Beyond Morse Theory
or
Instantons and Logarithms

IHES

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ИТЭФ

Strings 2006

Beijing

Based on the joint works in progress with

- Ed Frenkel and Andre Losev
- Andre Losev and Grigory Mikhalkin
- Nathan Berkovits

Some motivation

in random order

- Non-susy operators in susy theory
- Pure spinor formalism in superstring
- New invariants of 4-manifolds

  (beyond Donaldson)

- Tropical **Z**-theory

  towards nonperturbative topological string

N.Berkovits,
NN
Losev-Krotov

- **Most two and four dimensional quantum field theories have two types of couplings: the actual coupling constant g, which enters the perturbation theory, counts loops, and the q angle, which couples to the topological charge.**
- **One combines these two types of couplings into the complex couplings** t **and** t^*
- **Study the analytical dependence on** t **and** t^* **separately?**

For example, in four dimensional gauge theory it is customary to combine the gauge coupling g_{YM} and the theta angle ϑ:

$$t = \frac{\vartheta}{2\pi} + \frac{4\pi i}{g_{YM}^2}$$

Two dimensional sigma models

For Kahler target space X with the metric G_{ij} and the B-field one uses

$$t_{ij} = B_{ij} + i\,G_{ij}$$
$$t_{ij}^{*} = B_{ij} - i\,G_{ij}$$

When $dB=0$ the B-field plays the role of the q angle

(Supersymmetric) Quantum mechanics

$$L = \frac{\lambda}{2}\left(g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} - g^{\mu\nu}\partial_{\mu}f\partial_{\nu}f\right)$$

$$+ \text{ i } \vartheta\partial_{\mu}f\dot{x}^{\mu} \qquad + \text{fermions} \ldots$$

The holomorphic coupling is

$t = q \quad + i\, l$

For finite l this is the SQM used by E.Witten to prove Morse inequalities

In all these examples we expect

The correlators to be the functions of t and t^* :

$$Z(t, t^*)$$

They should simplify in the large t^* fixed t limit

Lagrangian implementation

Use the first order formalism (and drop the $l\, df$ *term)*

$$\mathcal{L} = ip_m(dx^m + \mathcal{V}^m\, dt) - l^{-1} g^{mn} p_m p_n\, dt + it\, df$$

$$\mathcal{V}^m = g^{mn} \frac{df}{dx^n}$$

As l z ∞ *the Lagrangian becomes simply*

$$\mathcal{L}_\infty = ip_m(dx^m + \mathcal{V}^m\, dt) + it\, df$$

Lagrangian implementation

Integrate out p_m *in the theory with Lagrangian*

$$\mathcal{L}_\infty = i p_m (dx^m + \mathcal{V}^m\, dt) + it\, df$$

the path integral becomes the finite dimensional integral over the moduli space $\mathcal{M}_{a,b}$ *of gradient trajectories:*

$$dx^m/dt + \mathcal{V}^m = 0$$

$$x(+\infty) = b, \; x(-\infty) = a$$

Lagrangian implementation: observables

General observables $\mathcal{O}(x^m, p_m, y^m, p_m)$ *in the supersymmetric theory correspond to the differential operators on the* PTX

Evaluation observables $\mathcal{O}(x^m, y^m) \longleftrightarrow$ *diff. forms on* **X** *are the simplest to study*

$$<_a \mathcal{O}_1(t_1)\, \mathcal{O}_2(t_2)\, \ldots\, \mathcal{O}_n(t_n) >_b \;=\; \int_{\mathcal{M}_{a,b}} ev(t_1)^* W_1 \wedge \ldots \wedge ev(t_n)^* W_n$$

$$\times \exp\left(i\, t\, (f(b) - f(a) \right)$$

Lagrangian implementation: correlators of observables

$$\mathcal{O}_n(x^m, y^m) \longleftrightarrow W_n(x^m, dx^m)$$

$$ev(t) : \mathcal{M}_{a,b} \longrightarrow X$$

$$ev(t)\,[x] = x(t)$$

evaluation of the gradient trajectory at the moment of time t

$$<_a \; \mathcal{O}_1(t_1)\, \mathcal{O}_2(t_2) \; \ldots \; \mathcal{O}_n(t_n) >_b \; = \int_{\mathcal{M}_{a,b}} ev(t_1)^* W_1 \wedge \ldots$$

$$\wedge\, ev(t_n)^* W_n$$

$$\times \exp\left(i\, t\, (f(b) - f(a)) \right)$$

Remark

For closed forms $W_n(x^m, dx^m)$

$$<_a O_1(t_1)\, O_2(t_2)\, \dots\, O_n(t_n) >_b \;=\; \int_{\mathcal{M}_{a,b}} ev(t_1)^* W_1 \wedge \dots \wedge ev(t_n)^* W_n$$

$$\times \exp\left(i\, t\, (\, f(b) - f(a)\,)\right)$$

Define the simplest Gromov-Witten invariants

We are not interested in them here, though

Hamiltonian implementation

Redefine wave functions

$$Y \rightarrow Y^{in} = Y \; \mathbf{exp} \, (+ \, l \, \mathbf{f} \,)$$

$$Y^{*} \rightarrow Y^{out} = Y^{*} \, \mathbf{exp} \, (- \, l \, \mathbf{f} \,)$$

Preserves pairing between (in) and (out) states but violates unitarity

The Hamiltonian becomes, after the conjugation

$$\mathbf{H} = \mathbf{L}_{v} - l^{-1} D$$

Hamiltonian implementation

Redefine wave functions

$$Y \Rightarrow Y^{in} = Y \ \mathbf{exp}\,(\, + \, l \, f\,)$$

$$Y^{*} \Rightarrow Y^{out} = Y^{*} \ \mathbf{exp}\,(\, - \, l \, f\,)$$

Preserves pairing between (in) and (out) states but violates unitarity

The Hamiltonian becomes, in the limit $l \ \underline{z} \ \infty$

$$\mathbf{H}_{\infty} = \mathbf{L}_{V}$$ *a first order operator!*

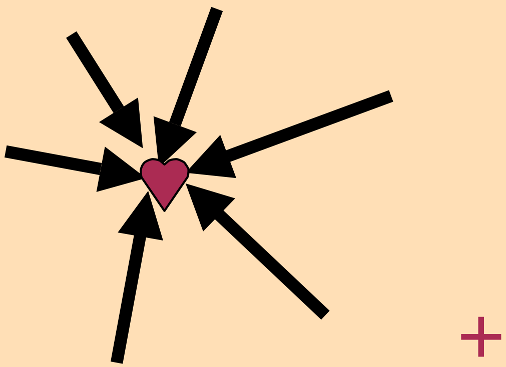
Local theory: harmonic oscillator

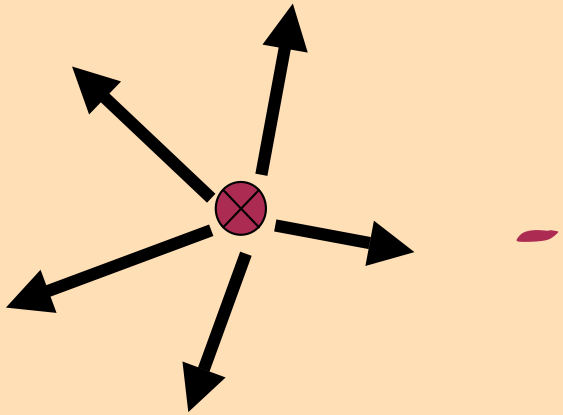
- *This is a situation near a critical point of* f
- *There are two basic cases:*

$$f = +\frac{w}{2}x^2 \quad \textit{or} \quad f = -\frac{w}{2}x^2$$

where we assume $w > 0$

Attractive critical point





Repulsive critical point

The fate of the states in the | z ∞ limit

$Y^{in} = P(x)$

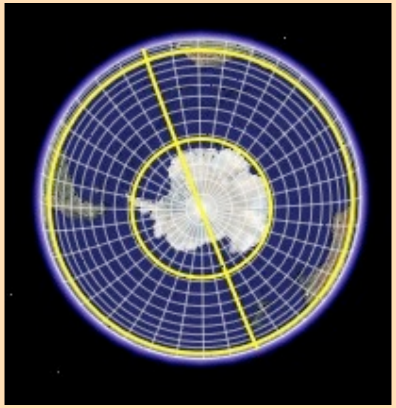
$Y^{out} = P(\partial_x)\, d(x)$

$Y^{out} = P(x)$

$Y^{in} = P(\partial_x)\, d(x)$

With P - polynomial(diff.form)s

Attractive critical point



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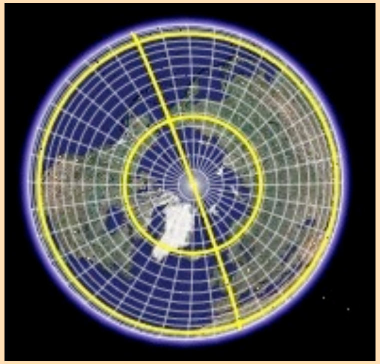

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Repulsive critical point

Remark

Here and below we drop the factors

$$e^{\pm itf}$$

in Y^{in} and Y^{out} ,

respectively

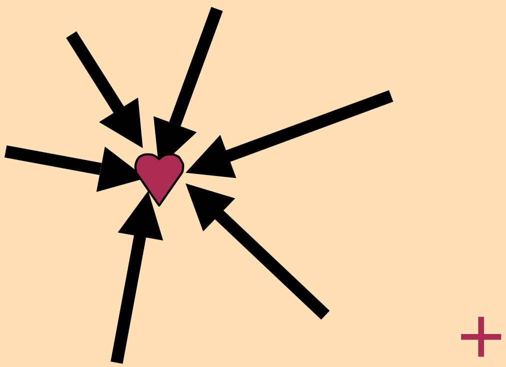
The spectrum of the Hamiltonian in the | z ∞ limit

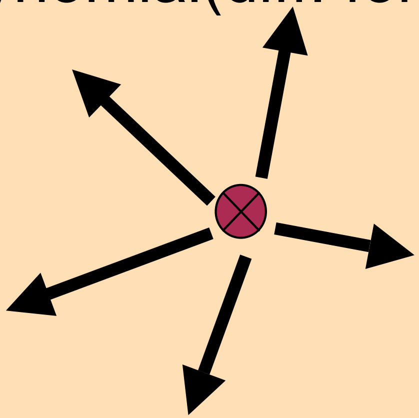
$$H^{in} = w\, L_{xd/dx}$$
$$H^{out} = -w\, L_{xd/dx}$$

$$H^{out} = -w\, L_{xd/dx}$$
$$H^{in} = w\, L_{xd/dx}$$

Acting on the corresponding polynomial(diff. form)s

Attractive critical point





Repulsive critical point

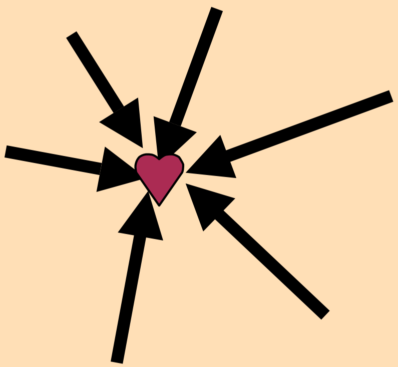
The spectrum of the Hamiltonian in the | z ∞ limit

$$E^{in}_{n} = |w|\, n$$
$$E^{out}_{n} = |w|\, n$$

n=0,1,2,..

In all cases!

Attractive critical point



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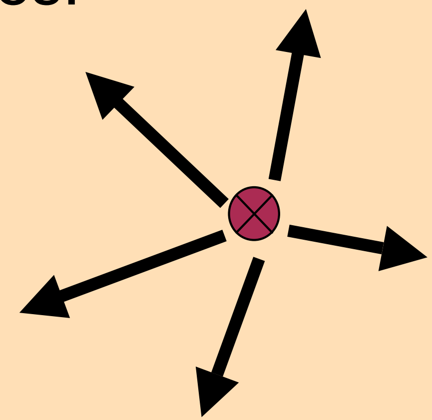


Repulsive critical point

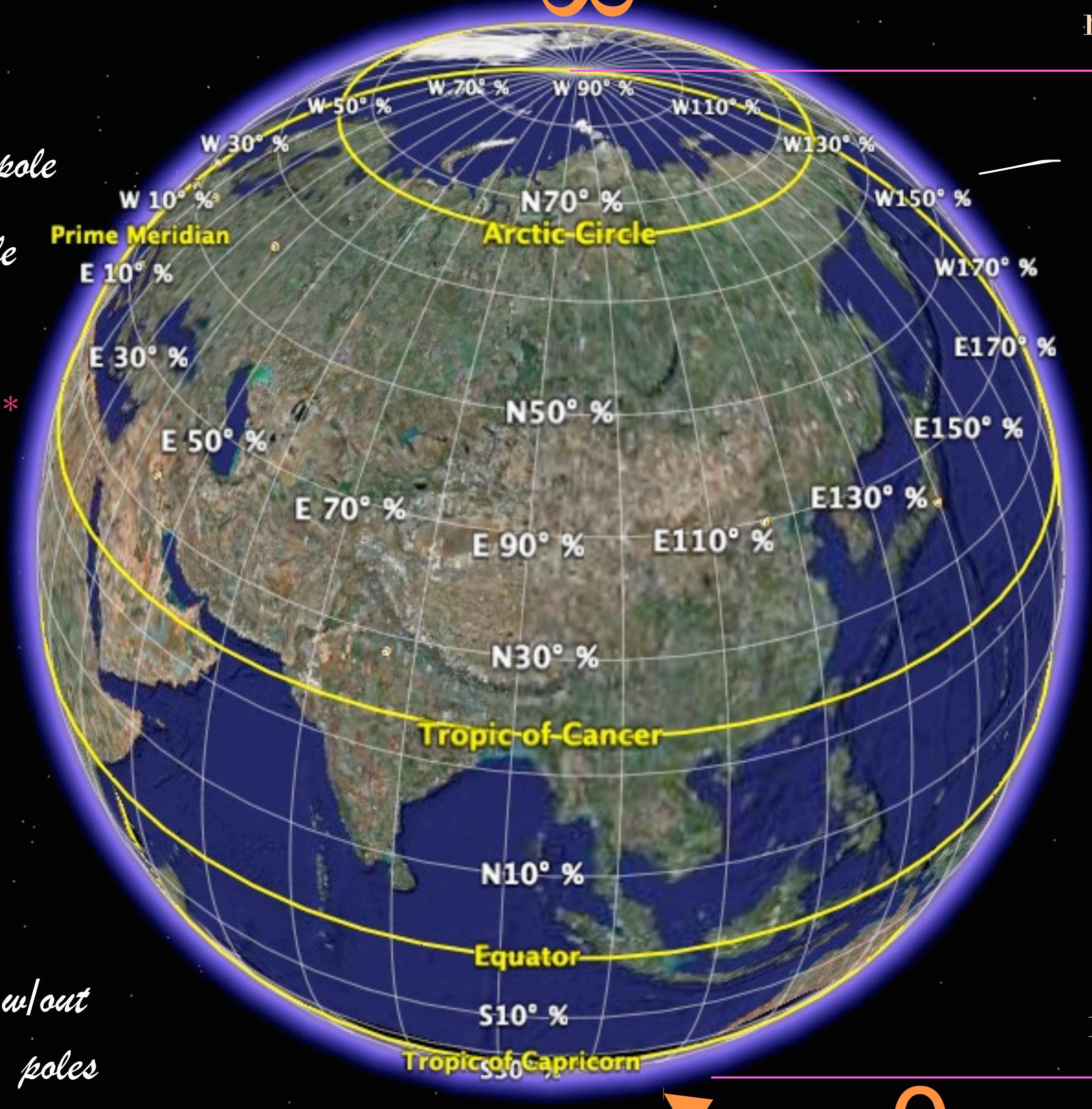
Example of global theory

$\mathcal{M}_{\infty,\infty}$ - *the North pole*

$\mathcal{M}_{0,0}$ - *the South pole*

$$\mathcal{V} = z\partial_z + z^*\partial_{z^*}$$

∞



max

f(x)

min

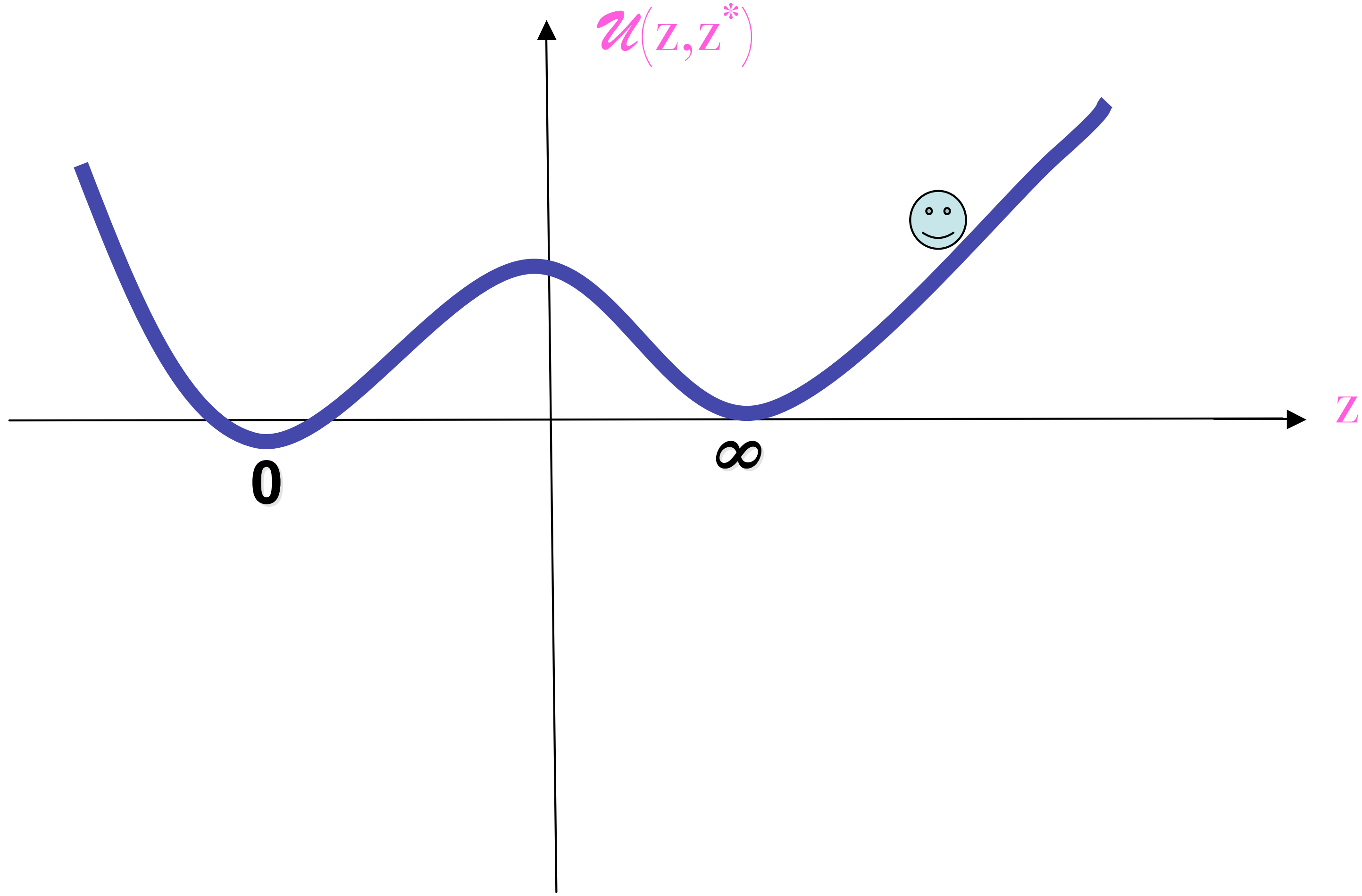
0

Image © 2006 NASA

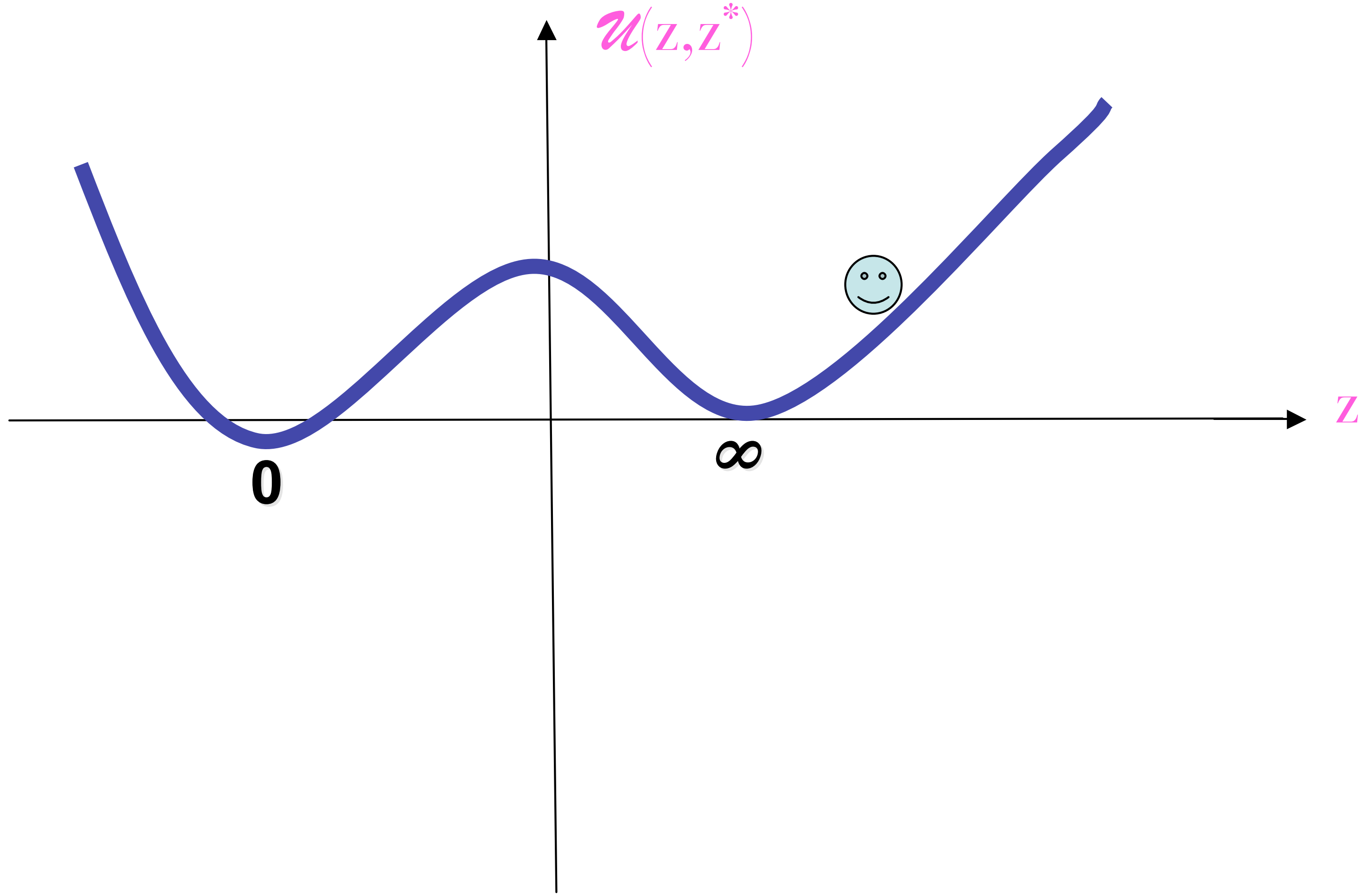
$\mathcal{M}_{0,\infty}$ - *empty*

$\mathcal{M}_{\infty,0}$ - *the Earth w/out poles*

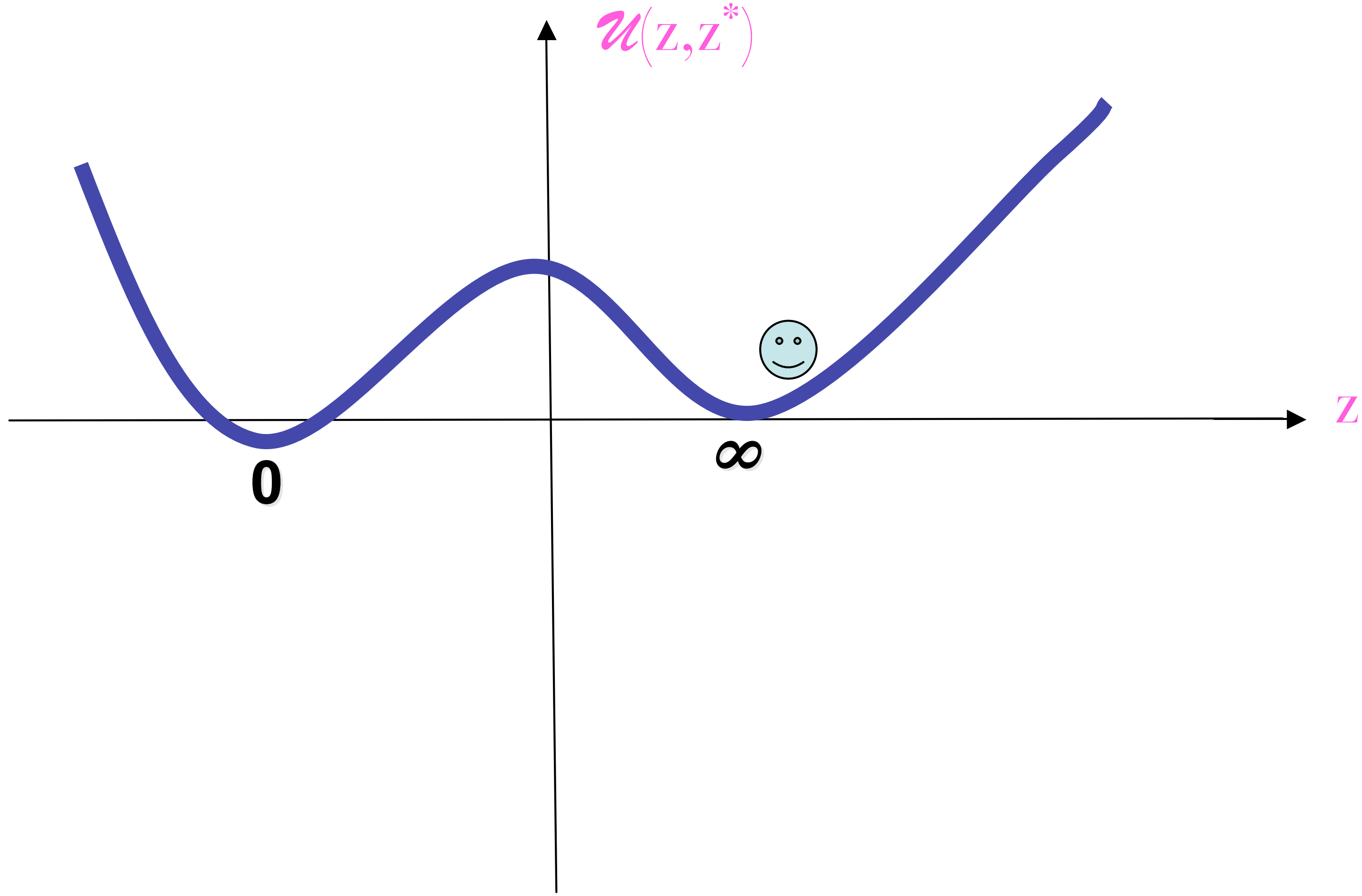
The globe viewed quantum mechanically



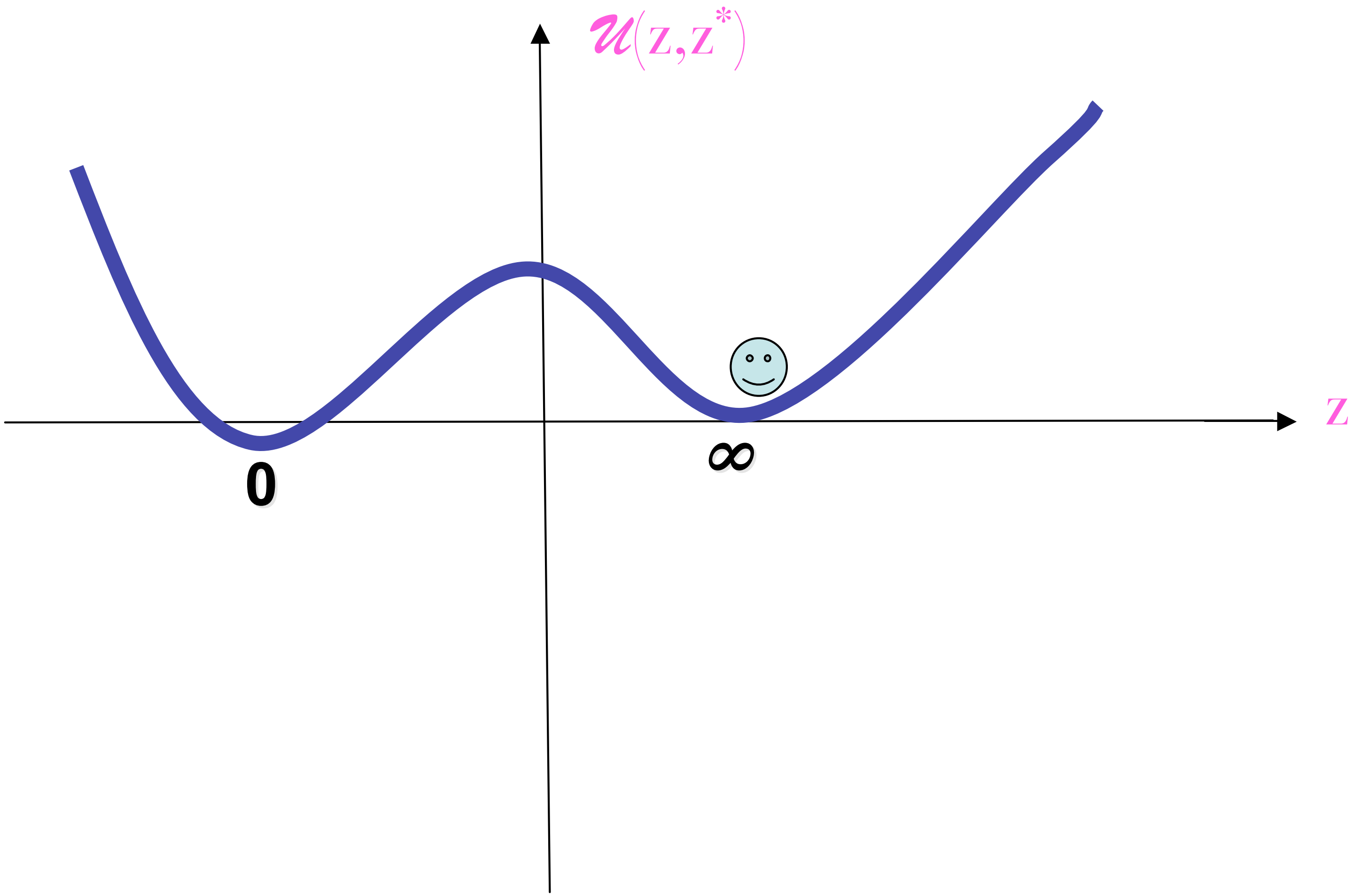
The globe viewed quantum mechanically



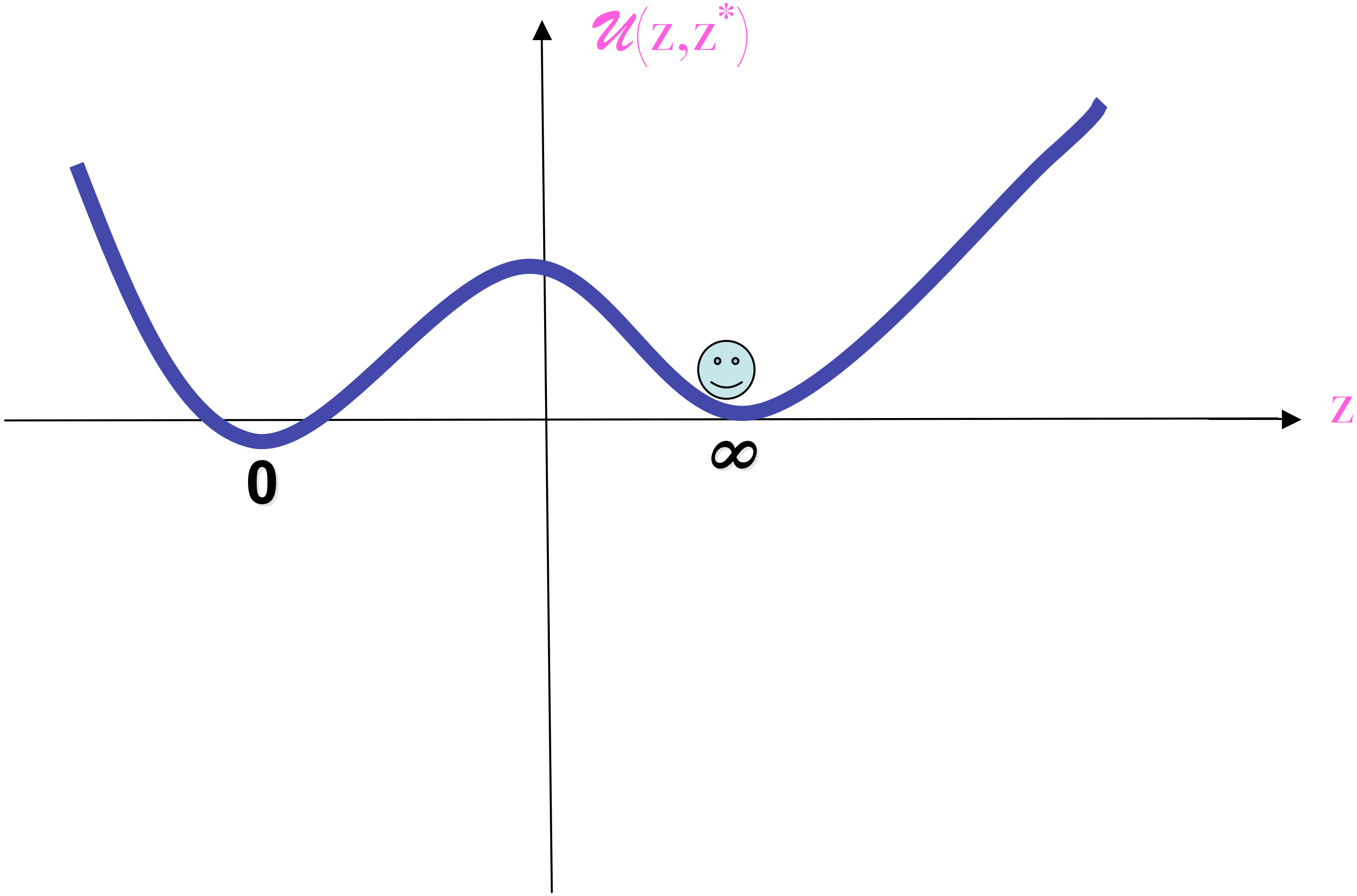
The globe viewed quantum mechanically



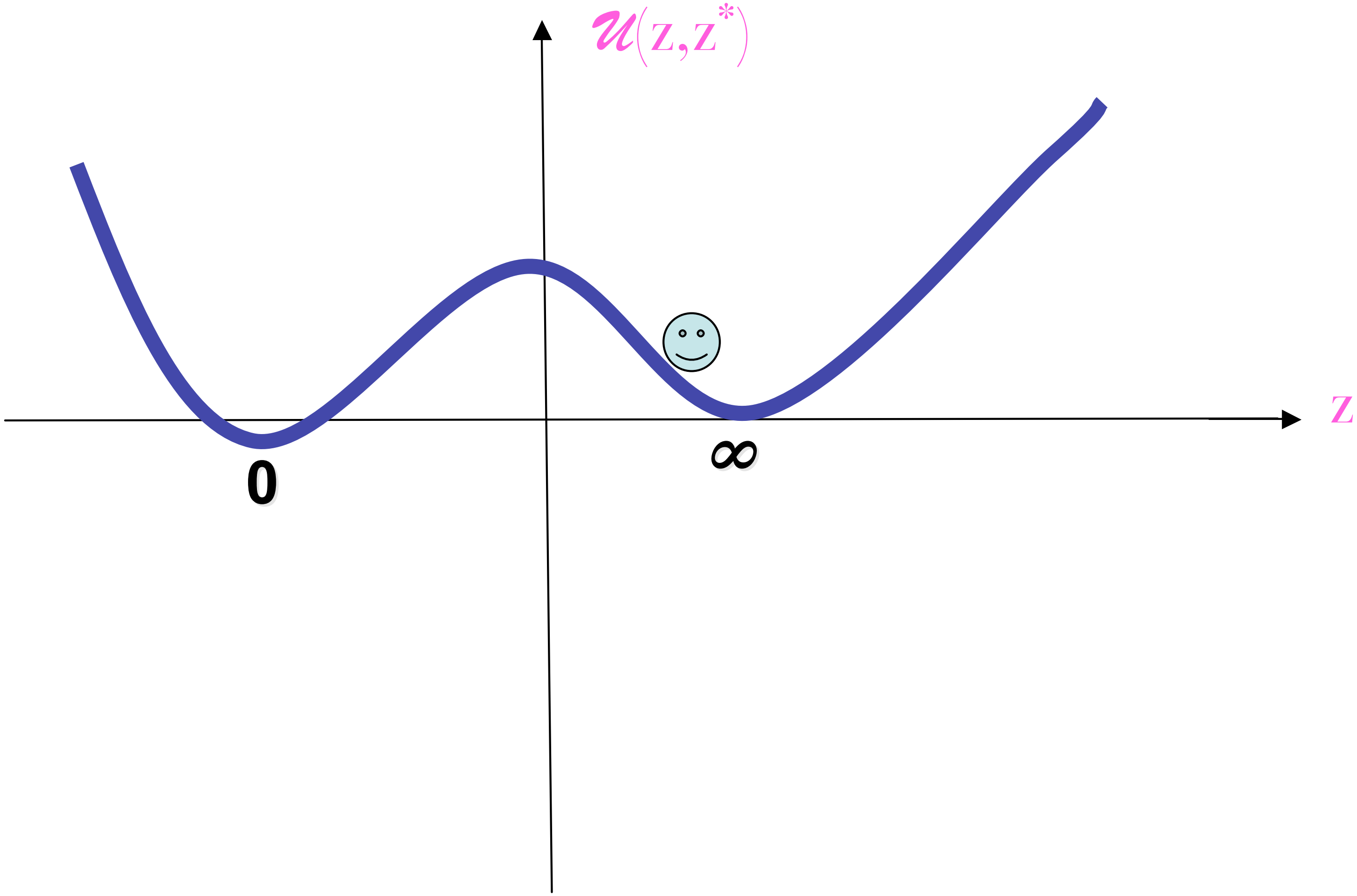
The globe viewed quantum mechanically



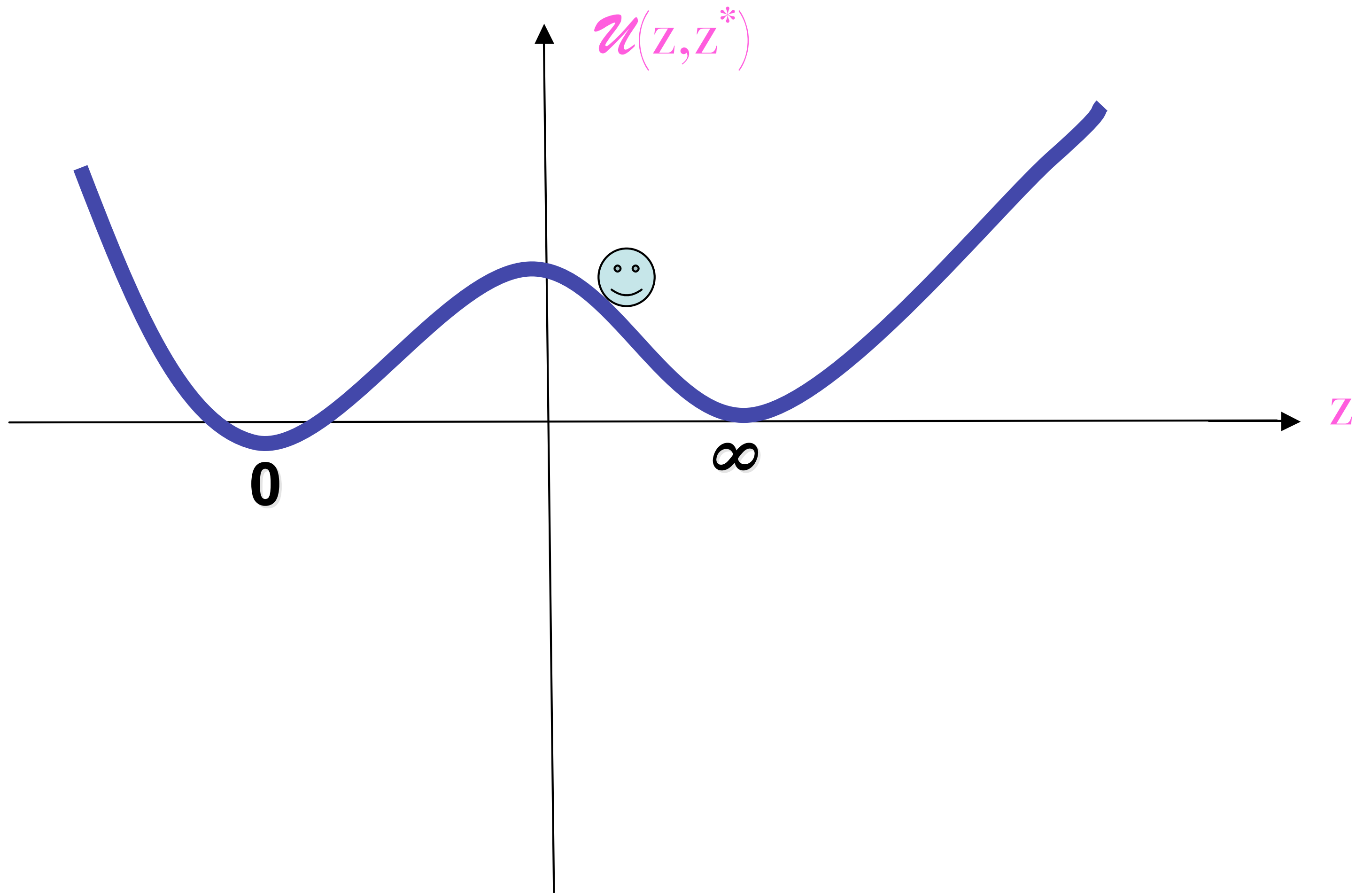
The globe viewed quantum mechanically



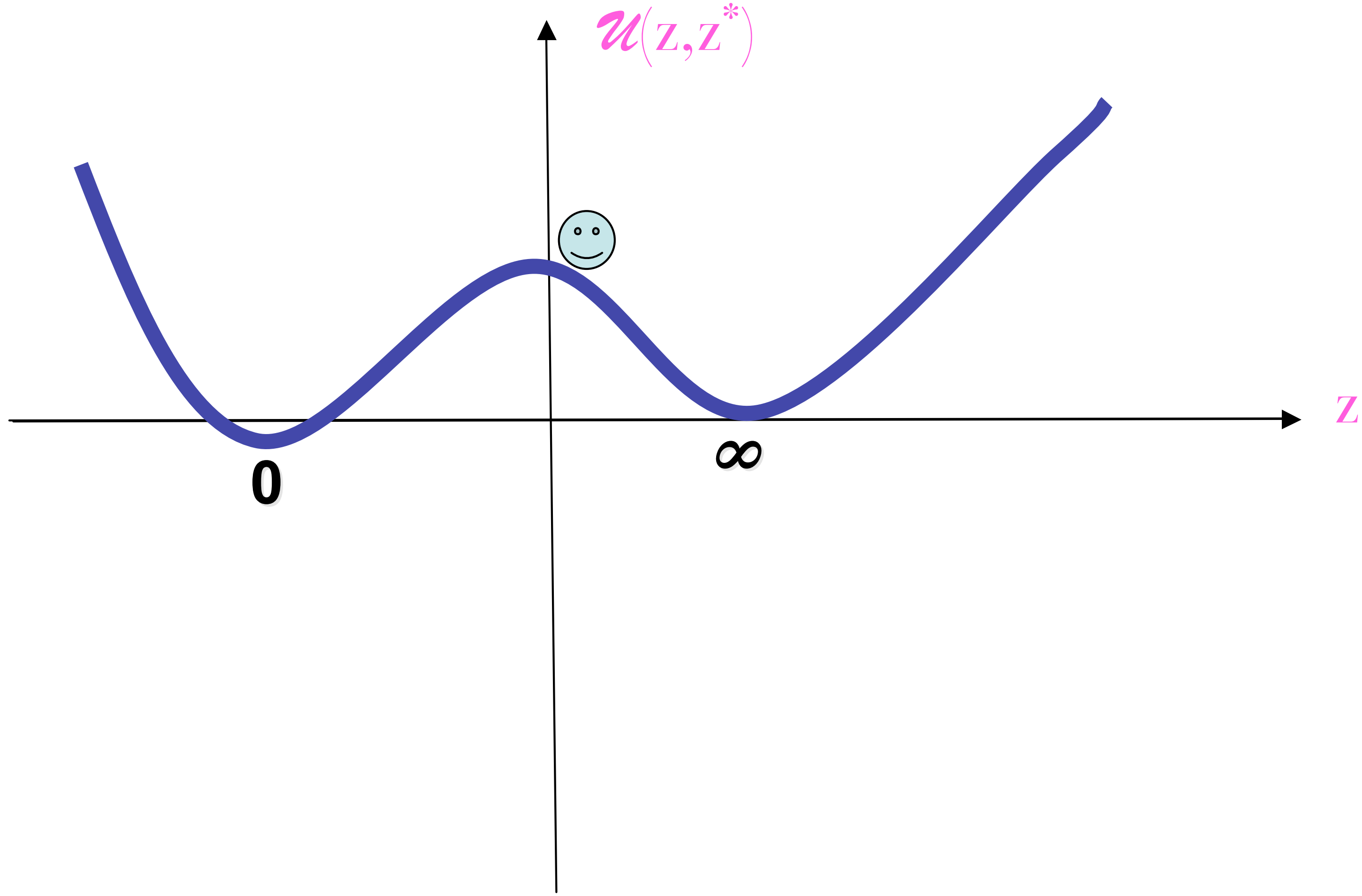
The globe viewed quantum mechanically



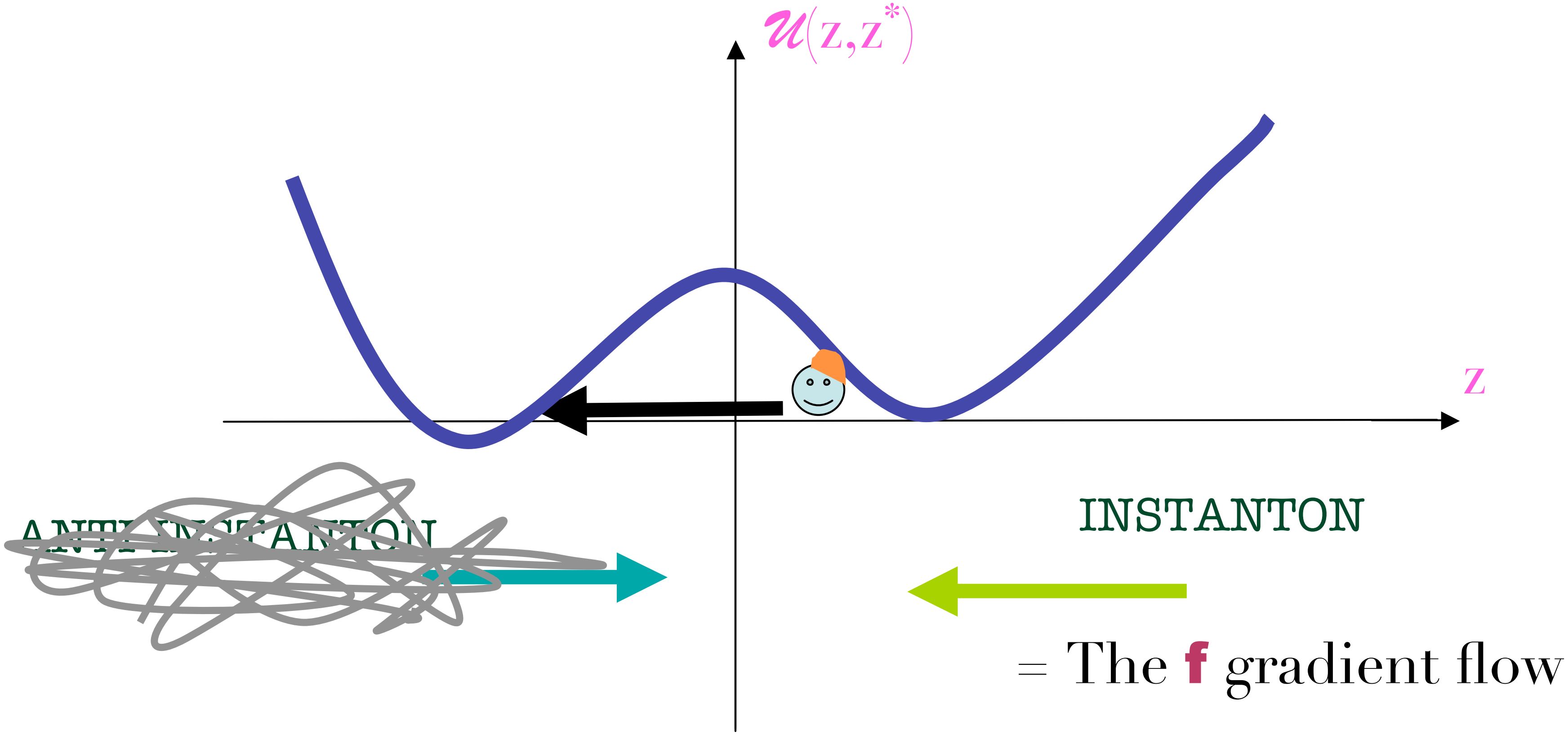
The globe viewed quantum mechanically



The globe viewed quantum mechanically

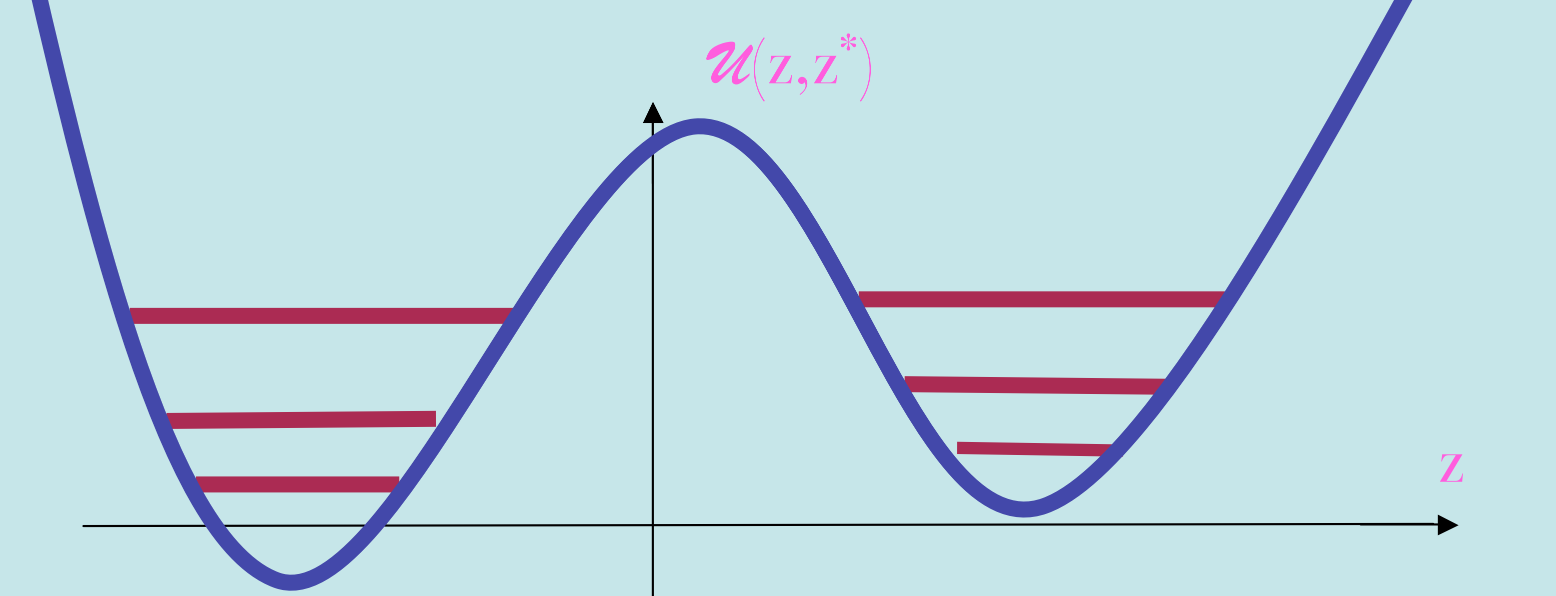


The globe viewed quantum mechanically



forbidden

The space of in-states



$\mathcal{H}^{\text{in}}{}_{\mathbf{0}}$ =

Polynomial forms on

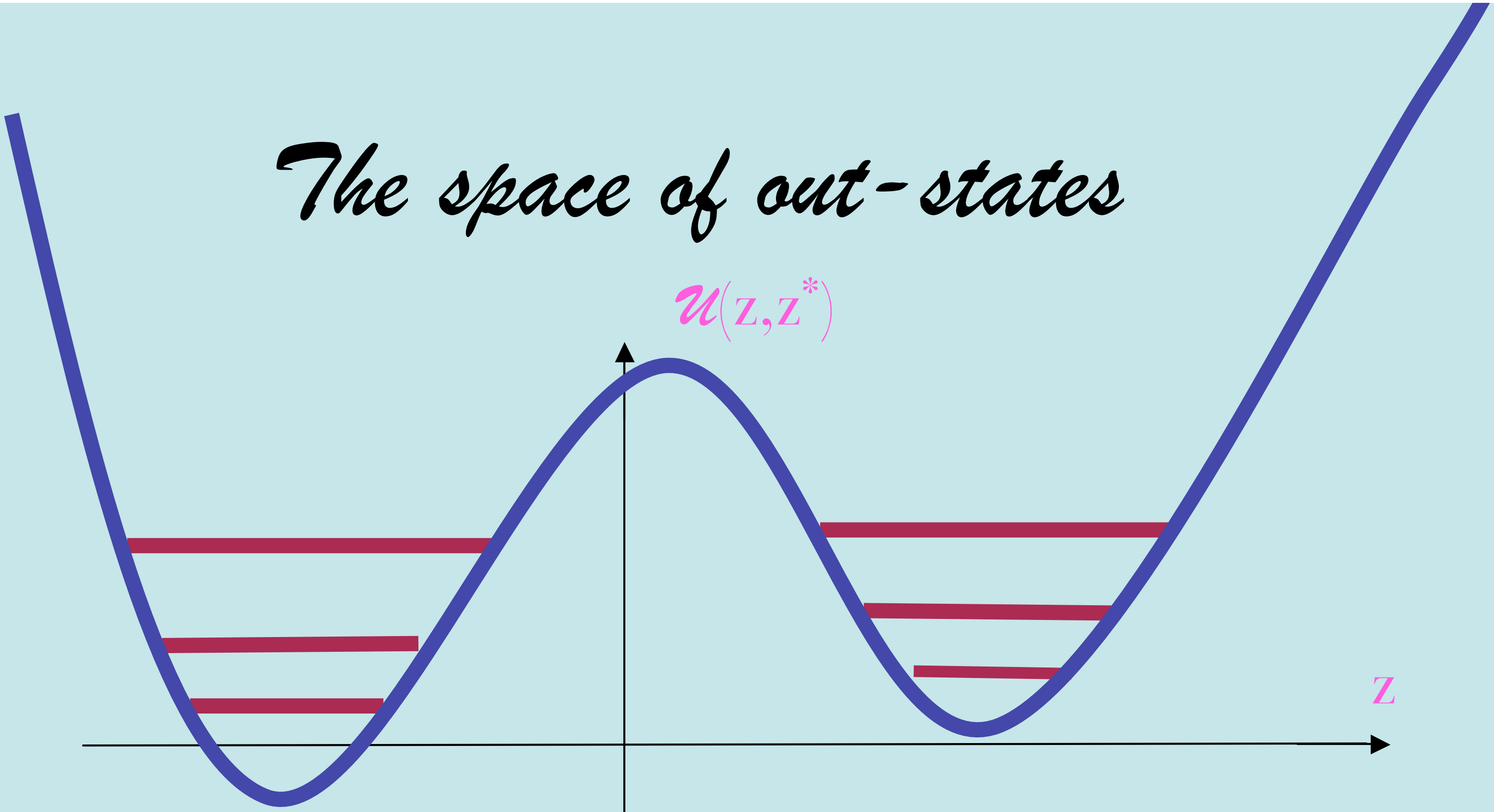
$\mathbf{C} = \mathbf{CP}^1 \setminus \{\infty\}$

$\mathcal{H}^{\text{in}}{}_{\infty}$ =

d-forms

supported at ∞

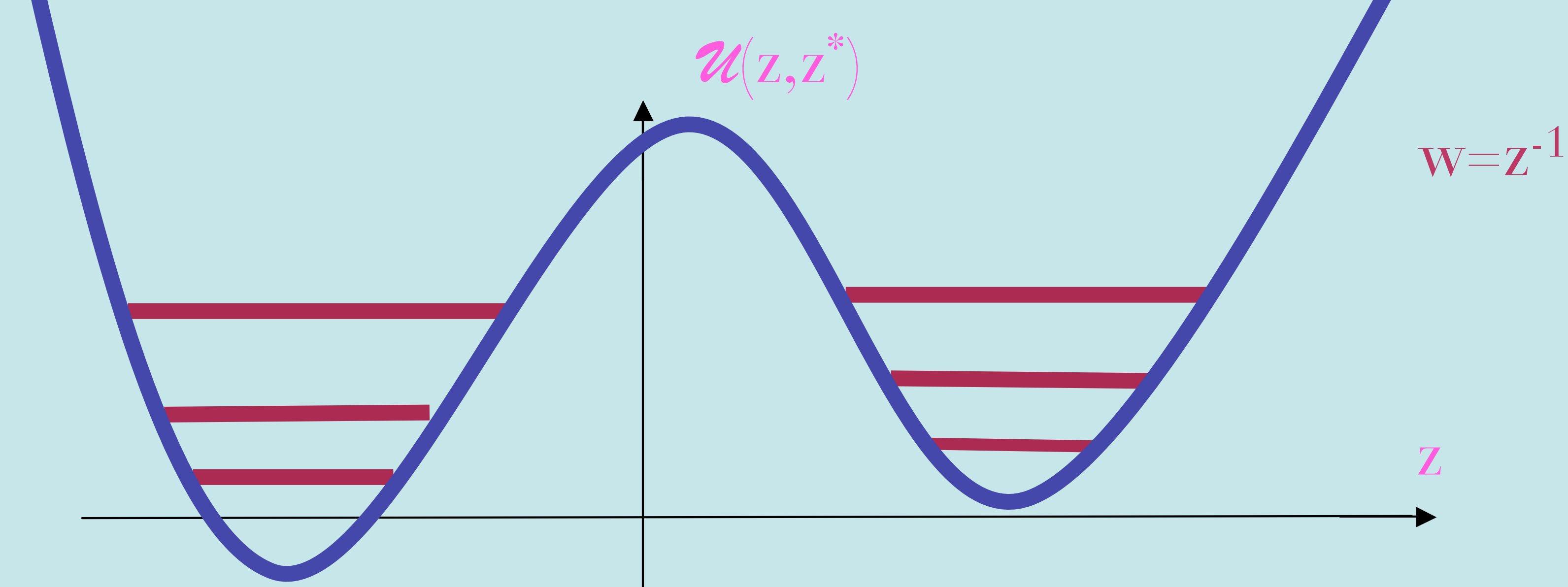
The space of out-states



$\mathcal{H}^{\text{out}}{}_{0}$ =
d-forms supported at 0

$\mathcal{H}^{\text{out}}{}_{\infty}$ =
Polynomial forms on
$C = CP^1 \setminus \{0\}$

The spectrum of Hamiltonian



On $\mathscr{H}^{\text{in}}{}_{0}$

n_1+n_2 on $z^{n_1} z^{*n_2}$

n_1+n_2+1 on $z^{n_1} z^{*n_2} dz$

n_1+n_2+1 on $z^{n_1} z^{*n_2} dz^*$

n_1+n_2+2 on $z^{n_1} z^{*n_2} dz dz^*$

On $\mathscr{H}^{\text{in}}{}_{\infty} =$

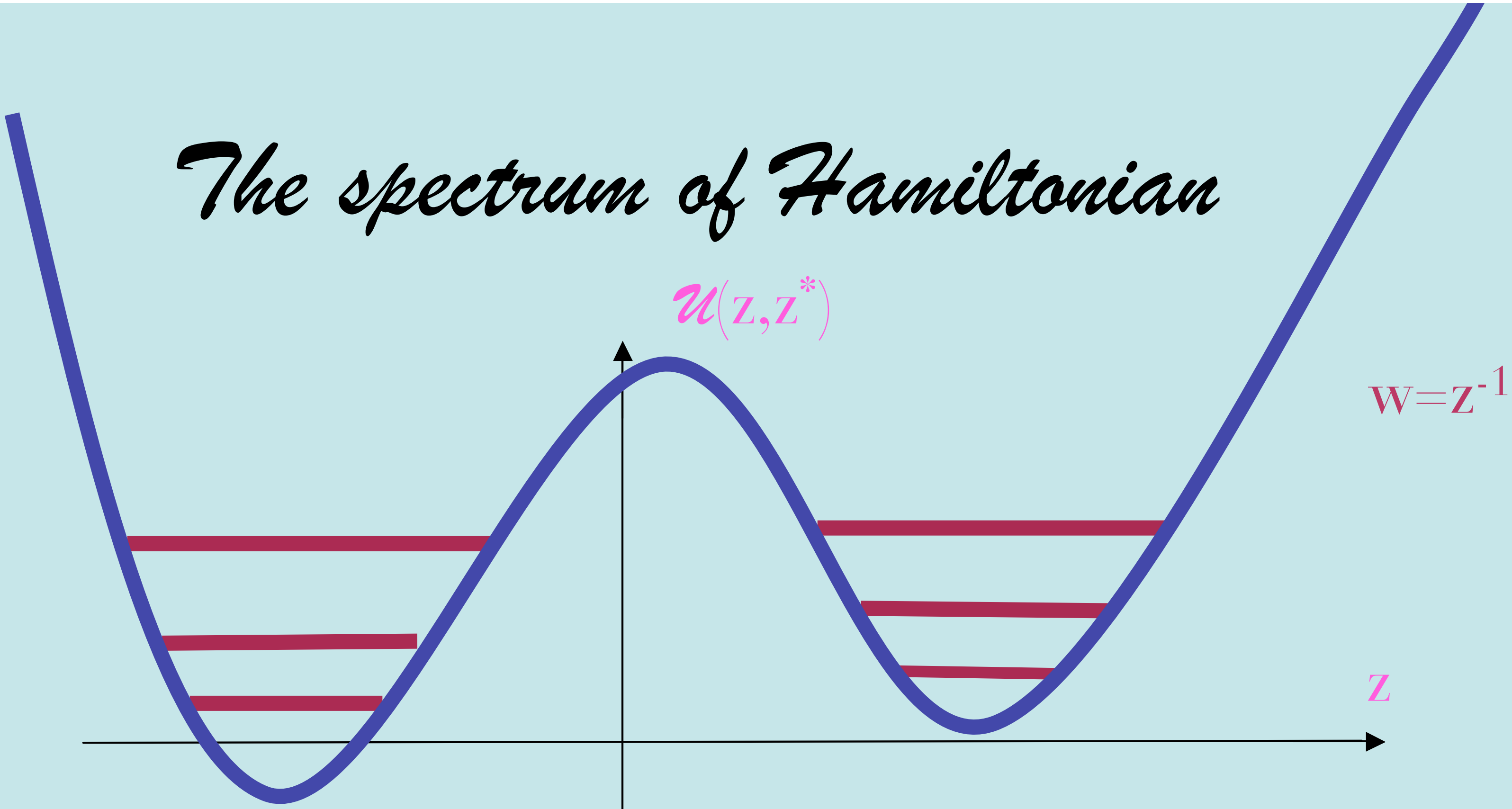
n_1+n_2 on $\partial_w{}^{n_1} \partial_{w^*}{}^{n_2}\, \mathbf{d}(w,w^*) dw dw^*$

n_1+n_2+1 on $\partial_w{}^{n_1} \partial_{w^*}{}^{n_2}\, \mathbf{d}(w,w^*) dw$

n_1+n_2+1 on $\partial_w{}^{n_1} \partial_{w^*}{}^{n_2}\, \mathbf{d}(w,w^*) dw^*$

n_1+n_2+2 on $\partial_w{}^{n_1} \partial_{w^*}{}^{n_2}\, \mathbf{d}(w,w^*)$

The spectrum of Hamiltonian



On $\mathscr{H}^{\mathrm{out}}{}_{\infty}$

n_1+n_2 on $w^{n_1}w^{*n_2}$

n_1+n_2+1 on $w^{n_1}w^{*n_2}dw$

n_1+n_2+1 on $w^{n_1}w^{*n_2}dw^*$

n_1+n_2+2 on $w^{n_1}w^{*n_2}dwdw^*$

On $\mathscr{H}^{\mathrm{out}}{}_{0}$ =

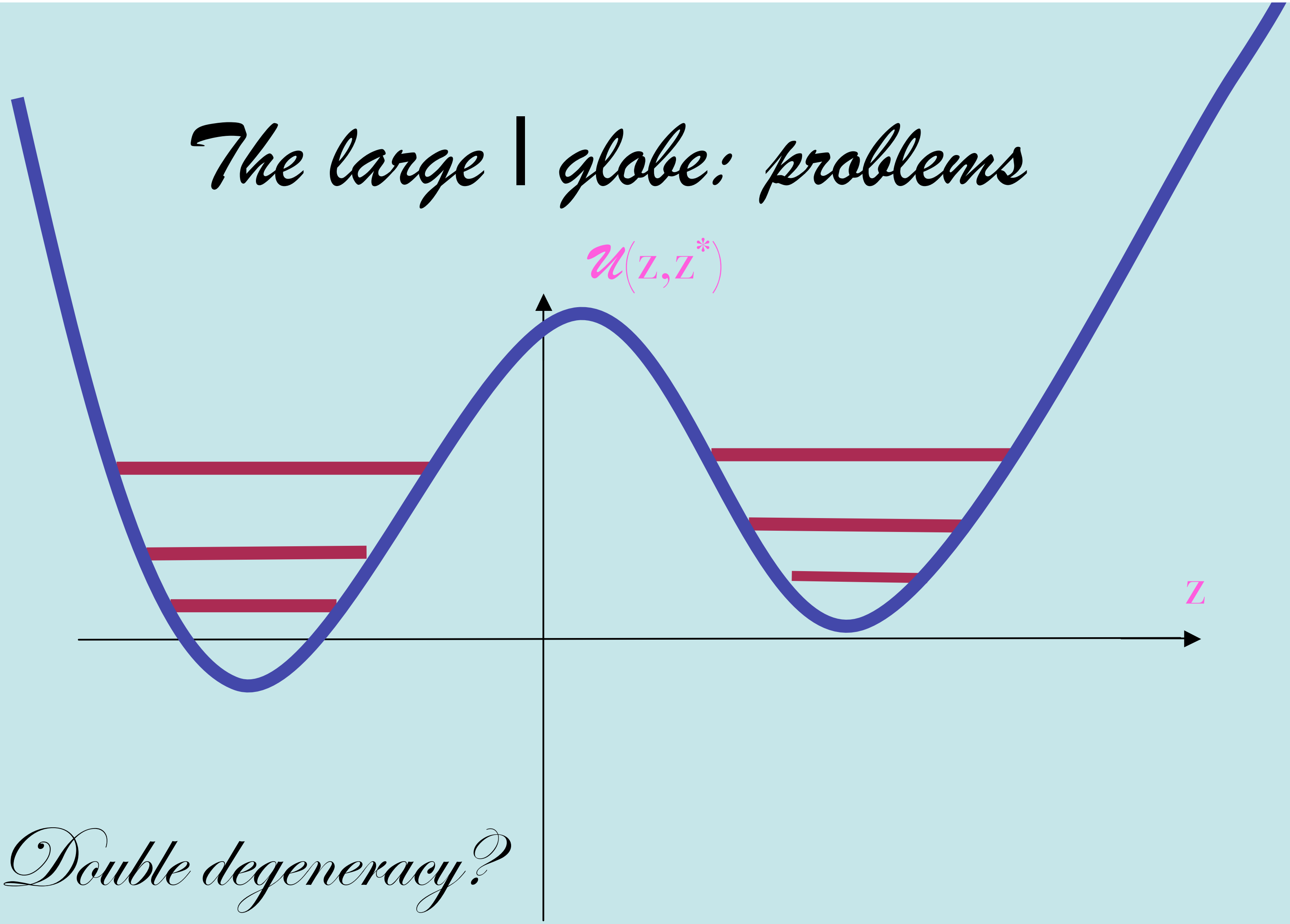
n_1+n_2 on $\partial_z^{n_1}\partial_{z*}^{n_2}\,\mathbf{d}(z,z^*)dzdz^*$

n_1+n_2+1 on $\partial_z^{n_1}\partial_{z*}^{n_2}\,\mathbf{d}(z,z^*)dz$

n_1+n_2+1 on $\partial_z^{n_1}\partial_{z*}^{n_2}\,\mathbf{d}(z,z^*)dz^*$

n_1+n_2+2 on $\partial_z^{n_1}\partial_{z*}^{n_2}\,\mathbf{d}(z,z^*)$

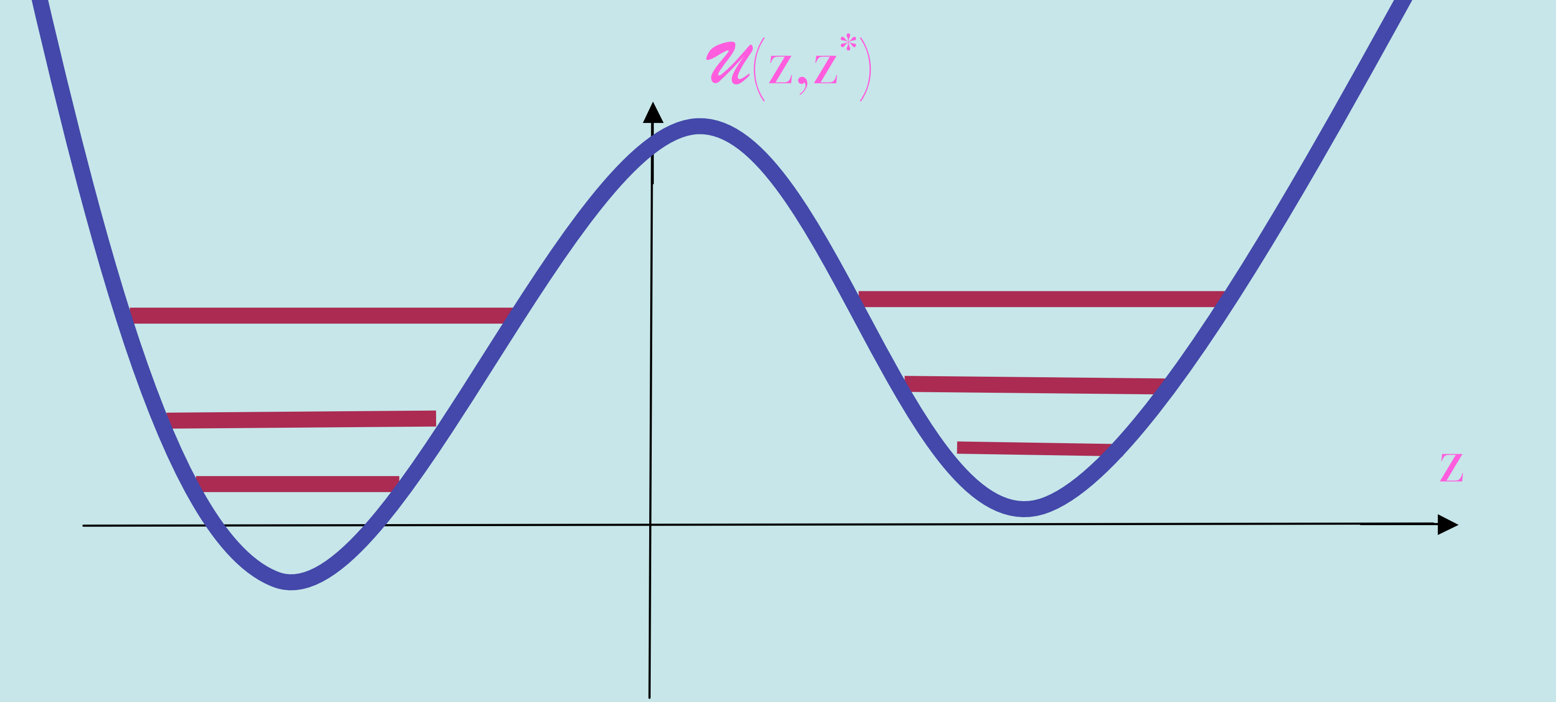
The large l globe: problems



1) Double degeneracy?

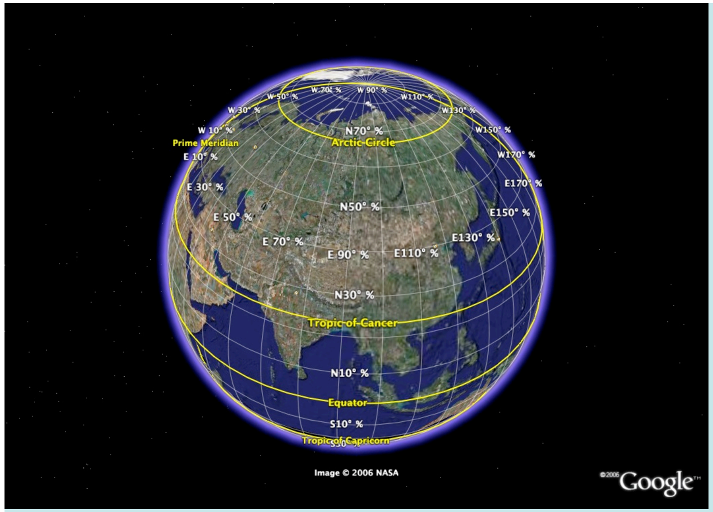
2) Polynomials are not good functions on the sphere.

The large l globe: problems



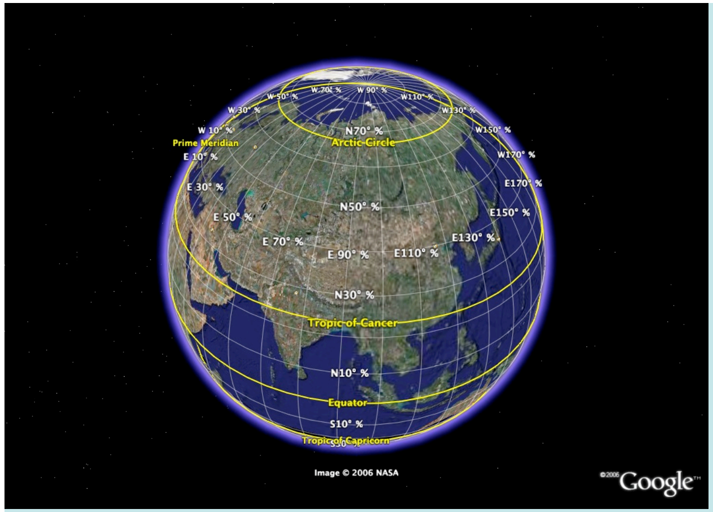
1) *Instantons should lift the degeneracy*

2) *Delta-functions are not functions either*



How to solve the problems

1) Read the spectrum off the correlation functions
2) Treat polynomials as generalized functions



Spectrum from correlators

Let F be a function and ω a two-form, then

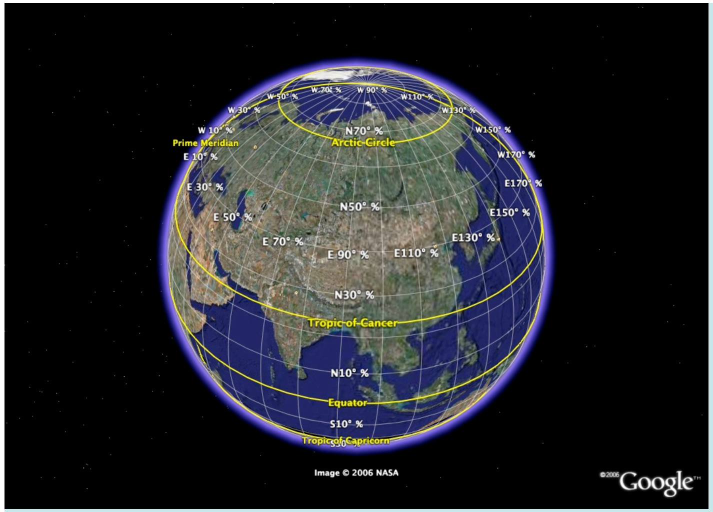
$$\langle_\infty\, \omega(t)\, F(0)\, \rangle_0 \stackrel{?}{=} \sum_E q^E c_E$$

$$q = \exp(-t)$$

already contains a lot of information about the spectrum

$$F = \frac{1}{1+|z|^2}, \qquad \omega = \frac{1}{(1+|z|)^2} d^2 z.$$

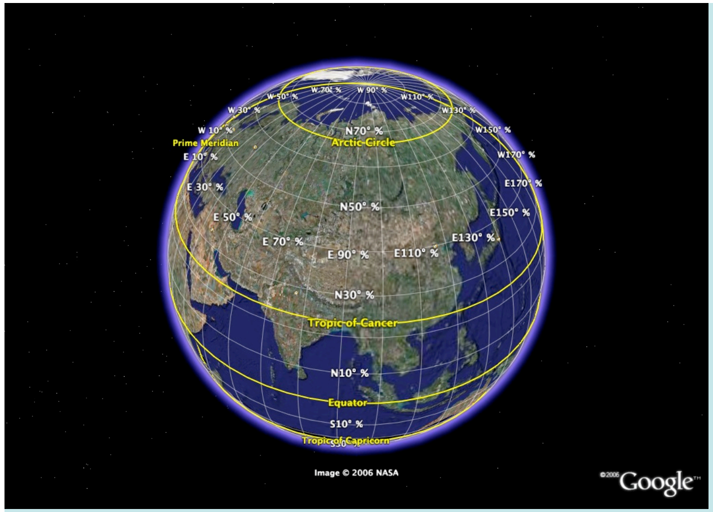
$$\int_{\mathcal{M}_{\infty,0}} \omega F(qz) = -\frac{1}{1-q^2} - \frac{2q^2}{(1-q)^2} \log q.$$



What? Logarithms?

$$<_{\infty} w\,(t)\;F\,(0)\;>_{0}\; = \;S_{E}\; q^{E}\; c_{E} + S_{E}\; q^{E}\; l_{E}\; \log(q)$$

implies the Hamiltonian is not diagonalizable, rather has the Jordan block form



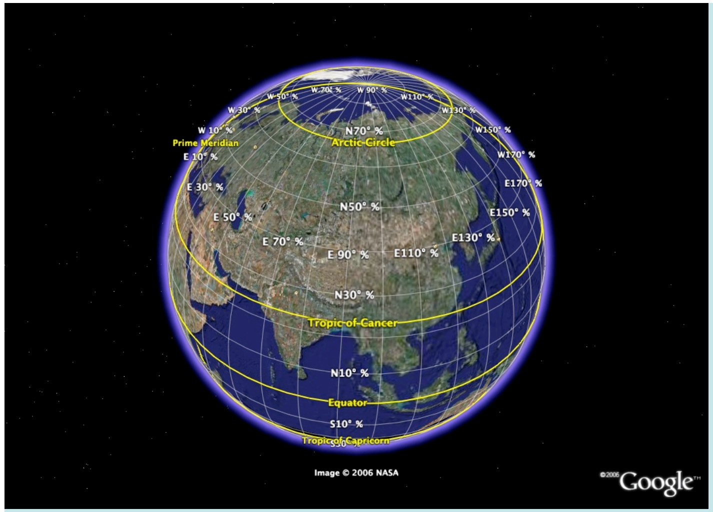
What? Logarithms?

Closer inspection shows the Hamiltonian acts as follows:

$$H\left(z^{n+1}(z^*)^{m+1}\right) = (n+m+2)\left(z^{n+1}(z^*)^{m+1}\right) + \partial_w{}^n \partial_{w^*}{}^m \, \mathbf{d}(w,w^*)$$

$$H\left(\partial_w{}^n \partial_{w^*}{}^m \, \mathbf{d}(w,w^*)\right) = (n+m+2)\left(\partial_w{}^n \partial_{w^*}{}^m \, \mathbf{d}(w,w^*)\right)$$

This reproduces all the correlation functions of (analytic) observables. This strange-looking action of dilatations on polynomials is actually quite natural.



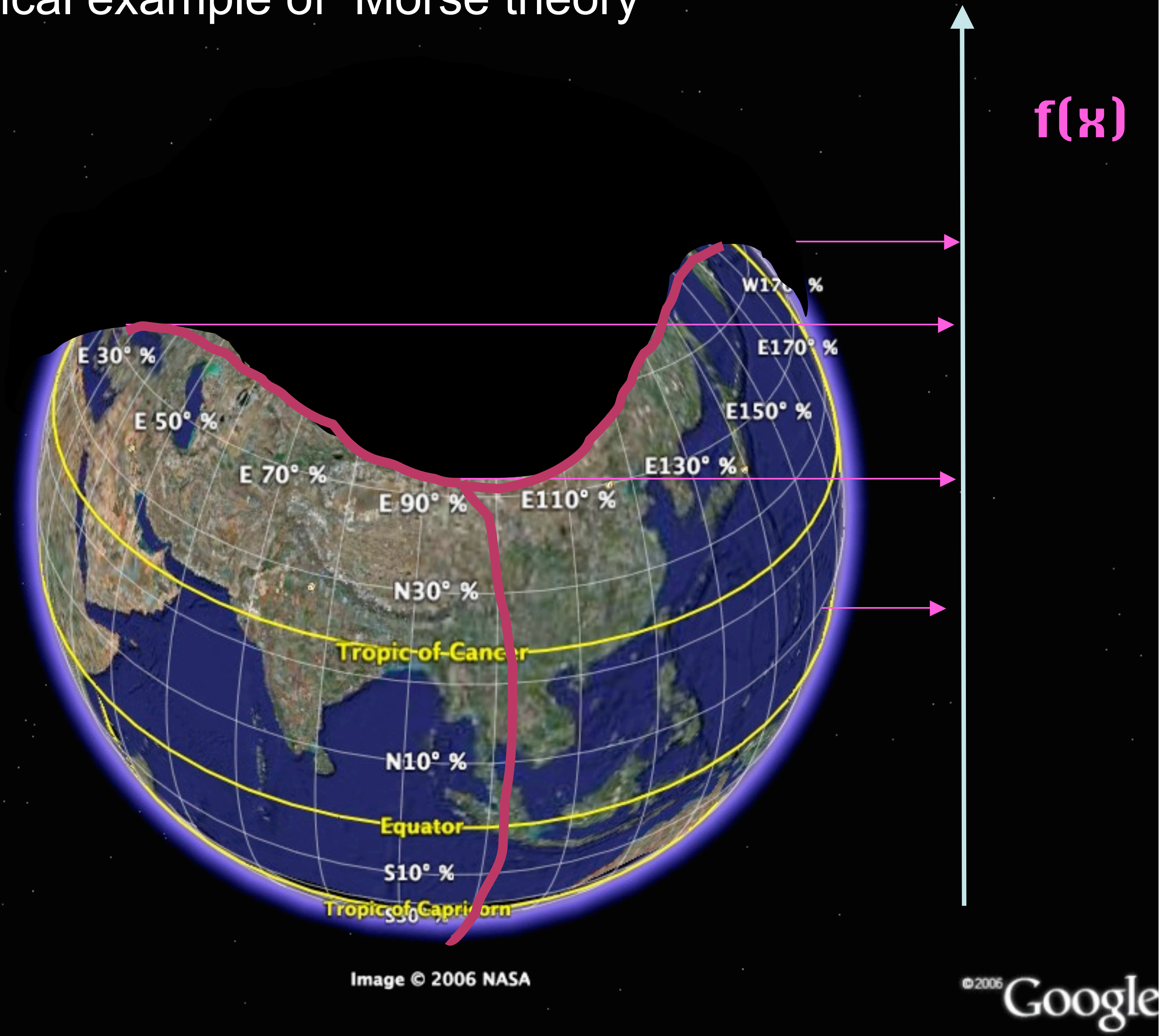
Hadamard, Epstein-Glazer, Hermander....

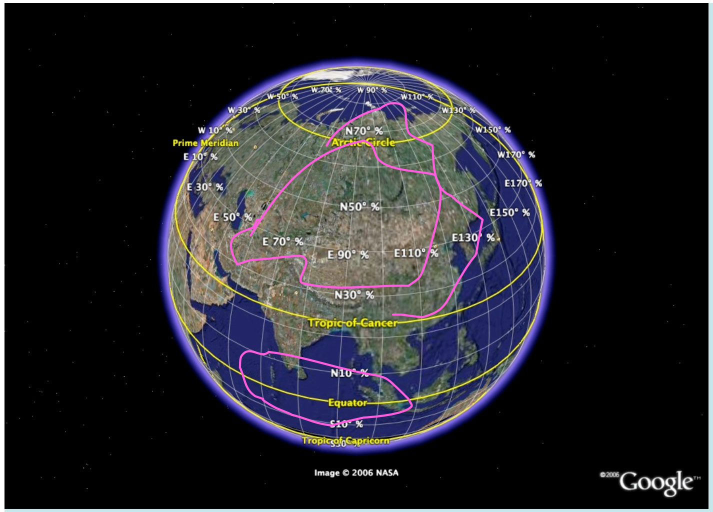
Generalized function:

$$\left(z^{n+1}\left(z^{*}\right)^{m+1}\right)(\mathbf{Y}) = \left[\int_{|z|<\mathbf{e}^{-1}} \left(z^{n+1}\left(z^{*}\right)^{m+1}\mathbf{Y}\right)\right]_{\mathbf{e}^{0}}$$

Dilatation changes the cutoff -- picks up the derivative of the function at infinity from the *log*e divergent terms, iff $n,m \geq 0$ $\Longrightarrow$ the importance of being excited!!!

More typical example of Morse theory



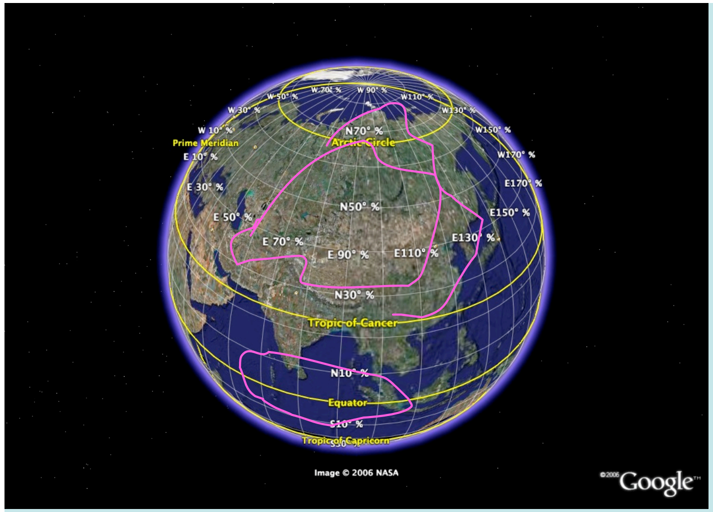


Infinite-dimensional versions: two and four dimensions

Sigma model = quantum mechanics on the loop space

Gauge theory = quantum mechanics on the space of three dimensional gauge fields

Novelty: the target spaces are not simply-connected, the functions **f** are multi-valued



Infinite-dimensional versions: two and four dimensions

Sigma model : $f = \int d^{-1}w$

Gauge theory : $f = CS(A)$

Novelty: the target spaces are not simply-connected, the functions f are multi-valued

For simplicity we consider twisted theories

- *Twisted (2,2) sigma model in two dimensions*
    - *Twisted $\mathcal{N}$=2 theory in four dimensions*

Study correlators of evaluation observables

- *Twisted (2,2) sigma model in two dimensions*

$y: S \longrightarrow X$

- *However (for* $X = \mathbb{CP}^1$*) already the simplest correlator*

$$< d^2(y(\infty))\, w(y(1))\, F(y(q))\, d^2(y(0)) >_{inst=1}$$

contains the same logarithms as the quantum mechanical model

- *Naively becomes* $c=0$ CFT *in our limit: curved* bcbg *system*

infinite radius, finite complex Kahler class

Chiral de Rham

Study correlators of evaluation observables

$$< d^2\,(y(\infty))\; w\,(y(1))\; F\,(y(q))\; d^2\,(y(0)) >_{\text{inst}=1}$$

$$= : -\frac{1}{1-q^2} - \frac{2q^2}{(1-q)^2}\log q.$$

Hence we are dealing with the

LOGARITHMIC CFT

Study correlators of evaluation observables

$$< d^2\,(y(\infty))\;w\,(y(1))\;F\,(y(q))\;d^2\,(y(0))>_{\text{inst}=1}$$

$$= : -\frac{1}{1-q^2} - \frac{2q^2}{(1-q)^2}\log q.$$

Hence we are dealing with the

LOGARITHMIC CFT

Introduced by V.Gurarie (1993)

Late Ian Kogan worked a lot on them

Study correlators of evaluation observables

The closest recent appearance of LCFT
in the paper of H.Saluer and V.Schomerus
on GL(1|1) WZW

LOGARITHMIC CFT

Four dimensional gauge theory (briefly)

$$C(x,y;z) = \langle \mathcal{O}(x)\mathcal{O}(y)\mathcal{S}(z)\rangle \ ,$$

$$\mathcal{O}(x) \ = \ \mathrm{tr}\phi^2(x) \quad , \quad \mathcal{S}(x) \ = \ \mathrm{tr}F_{mn}F^{mn}(x)$$

f is the adjoint Higgs field of N=2 theory

Four dimensional gauge theory (briefly)

$$C(x,y;z) = \frac{1}{|x-y|^4}\,\mathcal{C}\left(\frac{(\bar{x}-\bar{y})\cdot(x+y-2z)}{|x-y|^2}\right)$$

Four dimensional gauge theory (briefly)

$$\mathcal{C}(q) \propto$$

$$\frac{1}{|1-q|^6}\int_0^1 \frac{du}{M^8}\left(MP_4(M) - 3(1+M)^2(7+6M+M^2)\log(1+M)\right)$$

$$M = \frac{|(1-q)+u(1+q)|^2}{u(1-u)|1-q|^2}$$

$$P_4(M) = \frac{1}{5}M^4 + \frac{35}{4}M^3 + 37M^2 + \frac{99}{2}M + 21$$

So we get (poly) logarithms and hence LCFT_4

Remarks

1) It is important to study the correlators of non-closed forms, in the BPS sector only one does not see the log-structure

2) The correlators contain, in general, polylogs of degree bounded by the dimensions of the cells in the Morse decomposition of the manifold

3) The instanton corrections to the Hamiltonian are related to the action of the so-called Cousin-Grothendieck operator

4) In the $t^{*}=\infty$ limit the ground states are not necessarily annihilated by $Q=d$, only by $L_v = \{ d, i_v\}$

Remarks

5) Going beyond the ground states (beyond cohomology of the Q operator) can produce new invariants of four-manifolds (cf. Sullivan's deconstruction of the rational homotopy type from the minimal model of de Rham complex)
6) The "unusual realization" of the susy algebra $L_v = \{ d, i_v \}$ instead of more familiar $D = \{ d, d^* \}$ comes in handy in the tropical KS theory and Z-theory more generally

Remarks

7) In the quantum mechanical model on Kahler manifolds with U(1) isometry
(loop spaces of interest fall into this category)
the space of states (**in** or **out**) comes out
in the form (**f** runs over the fixed points)

$$\mathcal{H} = \bigoplus_{\mathbf{f}} \mathcal{H}^{hol}{}_{\mathbf{f}} \otimes \mathcal{H}^{anti}{}_{\mathbf{f}}$$

HOLOMORPHIC FACTORIZATION
POSSIBILITY FOR A CHIRAL THEORY

Possibly related to E.Witten's approach to holomorphic Morse inequalities

Summary and remarks

In the $t^{*}=\infty$ limit we get the theory with instantons but otherwise free (only one-loop is nontrivial)

The "free" theory spectrum is modified by instantons to make it that of LCFT

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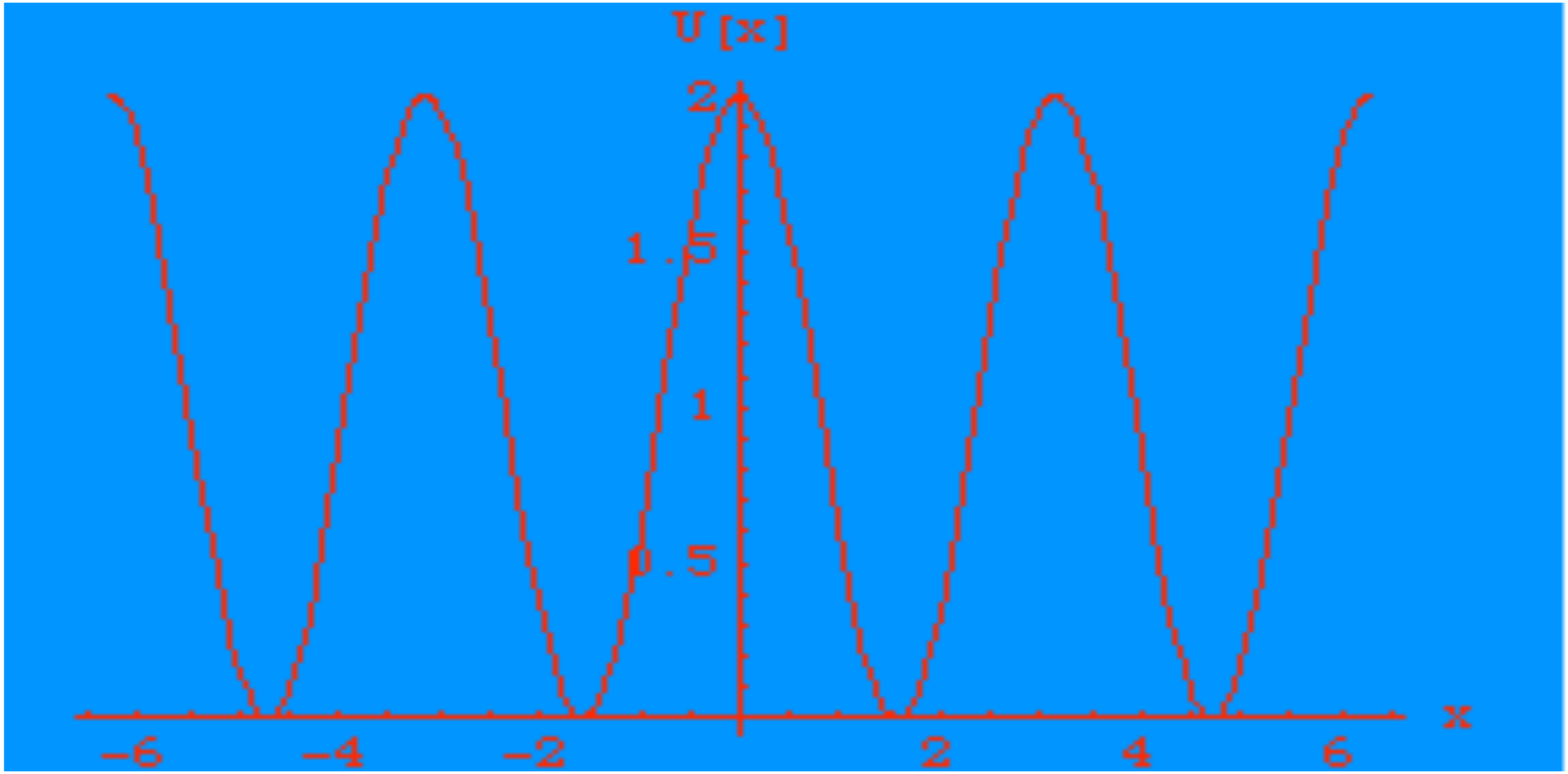
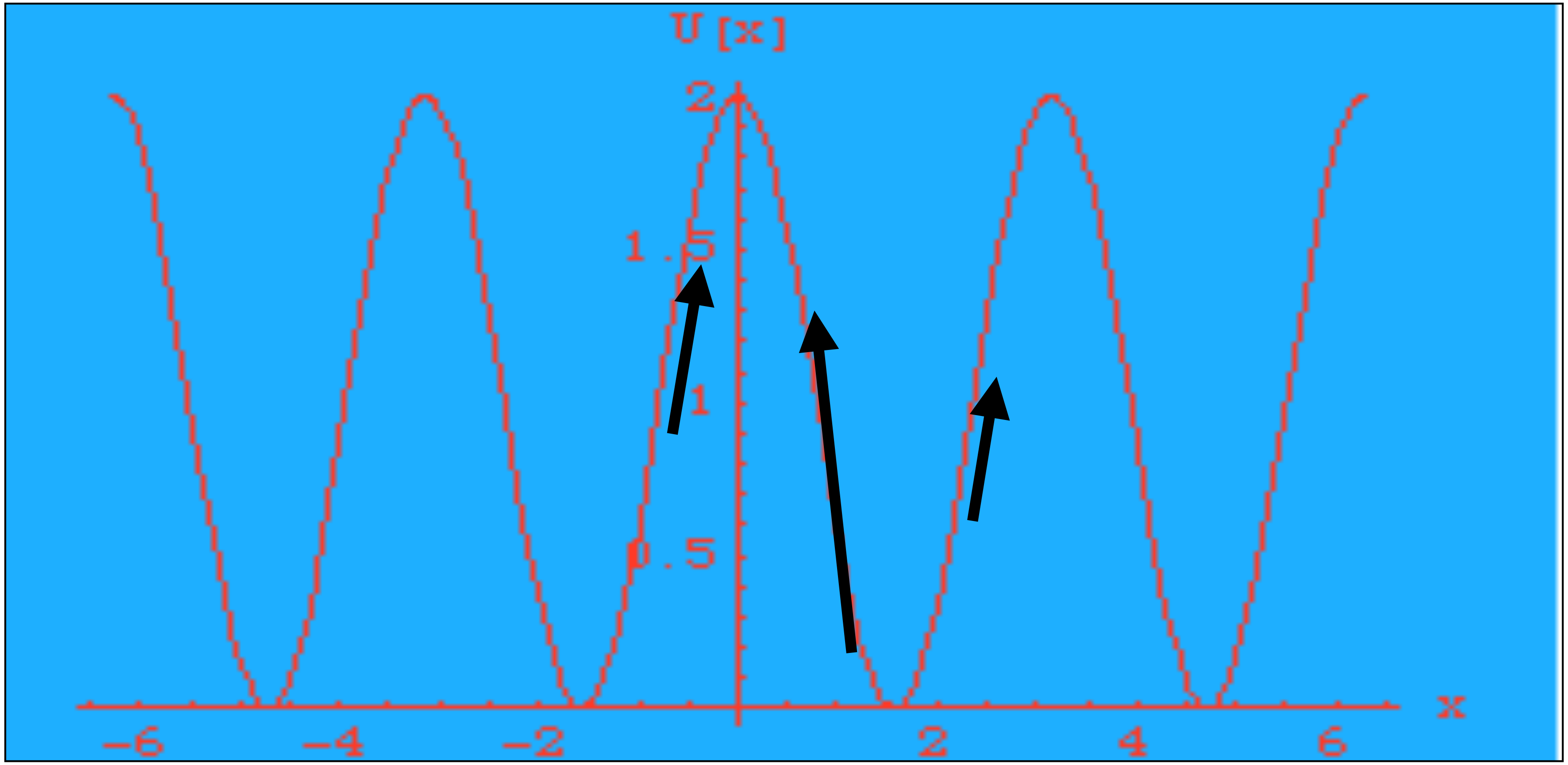
THANK YOU

Typical periodic potential from S.Coleman's lectures



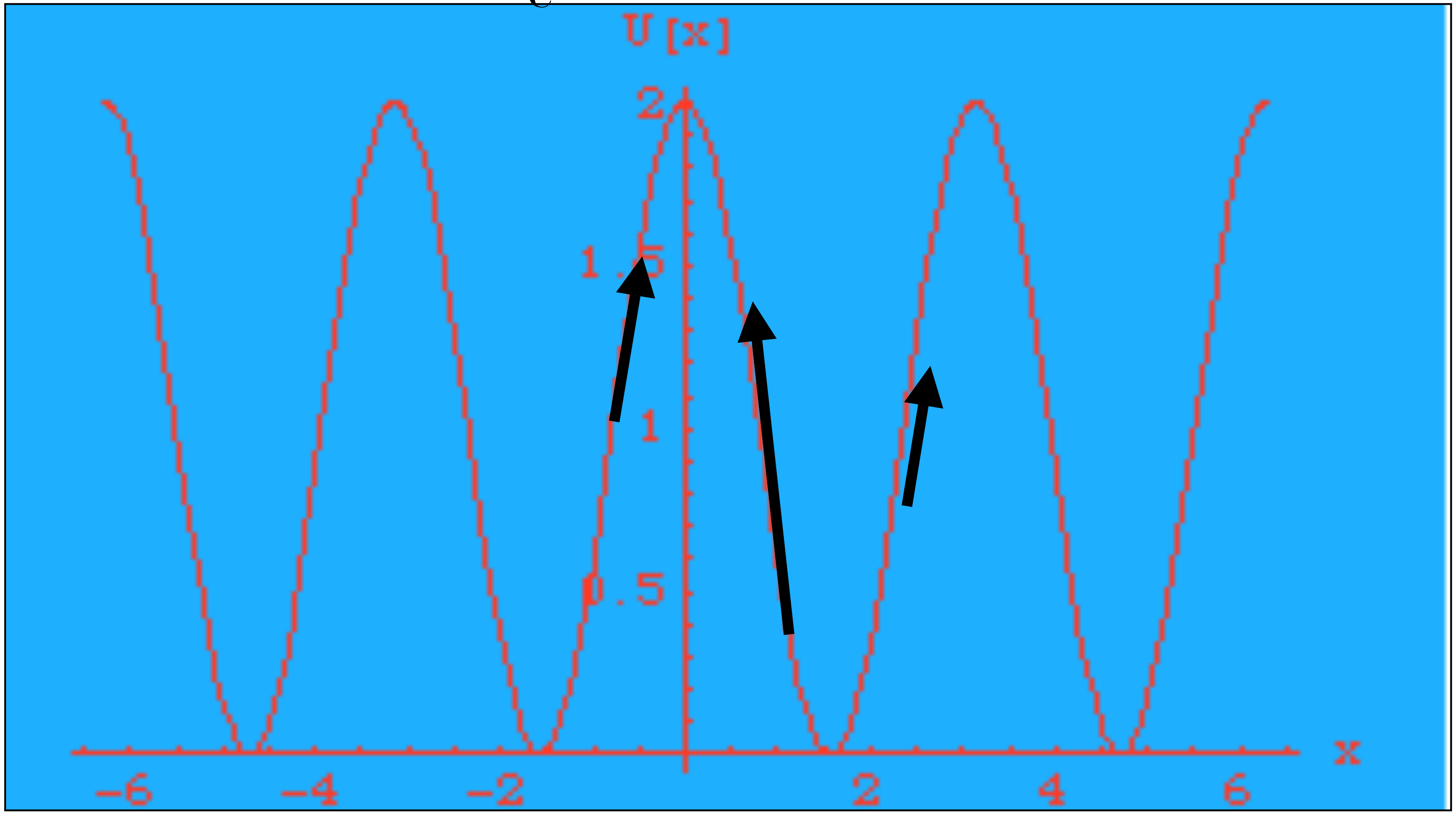
S. Coleman derives:

$$E = E_0 + Ke^{-S_0}\cos\vartheta$$



Shouldn't we get the complex spectrum in our limit?

$E = E_0 + K e^{it}$



The superpotential is not Morse!