GAUGE THEORY AND GEOMETRIC LANGLANDS

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The *Langlands program* is an attempt to unify many old and new results in number theory

– ranging from quadratic reciprocity, proved around 1800, to the modern proof of Fermat's Last Theorem.

Number theory is difficult because calculus is powerful ….

And so mathematicians have sought to find geometric analogs of problems in number theory.

Number field --- Riemann surface C

Prime number --- point on C

The Langlands program, too, has a geometric analog, which has been intensively developed.

Even in its geometric form, the Langlands program involves statements that at first sight are likely to look unrecognizable to physicists.

But, if one probes a little more deeply, the Langlands program has many obvious and less obvious analogies with quantum field theory.

For one thing, Langlands introduced (ca. 1970) a “duality” between a simple Lie group G and a “dual” group often called ^{L}G. However, this relationship, which pairs SU(N) with $SU(N)/Z_N$, E_8 with itself, SO(2n+1) with Sp(n), etc., also plays a very distinguished role in four-dimensional gauge theory…

In fact, the Langlands dual group ${}^{L}G$ is precisely the magnetic group introduced in 1976 by Goddard, Nuyts, and Olive to classify magnetic monopoles:

Their idea was that if electric charges are classified by representations of a group G, then the corresponding magnetic charges are classified by a representation of the dual group, which in fact coincides with the dual ${}^{L}G$ of Langlands.

The GNO work was of course the jumping off point for Montonen-Olive duality, an extremely fruitful idea in which the two groups G and $^L G$ enter in a completely symmetric fashion.

By contrast, in the Langlands program the roles of G and $^L G$ are at first sight bafflingly unlike. An object of one type associated with $^L G$ is classified by a completely different type of object associated with G.

That is a bit disconcerting at first sight, but on the positive side, the objects of study on the two sides are both objects that are familiar in QFT – or at least they are once we make the translation from number theory to geometry.

On the left hand side of the Langlands correspondence, we have a flat connection, on a Riemann surface C, with gauge group ${}^{L}G$. In gauge theory, flat connections are those with least energy, most supersymmetry, etc.

The right hand side of the Langlands correspondence involves an "automorphic representation" of G, a notion which when suitably translated to geometry is very closely related to the "conformal blocks" of current algebra, with symmetry group G, on a Riemann surface C.

Moreover, in recent years mathematicians (Beilinson, Drinfeld, Frenkel, Gaitsgory…) working on geometric Langlands have relied heavily on two-dimensional current algebra – albeit with a focus on aspects (such as the exceptional behavior at level $k=-h$) that appear extremely exotic from a physical point of view.

I will be reporting on a recent paper

Electric-Magnetic Duality And The Geometric Langlands Program.
Anton Kapustin (Caltech) , Edward Witten (Princeton, Inst. Advanced Study) . Apr 2006. 225pp.
e-Print Archive: **hep-th/0604151**

in which we tried to understand exactly what the geometric version of the Langlands program means from the point of view of quantum field theory.

To simplify a bit, this paper is based on six ideas:

(1) A certain twisting of N=4 super Yang-Mills theory gives a family of TQFT's parameterized by CP^1. S-duality acts on this CP^1

(2) If this theory is compactified to two dimensions on a Riemann surface C, one gets a sigma model whose target is "Hitchin's moduli space"; S-duality turns into, roughly speaking, mirror symmetry of the sigma model.

(3) Wilson and ‘t Hooft operators of the four-dimensional gauge theory act on branes of the sigma model. A brane that is mapped to a multiple of itself by the Wilson or ‘t Hooft operators is called an electric or magnetic eigenbrane, so S-duality automatically maps electric eigenbranes to magnetic eigenbranes.

(4) Electric eigenbranes correspond to representations of the fundamental group in ${}^LG$. This is the left hand side of the geometric Langlands correspondence.

(5) ‘t Hooft operators of gauge theory correspond to the Hecke operators of the geometric Langlands program.

(6) Magnetic eigenbranes can be given another interpretation because of the existence of a certain “coisotropic A-brane” on Hitchin’s moduli space. They are associated with the “D-modules” that appear on the right hand side of the geometric Langlands program.

(1) A certain twisting of N=4 super Yang-Mills theory gives a family of TQFT's parameterized by CP^1. S-duality acts on this CP^1

In general, twisting is achieved, starting with the theory on flat R^4, by defining a new Lorentz group – a combination of the standard Lorentz group and a subgroup of the group of R-symmetries, which for N=4 SYM is SU(4), such that one of the supercharges, which I will call Q, becomes Lorentz-invariant

Lorentz-invariance typically implies that $Q^2=0$, and if one works relative to the cohomology of Q, one typically gets a topological quantum field theory.

The key step is to show that – by virtue of the underlying supersymmetry algebra – the stress tensor T is trivial as an element of the cohomology of Q.

There are three ways to pick a suitable subgroup of $SU(4)_R$ and thereby "twist" N=4 super-Yang-Mills. Two of the twists give theories qualitatively similar to Donaldson theory of four-manifolds (and one was studied by Vafa and EW in 1994).

The third twist, though studied by e.g. Marcus 1995, has had no known applications until now. It is the one relevant to the geometric Langlands program.

When one makes this third twist, one finds that there is not one Lorentz-invariant supercharge Q, but *two* of them, say Q_1 and Q_2. So we can define Q to be an arbitrary complex linear combination of these two

$$Q = u\, Q_1 + v\, Q_2$$

with coefficients u,v that we require to be not both zero.

So working relative to the cohomology of

$$Q = u\, Q_1 + v\, Q_2$$

we get a family of topological field theories, parameterized by a copy of CP^1 with homogeneous coordinates u,v.

There are essentially no trivial equivalences among these theories, but there are non-trivial Montonen-Olive dualities that lead to the geometric Langlands program.

There actually is one further trick: the parameter $t = v/u$ combines with the gauge coupling parameter $\tau = \theta/2\pi + 4\pi i/e^2$ to make a parameter ψ which is the really important one.

Montonen-Olive duality acts on ψ by

$$\Psi \to (a\Psi+b)/(c\Psi+d)$$

and the basic statement of geometric Langlands relates $\Psi=0$ to $\Psi=\infty$

(2) *If this theory is compactified to two dimensions on a Riemann surface C, one gets a sigma model whose target is "Hitchin's moduli space"; S-duality turns into, roughly speaking, mirror symmetry of the sigma model.*

This step largely follows BJSV and HMS (1995). We take the four-manifold M_4 to be a product of Riemann surfaces Σ and C. In the limit that Σ is very large compared to C, we get an effective two dimensional σ-model on Σ.

The target space of the σ-model is Hitchin's moduli space of stable Higgs bundles, which I will call M_H.

M_H is, roughly speaking, the cotangent bundle of the moduli space of flat bundles on C. It is a hyper-Kahler manifold, and the sigma model with target M_H has (4,4) supersymmetry.

Our four-dimensional topological field theory reduces in two dimensions essentially to a familiar type of A- or B-model of the sigma model (or a hybrid made possible by the fact that in this case the target is hyper-Kahler).

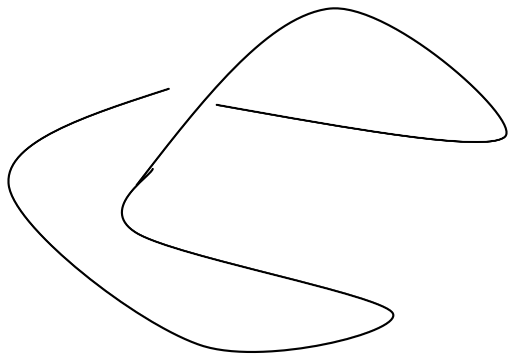
In one of the complex structures, the S-duality reduces essentially to a mirror symmetry.

(3) *Wilson and ‘t Hooft operators of the four-dimensional gauge theory act on branes of the sigma model. A brane that is mapped to a multiple of itself by the Wilson or ‘t Hooft operators is called an electric or magnetic eigenbrane, so S-duality automatically maps electric eigenbranes to magnetic eigenbranes.*

In the twisted topological field theory related to Donaldson theory, the important operators are local operators and their “descendants,” but in the present problem…

The important operators are not local operators, but rather Wilson and ‘t Hooft “line operators”

S

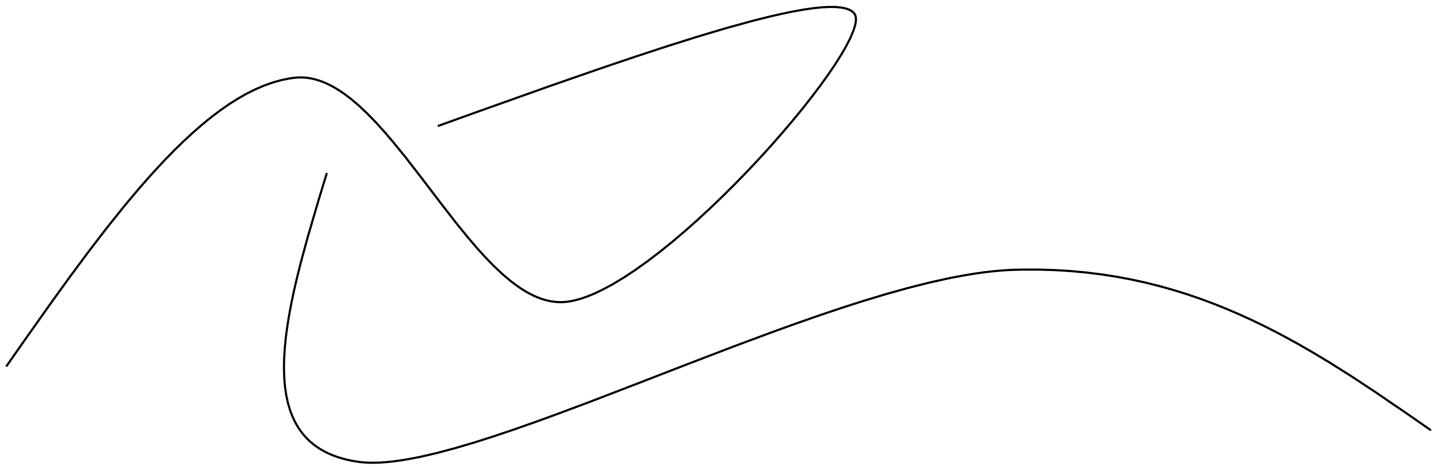


associated with an open or closed loop S in spacetime.

To perform a gauge theory calculation with a Wilson loop, one simply performs the path integral with an insertion of an extra factor, which is essentially the trace of the holonomy around the loop S, taken in a representation R of the gauge group G.

In particular, a Wilson loop is labeled by a representation R of G.

By contrast, the ‘t Hooft operator is a “disorder operator.” It is defined not by giving a classical expression (such as the holonomy) that should be quantized, but by giving a recipe, by describing a singularity that the fields should have along (in this case) a curve in spacetime



We require that along the specified curve, S, the fields should have a singularity which is most easily described by saying that in the directions transverse to S, one has a Dirac monopole singularity:

$$F = \star d \left(\frac{1}{|\vec{x}|} \right)$$

To be more exact, what I've written is the basic Dirac monopole for U(1)…

We need to supersymmetrize and to embed it from U(1) into the gauge group G. The basic observation of Goddard,Nuyts, and Olive (or Langlands) is in effect that the embeddings of U(1) in G, up to conjugacy, are classified by a choice of representation $^L R$ of the dual group $^L G$. So ‘t Hooft operators are classified by a choice of such a representation $^L R$.

Electric-magnetic duality maps ^{L}G to G while also mapping Wilson operators to 't Hooft operators.

So a Wilson operator of ^{L}G, classified by a representation ^{L}R, is mapped to an 't Hooft operator of G, also classified by the representation ^{L}R.

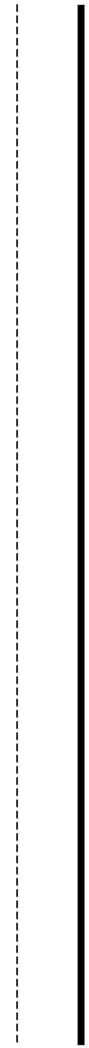
(4) Electric eigenbranes correspond to representations of the fundamental group in ${}^L G$. This is the left hand side of the geometric Langlands correspondence.

Now to get to the Langlands program, we need to consider *branes* in the gauge theory – or in the effective two-dimensional sigma model.

The branes are defined in the four-dimensional gauge theory by local boundary conditions at the boundary of the four-manifold M.

In two dimensions, they reduce to the ordinary branes of the sigma model, including the usual A-branes and B-branes of the appropriate topological field theories.

Now the key fact is that line operators behave as operators acting on branes… A brane (the solid line) with a line operator (the dashed line) gives us a new "composite" brane.



This gives an action of line operators on "theories," more abstract than the usual action of operators on states.

If L is a line operator and B is a brane, we say that B is an “eigenbrane” of L if L acting on B gives us back several copies of B.

In formulas

$$LB = B \otimes V$$

where V is a vector space.

An eigenbrane of the Wilson operators is called an electric eigenbrane, and an eigenbrane of the ‘t Hooft operators is called a magnetic eigenbrane.

Clearly, S-duality will map electric eigenbranes to magnetic eigenbranes.

The electric eigenbranes turn out to be simply zerobranes … (which are B-branes in each complex structure) … Since the target space of the sigma model is the Hitchin moduli space M_H, a zerobrane, being a point in M_H, corresponds to a flat bundle on our Riemann surface C with gauge group ${}^L G_{\mathbf{C}}$, the complexification of the gauge group ${}^L G$

The complexification comes in because of the way M_H parameterizes pairs (A, ϕ)

Such a homomorphism of the fundamental group of C into ${}^L G$ appears on the left hand side of the usual geometric Langlands correspondence.

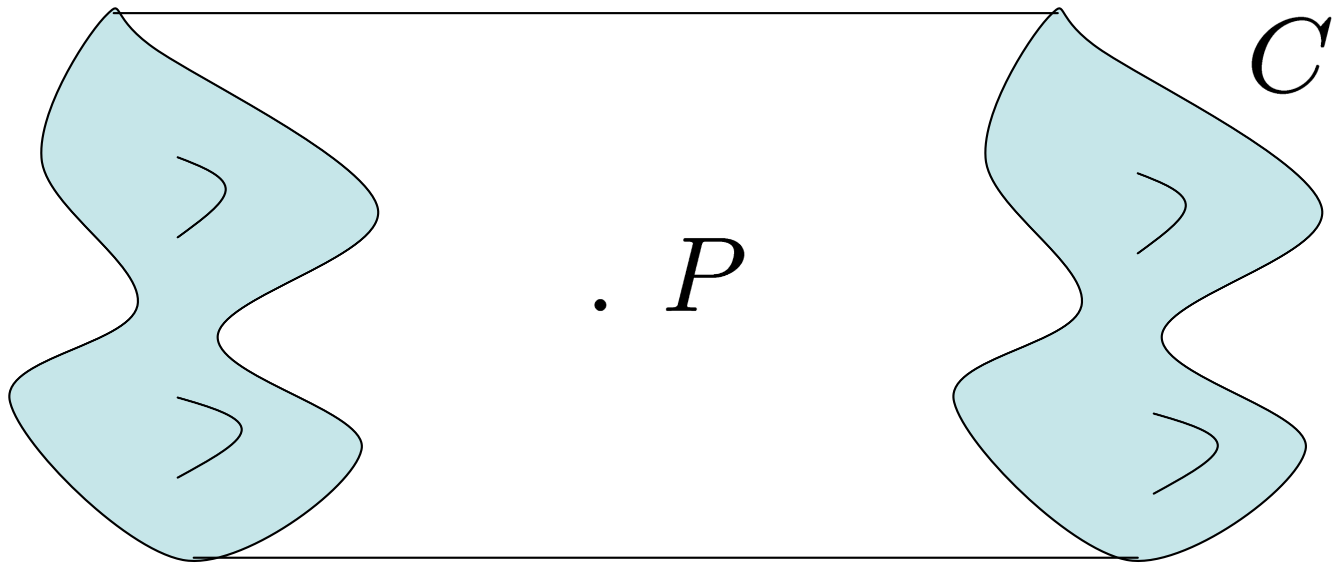
The right hand side involves whatever we are going to get by applying electric-magnetic duality to the zerobranes

Here, some more ideas are involved…

Once we know the electric eigenbranes, we can find what the magnetic eigenbranes are by applying duality (which acts via an SYZ torus fibration)…

They are A-branes of the usual sort (supported on a Lagrangian submanifold that is endowed with a flat connection)

But to understand what it means concretely for a brane to be a magnetic eigenbrane takes more work ….



This is a time zero slice that looks like I times C where I is an interval and C is the Riemann surface on which we compactified …
P is the point at which a time-independent ‘t Hooft operator is inserted….

Reading the picture from right to left, we start with a "brane" in which the gauge bundle on C is specified – this determines a particular holomorphic G-bundle E over C.

Proceeding to the left, nothing happens (to the holomorphic structure) until it "jumps" when one crosses the position of the 't Hooft operator.

Mathematically, this jump in the bundle is called a "Hecke transformation," so we arrive at our fifth main point:

(5) 't Hooft operators of gauge theory correspond to the Hecke operators of the geometric Langlands program.

Accordingly, the magnetic eigenbranes are called "Hecke eigensheaves" in the mathematical literature.

There is one more key step to arrive at the usual statement of the geometric Langlands duality:

(6) Magnetic eigenbranes can be given another interpretation because of the existence of a certain "coisotropic A-brane" on Hitchin's moduli space. They are associated with the "D-modules" that appear on the right hand side of the geometric Langlands program.

The key point here is that in general, in two-dimensional A-models, in addition to the usual A-branes (supported on a Lagrangian submanifold) there can be additional "coisotropic" A-branes, first described by Kapustin and Orlov, whose support is above the middle dimension.

They don't exist on Calabi-Yau threefolds, but they do exist on hyper-Kahler manifolds such as M_H

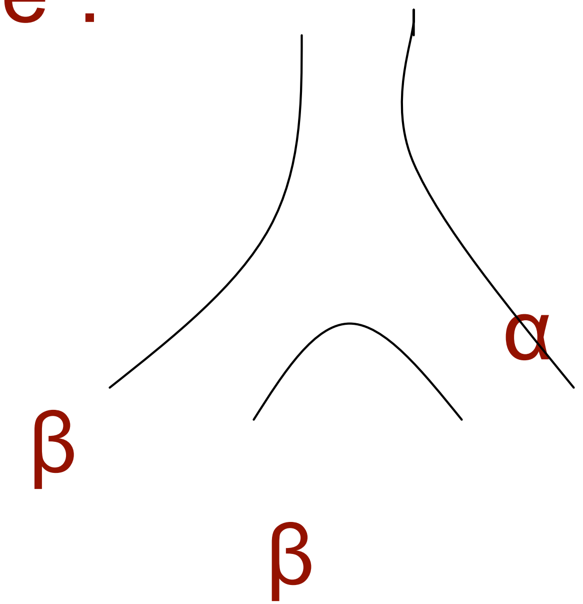
On M_H, there is a “canonical coisotropic brane,” which I will call β, with the property that the β – β strings are described by a (sheaf of) noncommutative algebras – the algebra of differential operators on M_H

The mechanism by which noncommutativity comes in is similar to the usual mechanism with the B-field

Now something funny happens: Just because the brane β exists, if we are given any other A-brane α, we can consider the

"sheaf" of β-α strings, and this gives us a

"module" for the β-β strings, i.e. a "D-module":



So every representation of the fundamental group, i.e. electric eigenbrane, gives us a D-module, as claimed in the usual statement of geometric Langlands duality.

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