such states are annihilated by at least one supersymmetry + HC.

Can restrict attention to states annihilated by one particular Q (we choose it to have quantum numbers $J_1 = 1/2, (1/2, 1/2, 1/2)$) or by one particular $\bar{Q}$ (e.g. with charges $j_2 = 1/2$, $(-1/2, -1/2, -1/2)$) . Such states obey $H = 2J_1 + R_1 + R_2 + R_3$. All other susy states obtained from these by $PSU(4/4)$.

States annihilated by exactly one Q + HC preserve $\frac{1}{16}$ of the gauge theory susy.

In much of this talk we will be interested 1/8 BPS states annihilated by two Qs, with charges $j_1 = \pm 1/2, (1/2, 1/2, 1/2)$; such states have $j_1 = 0$, $H = R_1 + R_2 + R_3$ and characterize their multiplet.

1/4 BPS states are annihilated by Qs with $j_1 = \pm 1/2, (1/2, 1/2, 1/2)$ and $\bar{Q}$ with $j_2 = \pm 1/2$ $(-1/2, 1/2, 1/2)$ + HC. Such states have $J_1 = J_2 = R_1 = 0$ and $H = R_2 + R_3$.

Finally half BPS states are annihilated by Qs with $j_1 = \pm 1/2$, $(\pm 1/2, \pm 1/2, 1/2)$ and by $\bar{Q}$ with $j_2 = \pm 1/2$, $(\pm 1/2, -\pm 1/2, 1/2)$; they have $J_1 = J_2 = R_1 = R_2 = 0$ and $H = R_3$.

There also exist more exotic multiplets of $PSU(4/4)$ that preserve - for instance - 3 supersymmetries. We will not have anything to say about these here.

We are interested in states annihilated by a given set of Qs +HC. Standard arguments demonstrate that such states are in 1-1 correspondence with Q cohomology ($Q|\psi\rangle = 0$ for all Q and $|\psi_1\rangle \sim |\psi_2\rangle$ if $|\psi_1\rangle - |\psi_2\rangle = Q|\phi\rangle$ for some $|\phi\rangle$ and any Q.)

Using the state-operator map, the partition function over supersymmetric states of $\mathcal{N} = 4$ theory on S^3 may be reinterpreted as a partition function over operators in the same theory on R^4, in Q cohomology.

This description, in turn, is useful because the action of supersymmetries on operators in R^4 is relatively familiar

Classical Q Cohomology and a Conjecture

$$Q^i_\alpha \Phi_{jk} \sim \delta^i_j \Psi_{k\alpha} - \delta^i_k \Psi_{j\alpha} \, ,$$

$$Q^i_\alpha \Psi_{j\beta} \sim \delta^i_j f_{\alpha\beta} + g_{YM} \epsilon_{\alpha\beta} [\Phi^{ik}, \Phi_{kj}] \, ,$$

$$Q^i_\alpha \overline{\Psi}^j_{\dot\beta} \sim D_{\alpha\dot\beta} \Phi^{ij} \, ,$$

$$Q^i_\alpha A_{\beta\dot\gamma} \sim \epsilon_{\alpha\beta} \overline{\Psi}^i_{\dot\gamma}$$

classically , where

$$D_{\alpha\dot\beta} = \partial_{\alpha\dot\beta} + g_{YM} [A_{\alpha\dot\beta},]$$

Quantum mechanically these relations are presumably modified. Nonetheless we conjecture that

$Z_{\mathcal{SUSY}}$ is given by the partition function over the classical Q cohomology at every finite value of λ and N

At $\lambda = 0$ this is trivially true.

At nonzero, non-infinite λ it implies that $Z_{\mathcal{SUSY}}$ is independent of λ (follows from scaling)

Computation of Classical Cohomology

The classical half BPS cohomology made up of $F(Z)$ at all λ including $\lambda = 0$. Correct answer as half BPS multiplets are absolutely protected by $PSU(4/4)$ representation theory.

One fourth BPS states are more interesting. At $\lambda = 0$ we have arbitrary gauge invariant functions of Y and Z

$$Z_{\mathcal{SUSY}} = \int dU \exp\left[-\frac{e^{n\beta_2} + e^{n\beta_3}}{n} \mathrm{Tr} U^n \mathrm{Tr} U^{-n}\right]$$

Phase transition between $\ln Z = \mathcal{O}(1)$ and $\ln Z = \mathcal{O}(N^2)$ at β of unit order.

At nonzero λ the classical cohomology is discontinuously different. The cohomology consists of functions of Y and Z with $[Y, Z] \sim 0$, i.e. of gauge invariant functions of diagonal Y and Z.

This is the same as holomorphic functions in $(C^2)^N/S_N$, i.e. the wave functions (in holomorphic or coherent state quantization) of N bosons in a 2d harmonic oscillator. The partition function is the power of p^N in

$$Z_{1/4} = \prod_{n_1,n_2=0}^{\infty} \frac{1}{1 - p \exp\left(-n_1\beta_1 + n_2\beta_2\right)}$$

1/8 BPS cohomology qualitatively similar. 3 Bosonic and 2 Fermionic diagonal matrices. Similar formula for partition function

At fixed β_i and $N \to \infty$ $Z_{1/8}$ is identical to the partition function over 1/8 BPS multi gravitons in supergravity on $AdS_5 \times S^5$.

However at every N, no matter how large, $Z_{1/4}$ and $Z_{1/8}$ deviate significantly from the KK partition function at small enough β. In fact as $N \to \infty$ $Z_{1/8}$ undergoes a Bose Einstein phase transition at $\tilde{\beta} = N^{\frac{1}{3}}\beta$ of unit order. Large $\tilde{\beta}$, $\ln Z_{1/8} = \mathcal{O}(1)$; small $\tilde{\beta}, \ln Z_{1/8} = \mathcal{O}(N)$.

$Z_{1/8}$ from Giant Gravitons

At $E \sim N$, Kalutza Klein reduction incorrect because gravitons blow up (Myers Effect). Must quantize giant gravitons.

Most general 'scalar' Giant Graviton given by the intersection of surface $F(z^i) = 0$ in C^3 with unit sphere (Mikhailov). Relevant part of surface arbitrarily well approximated by $P(z^i) = 0$ where P is polynomial. Regulate by polynomials generated by linear span of any n_C monomials.

Generalization of intersection of $z = z_0$ with S^5 (S^3 of squared radius $1 - 1/|z_0|^2$)

Space of distinct surfaces $P(z^i) = 0$ is $\mathcal{CP}^{n_C-1}$. Motiviated by this observation, Beasley suggested that giant gravitons are quantized by quantizing $\mathcal{CP}^{\infty}$. We demonstrate that, though the story has twists and turns, this guess is correct.

Chief Complication: The $\mathcal{CP}^{n_C-1}$ of coefficients does not label intersections of $P(z) = 0$ with the unit sphere in a one-one fashion. The labeling has ‘holes’ and redundancies.

Example: $P(z) = c + c_i z^i$. Set gauge $c = 1$. Space of intersections labeled by c_i with $|c_i|^2 > 1$.

Symplectic Form from Born Infeld + WZ

$$\omega = 2N\left[\left(\frac{1}{|c_i|^2} - \frac{1}{|c_i|^6}\right)\frac{d\bar{c}_i \wedge dc^i}{2i} - \left(\frac{1}{|c_i|^2} - \frac{2}{|c_i|^4}\right)\frac{\bar{c}_i c_j}{|c_i|^2}\frac{d}{}\right.$$

$\mathcal{CP}^3$ with a hole $|c_i|^2 < 1$ 'eaten out'. Also the symplectic form degenerates at $|c_i|^2 = 1$. However set

$$c_i = \sqrt{\frac{1+|w_i|^2}{|w_i|^2}}w_i$$

(note $|w|$=0 when $|c| = 1$) after which

$$\omega = \frac{2N}{1+|w_i|^2}\left[\frac{d\bar{w}_i \wedge dw^i}{2i} - \frac{\bar{w}_i w_j}{1+|w_i|^2}\frac{d\bar{w}_i \wedge dw_j}{2i}\right]$$

In the w^i variables the symplectic form is just N times the Kalher or Fubini Study form on $\mathcal{CP}^3$.

This result generalizes. The parameterization by coefficients of the intersections of $P(z) = 0$ with the unit sphere is bad (has holes and gauge equivalences).

However we have proved that it is always possible to find a variable change to good coordinates that parameterize a genuine $\mathcal{CP}^{n_C-1}$.

In new coordinates the symplectic form is a non degenerate current (distribution) in the cohomology class of $N\omega_{\mathcal{FS}}$.

Our general abstract proof establishes existence; however it is possible to perform explicit constructions for specific choices of P_C.

Quantization is now straightforward. In a choice of polarization, the Hilbert Space is the set of holomorphic functions of degree N in the new 'good' projective variables w. The partition function over this Hilbert Space equals $Z_{1/8}^{Scalar}$.

$Z_{1/8}$ From Dual Giant Gravitons

1/8 BPS dual giant gravitons carry no angular momentum in AdS_5 and so are simply concentric spheres in AdS at points on the S^5.

G. Mandal and N. Suryanarayana have demonstrated that the phase space of a single dual giant graviton is, infact, C^3; the cone over S^5, with symplectic form $\omega = dz_i \wedge d\bar{z}^i$. Quantization yields the 3d Harmonic oscillator.

It is thus tempting to interpret $Z_{1/8}$ as the partition function of exactly N dual giant (G.Mandal. and N.Suryanarayana). The restriction to N is motivated by the requirement that the flux at the origin be non negative

Bulk Interpretation of the Phase Transition

Recall that, at large N $Z_{1/8}$ undergoes a Bose Einstein Phase transition. Low temperature phase a gas of gravitons in $AdS_5 \times S^5$.

The high temperature phase is one in which all N dual giant gravitons are out of their ground state and so have finite radius

Thus the 5-form flux is zero in the center of AdS in this phase. The corresponding gravity solution should, have a bubble of flat space in the center; analogeous to the enhancon solution.

An Aside: Generalization to L^{abc} Spaces

Over the last 3 years a large number of $\mathcal{N} = 1$ examples of the AdS/CFT duality have been discovered. On the bulk S^5 is replaced by L^{abc}, the base over the cone of the CY obtained from the $U(1)$ GLSM with 4 fields of charges $a, b, c, -a-b-c$ and zero FI term.

The set of cohomologically trivial giant gravitons in these models are simply given by the intersection of the unit sphere with those holomorphic functions in C^4 that are invariant under the GLSM projection (provided L^{abc} is geometrically nonsingular)

Running through the quantization procedure behind, we find that the partition function is the power of p^N in

$$Z = \prod_{n_1, n_2, n_3, n_4 = 0}^{\infty} \frac{1}{1 - p \exp\left(-\sum_i \beta_i n_i\right)}$$

where the product is restricted to integers that obey

$$n_1 a + n_2 b + n_3 c - (a + b + c) n_4 = 0.$$

The chemical potentials β_i couple to rotations of z_i.

Z may also presumably be interpreted as the Hilbert Space of N dual giant gravitons if H_1=holomo
functions on CY

Let us compare Z with the classical 2 supercharge cohomology (chiral ring) of the dual gauge theories. I now describe this cohomology in every geometrically nonsingular example we have analyzed - including the infinite class $(a, b, c) = (1, 2p - 1, -p)$.

Gauge invariant chiral operators uncharged under the global $U(1) \subset U(N)$ (this restriction maps to the cohomological triviality of the giant gravitions) may be written as gauge invariant functions ('traces of products') of a finite set of commuting adjoint matrices at any chosen node of the quiver

These adjoint matrices are in one-one correspondence with the generators of gauge invaraint monomials of the GLSM. To the extent we have checked they also satisfy their constraint equations. Implies that the cohomological partition function is Z.

In general this match would work whenever the classical (non Baryonic) moduli space of the gauge theories is the symmetric product of N copies of the Calabi Yau Space, a statment that may already have been established for nonsingular L^{abc} by experts.

Singular L^{abc} and cohomologically nontrivial states : work in progress.

1/16 BPS Cohomology and Black Holes

The free cohomological partition function counts the number of gauge invariant states formed from two unrestricted derivatives of 4 bosonic and (effectively) 4 fermionic letters. Like 2+1 dimensional gas at high energies. The exact formula for Z is given in terms of an integral over a unitary matrix - will not write

I have not yet been able to compute the partition function over the interacting classical cohomology. This is a very interesting but quite difficult problem. Work in progress

Rest of talk: Some remarks about $Z_{1/16}$.

An Index over 1/16 Cohomology

$Z_{1/16}$ is easy to compute on a 4 dimensional slice of 5 dimensional parameter space, because it is an index at those special parameters. Specifically

$$Z_{\text{Index}} = \text{Tr} \exp \left(i\pi J_1 + i\zeta J_2 - \gamma_i L_i \right)$$

where

$$L_1 = J_1 - R_1 + R_2 + R_3$$

Z_{Index} may be computed in the free theory as it is guaranteed not to be renormalized.

An aside: Z_{Index} is the most general index whose protection is guaranteed by the superconformal algebra alone; has counterparts for all superconformal theories.

In the large N limit

$$Z_{\text{Index}} = \prod_{n=1}^{\infty} \frac{\left(1 - e^{-in\zeta - n\gamma_1 - n\gamma_2 - n\gamma_3}\right)\left(1 - e^{in\zeta - n\gamma_1 - n\cdots}\right.}{(1 - e^{-2n\gamma_1})(1 - e^{-2n\gamma_2})(1 - e^{-2n\gamma\cdots}}$$

Agrees with supergravity index. Qualitatively different behaviour from free $Z_{1/16}$ at other values of coupling. No phase transition. Logarithm never of order N^2. Disappointing.

Exact expression for Z_{Index} at finite N in terms of an integral over unitary matrices. Would be interesting to evaluate.

$Z \sim$ Poincare Polynomial at $t = 1$

$Z_{Index} \sim$ Euler Character

Free $\rightarrow$ Interacting $\sim$ Blow up?

If so can we identify the singular space whose Poincare Polynomial is $Z_{1/16}^{\text{Free}}$?

Giant Gravitons

As for the 1/8 BPS states, we expect, atleast in some energy regime, to be able to reproduce $Z_{1/16}$ by quantizing giant gravitons.

As a first step in this programme we must generalize Mikhailov's construction to 1/16 BPS giant gravitons. We have partially achieved this. One set of 1/16 giant gravitons are given by the intersection of the zero sets of 3 homogeneous, holomorphic functions in $C^{1,5}$ with $AdS_5 \times S^5$. May be rewarding to study, complete and quantize these solutions.

Black Holes

A number of authors have constructed extremal black hole solutions in $AdS_5 \times S^5$, within gauge supergravity.

These black holes exist on a 5 dimensional manifold parameterized by charges, but are supersymmetric on only a 4 dimensional submanifold. Very puzzling Feature

The entropy of the supersymmetric black holes is qualitatively, but not quantitatively reproduced by $Z_{1/16}^{\text{Free}}$.

Phenomenological Observation: Entropy of small black holes reproduced by the 1+1 d CFT of 'waves' on Mikhailov's static giant gravitons. Why

Extremal, even supersymmetric Black holes in $AdS_5 \times S^5$ very poorly understood. An accounting of the entropy of these black holes will be an important and an interesting application of $Z_{1/16}$.