Recent Applications of the Pure Spinor Formalism

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There are 4 sectors of (open) superstring theory:

	Neveu-Schwarz	Ramond
GSO(+)	gluon, ...	Maj-Weyl gluino, ...
GSO(−)	tachyon, ...	Maj-anti-Weyl gluino, ...

RNS formalism is convenient for describing NS GSO(+) and NS GSO(−) sectors.
Worldsheet supersymmetry is manifest: $\mathcal{X}^m = x^m + \kappa \psi^m$

Pure spinor formalism is convenient for describing NS GSO(+) and R GSO(+) sectors.
Spacetime supersymmetry is manifest: (x^m, θ^α) where $\delta\theta^\alpha = \epsilon^\alpha$, $\delta x^m = \epsilon\gamma^m\theta$

- Using pure spinor formalism, multiloop scattering amplitudes are easier to compute since there is no need to sum over spin structures.
- Supergravity backgrounds including R-R fields can be covariantly described using d=10 superspace.
- Unlike GS formalism, quantization is straightforward.

Outline

I. Basics of Pure Spinor Formalism

Siegel '86, Howe '91, Tonin '91

NB '00

Aisaka, Chandía, Chesterman, Grassi, Howe, Kazama, Kluson, Losev, Mafra, Mazzucato, Movshev, Mukhopadhyay, Nekrasov, Oda, Oz, Policastro, Schiappa, Stahn, Tonin, Tsimpis, Vallilo, Vanhove, van Nieuwenhuizen, Wyllard, ...

II. Higher Derivative R^4 Theorems

When $1 < g \leq 6$, low-energy contribution to 4-point g-loop superstring amplitude starts at $\partial^{2g} R^4$.

Suggests that $N=8$ $d=4$ supergravity is finite up to 8 loops, but not beyond.

NB '06, NB + Nekrasov '06, Green, Russo, Vanhove '06

III. Topological A-model for $AdS_5 \times S^5$?

Superparticle on $AdS_5 \times S^5$ can be described as $t \to \infty$ limit of topological A-model.

May be useful for generalizing proof of conifold duality to Maldacena conjecture.

Gopakumar + Vafa '99, Ooguri + Vafa '02, NB '07, NB + Vafa (work in progress)

I. Basics of Pure Spinor Formalism

Worldsheet variables : x^m $\quad \underbrace{(\theta^\alpha, p_\alpha)}_{\text{fermions}} \quad \underbrace{(\lambda^\alpha, w_\alpha)}_{\text{bosonic ghosts}}$ $\qquad m = 0$ to 9, $\alpha = 1$ to 16
(ignore right-movers)

Worldsheet action in flat background : $S = \int d^2 z \left[\frac{1}{2} \partial x^m \bar\partial x_m + p_\alpha \bar\partial \theta^\alpha + w_\alpha \bar\partial \lambda^\alpha \right]$

λ^α is a $d=10$ "pure spinor" satisfying $\lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta = 0$ for $m = 0$ to 9

$\Rightarrow \lambda^\alpha$ has 11 independent (complex) components which parameterize $\mathbb{C} \times \frac{SO(10)}{U(5)}$

$(w_\alpha, \lambda^\alpha)$ fields can be quantized as "curved β-γ system" (Nekrasov, '06)

- There is no conformal anomaly since $c = +10 - 32 + 22 = 0$.
  Formalism is in conformal gauge, but there are no explicit (b, c) ghosts.
- Lorentz current $M_{mn} = \frac{1}{2}(p \gamma_{mn} \theta) + \frac{1}{2}(w \gamma_{mn} \lambda)$ generates $SO(10)$ Kac-Moody algebra with same level as RNS Lorentz current $M_{mn} = \psi_m \psi_n$.

BRST operator

Physical open string states are defined as states of +1 ghost-number in cohomology of BRST operator $Q = \int dz\, \lambda^\alpha d_\alpha$ where

$d_\alpha = p_\alpha + (\gamma^m\theta)_\alpha \partial x_m + \frac{1}{8}(\theta\gamma^m\partial\theta)(\gamma_m\theta)_\alpha$ is the Dirac constraint in GS formalism

d_α has 8 first-class and 8 second-class constraints.

Nevertheless, Q is nilpotent since

$$Q^2 = \frac{1}{2}\lambda^\alpha\lambda^\beta\{d_\alpha, d_\beta\} = \lambda^\alpha\lambda^\beta\gamma^m_{\alpha\beta}(\partial x_m + \theta\gamma_m\partial\theta) = 0$$

using the pure spinor constraint $\lambda\gamma^m\lambda = 0$.

Can derive BRST operator Q by gauge-fixing GS formalism, but derivation is messy. (NB+Marchioro '04, Aisaka+Kazama '05)

Massless open superstring vertex operator:

Ghost-number: +1 $\Rightarrow$ $V = \lambda^\alpha A_\alpha(x,\theta)$ for some $A_\alpha(x,\theta)$

$QV=0$ and $V \neq Q\Omega$ $\Rightarrow$ $A_\alpha(x,\theta) = (\gamma^m\theta)_\alpha a_m(x) + (\theta\gamma_{mnp}\theta)(\gamma^{mnp}\chi(x))_\alpha + \ldots$

where $a_m(x)$ and $\chi^\alpha(x)$ describe on-shell gluon and gluino and $\ldots$ only involves derivatives of $a_m(x)$ and $\chi^\alpha(x)$.

So massless cohomology describes linearized $N=1$ $d=10$ super-Yang-Mills.

Massless closed superstring vertex operator:

$$V = \lambda^\alpha \bar\lambda^\beta A_{\alpha\beta}(x,\theta,\bar\theta)$$

where $(\lambda^\alpha,\theta^\alpha)$ are left-moving and $(\bar\lambda^\alpha, \bar\theta^\alpha)$ are right-moving.

$QV = \bar{Q}V = 0$ $\Rightarrow$ $A_{\alpha\beta}(x,\theta,\bar\theta) = (\gamma^m\theta)_\alpha(\gamma^n\bar\theta)_\beta\,(g_{mn}(x) + b_{mn}(x) + \eta_{mn}\varphi(x)) + \ldots$

Massless closed string cohomology describes linearized $N=2$ $d=10$ supergravity.

To compute scattering amplitudes, also need integrated vertex operators $\int dz\, U(z)$ which satisfy $QU = \partial V$.

Integrated open superstring massless vertex operator:

$$\int dz\, U = \int dz \left[A_M(x,\theta)\, \partial Y^M + W^\alpha(x,\theta)\, d_\alpha + F_{mn}(x,\theta)\, w\gamma^{mn}\lambda \right]$$

$Y^M = (x^m, \theta^\alpha)$, $A_M = (A_\alpha, A_m)$ are sYM gauge superfields

W^α, F_{mn} are sYM field-strengths where $W^\alpha(x,\theta) = \chi^\alpha(x) + (\gamma^{mn}\theta)^\alpha \partial_m a_n(x) + \ldots$

Integrated closed superstring massless vertex operator:

$$\int d^2z\, U = \int d^2z \left[\left(G_{MN}(x,\theta,\bar\theta) + B_{MN}(x,\theta,\bar\theta)\right) \partial Y^M \bar\partial Y^N + W^{\alpha\beta}(x,\theta,\bar\theta)\, d_\alpha \bar d_\beta + \ldots \right]$$

$Y^M = (x^m, \theta^\alpha, \bar\theta^\beta)$, $W^{\alpha\beta}(x,\theta,\bar\theta) = \underset{\substack{\uparrow \\ \text{R-R field strength}}}{f^{\alpha\beta}(x)} + (\gamma^{mn}\theta)^\alpha (\gamma^{pq}\bar\theta)^\beta \underset{\substack{\uparrow \\ \text{Riemann tensor}}}{R_{mnpq}(x)} + \ldots$

Tests of Pure Spinor Formalism:

- BRST cohomology reproduces GSO(+) superstring spectrum.

- Tree-level, one-loop and two-loop amplitudes agree with RNS computations. Two-loop computations are much simpler than in RNS and can include external fermions. (Mafra, '05)

- BRST invariance in curved supergravity background implies low-energy superspace equations of motion for background superfields.

- $AdS_5 \times S^5$ sigma model was shown to be consistent to all orders in α' with infinite set of non-local conserved currents. Recent explicit one-loop computations confirm quantum integrability.
  (Mikhailov + Schäfer-Nameki, '07)

II. Higher-Derivative R^4 Theorems

Consider massless 4-point g-loop closed superstring amplitude:

$$\mathcal{A}_g = \int d^{6g-6}\tau \left\langle \left(\int d^2 z_r \, U_r \right)^4 \left| \left(\int b \right)^{3g-3} \mathcal{N} \right|^2 \right\rangle$$

$\langle \; \rangle$ is genus-g correlation function,

$b(z)$ is composite operator satisfying $\{Q, b(z)\} = T(z)$,

$\mathcal{N}$ is BRST-invariant regulator for integrating over $(\lambda^\alpha, w_\alpha)$ zero modes.

$\int_{-\infty}^{\infty} d^{11}\lambda \left(\int_{-\infty}^{\infty} d^{11} w \right)^g$ diverges because of non-compact bosonic zero modes.

$\int d^{16}\theta \left(\int d^{16} p \right)^g$ gives zero if all fermionic zero modes are not present.

$\mathcal{N}$ cancels ∞'s against 0's in a BRST-invariant manner.

Since $\mathcal{N} = 1 + \{Q, \Omega\}$ for some Ω, $\mathcal{A}_g$ is independent of choice and location of $\mathcal{N}$. Similar to picture-changing operators in RNS for integration over (β, γ) zero modes.

For a_g to be non-vanishing, integrand must contain 16 θ^α zero modes and $16g$ d_α zero modes which can come from either $(\int b)^{3g-3}$, $\mathcal{N}$, or $(\int d^2 z_r U_r)^4$.

When $g \leq 6$, simple counting argument implies that $(\int d^2 z_r U_r)^4$ must contribute at least $(4+2g)$ θ^α and 4 d_α zero modes.
When $g > 6$, regularization prescription is modified and $(\int d^2 z_r U_r)^4$ must contribute 16 θ^α and 4 d_α zero modes.

Since $U_r = \ldots + d_\alpha \bar{d}_\beta W_r^{\alpha\beta}(x, \theta, \bar{\theta}) + \ldots$, low-energy contribution starts at

$$a_g \sim \left(\frac{d}{d\theta}\right)^{4+2g} \left(\frac{d}{d\bar{\theta}}\right)^{4+2g} (W^{\alpha\beta})^4 \sim \partial^{2g} R^4 + \ldots$$

where $W^{\alpha\beta}(\theta, \bar{\theta}) = f^{\alpha\beta} + (\theta\bar{\theta}R)^{\alpha\beta} + (\theta^3\bar{\theta}\,\partial R)^{\alpha\beta} + \ldots$

So when $1 < g \leq 6$, low-energy contribution to g-loop amplitude starts at $\partial^{2g} R^4$.

But when $g \geq 6$, low-energy contribution (naively) starts at $\partial^{12} R^4$!

After dimensional reduction to $d=4$, this result suggests that $N=8$ $d=4$ supergravity is finite up to 8 loops, but not beyond.

(Green, Russo, Vanhove, '06)

8-loop finiteness can be explained if one assumes that $W^{\alpha\beta}$ is the gauge-invariant superfield of lowest dimension, so $\int d^{16}\theta\, d^{16}\bar{\theta}\, (W)^4 \sim \partial^{12} R^4$ is the D-term of lowest dimension which could act as a counterterm.

Assumption is natural if one starts in $d=10$ and requires that $N=2$ $d=10$ superfields are "left-right" product of $N=1$ $d=10$ superfields.

But need to better understand "off-shell" $N=8$ $d=4$ supersymmetry in order to verify this assumption.

III. TOPOLOGICAL A-MODEL FOR $AdS_5 \times S^5$?

Pure spinor formalism for $N=2$ $d=10$ superparticle has
32 bosons $(x^m, \lambda^\alpha, \bar\lambda^\alpha)$ and 32 fermions $(\theta^\alpha, \bar\theta^\alpha)$.

$\theta^\alpha = \theta_1^\alpha + i\theta_2^\alpha$
$\bar\theta^\alpha = \theta_1^\alpha - i\theta_2^\alpha$

BRST operator $Q + \bar Q = \lambda^\alpha d_\alpha + \bar\lambda^\alpha \bar d_\alpha$ transforms $\delta\theta^\alpha = \lambda^\alpha$, $\delta\bar\theta^\alpha = \bar\lambda^\alpha$.

Does formalism have twisted worldline supersymmetry?
(e.g. Sorokin, Tkach, Volkov, Zheltukhin '88)

In flat background, b ghost satisfying $\{Q, b\} = H$ is complicated.

However, for superparticle in $AdS_5 \times S^5$ background, worldline supersymmetry and b ghost can be naturally constructed by combining $(x^m, \lambda^\alpha, \bar\lambda^\alpha)$ into 32 unconstrained twistor-like variables $(Z^\alpha, \bar Z^\alpha)$ such that $\delta\theta^\alpha = Z^\alpha$, $\delta\bar\theta^\alpha = \bar Z^\alpha$ under twisted worldline susy.

To construct $(Z^\alpha, \bar{Z}^\alpha)$, parameterize $AdS_5 \times S^5$ using supercoset

$$g \in \frac{PSU(2,2|4)}{SO(4,1)\times SO(5)} \quad \text{where} \quad g(x^c, y^{c'}, \theta^\alpha, \bar{\theta}^\alpha) = G(\theta^\alpha, \bar{\theta}^\alpha)\, H(x^c)\, \tilde{H}(y^{c'})$$

$\alpha: 1$ to 16
$c: 0$ to 4
$c': 5$ to 9

$$G(\theta^\alpha, \bar{\theta}^\alpha) \in \frac{PSU(2,2|4)}{SU(2,2)\times SU(4)}, \quad H(x^c) \in \frac{SU(2,2)}{SO(4,1)} = AdS_5, \quad \tilde{H}(y^{c'}) \in \frac{SU(4)}{SO(5)} = S^5$$

Using $SO(4,2)$ and $SO(6)$ σ-matrices, can express $H(x^c)$ and $\tilde{H}(y^{c'})$ as 4×4 bosonic matrices $H^A_B(x)$ and $\tilde{H}^J_K(y)$ where $A: 1$ to 4 is $SU(2,2)$ index and $J: 1$ to 4 is $SU(4)$ index.

Using this $SU(2,2)\times SU(4)$ notation,

$$\theta^\alpha \to \theta^A_J, \quad (\gamma^{01234}\bar{\theta})_\alpha \to \bar{\theta}^J_A$$

$$\lambda^\alpha \to \lambda^A_J, \quad (\gamma^{01234}\bar{\lambda})_\alpha \to \bar{\lambda}^J_A$$

To construct $(Z^A{}_J, \bar{Z}^J{}_A)$ out of $(x^c, y^{c'}, \lambda^A{}_J, \bar{\lambda}^J{}_A)$,
use $H(x) \in \frac{SU(2,2)}{SO(4,1)}$ and $\tilde{H}(y) \in \frac{SU(4)}{SO(5)}$ cosets to convert
constrained $SO(4,1) \times SO(5)$ pure spinors $(\lambda^A{}_J, \bar{\lambda}^J{}_A)$ into
unconstrained $SU(2,2) \times SU(4)$ spinors $(Z^A{}_J, \bar{Z}^J{}_A)$ by defining

$$Z^A{}_J = H^A{}_B(x)\, \tilde{H}^K{}_J(y)\, \lambda^B{}_K \,, \quad \bar{Z}^J{}_A = (H^{-1}(x))^B{}_A\, (\tilde{H}^{-1}(y))^J{}_K\, \bar{\lambda}^K{}_B \,.$$

Since $(H(x), \tilde{H}(y), \lambda, \bar{\lambda})$ describe 32 independent variables,
$Z^A{}_J$ and $\bar{Z}^J{}_A$ are generically unconstrained.
Like twistor variables, $Z^A{}_J$ and $\bar{Z}^J{}_A$ transform linearly under $SU(2,2)$ conformal transf.'s

Can similarly construct conjugate variables $(Y^A{}_J, \bar{Y}^J{}_A)$ as

$$Y^A{}_J = H^A{}_B(x)\, \tilde{H}^K{}_J(y)\, \bar{w}^B{}_K \,, \quad \bar{Y}^J{}_A = (H^{-1}(x))^B{}_A\, (\tilde{H}^{-1}(y))^J{}_K\, w^K{}_B$$

$w_\alpha = w^K{}_B$ and $(\bar{w}\gamma^{01234})^\alpha = \bar{w}^B{}_K$ are conjugate momentum to λ^α and $(\gamma^{01234}\bar{\lambda})_\alpha$.

In terms of supercoset $G(\theta,\bar\theta) \in \dfrac{U(2,2|4)}{U(2,2)\times U(4)} = \dfrac{PSU(2,2|4)}{SU(2,2)\times SU(4)}$

and twistor-like variables $(Z^A{}_J, \bar Z^J{}_A)$ and $(Y^A{}_J, \bar Y^J{}_A)$, one finds that

$Q = \lambda^\alpha d_\alpha$ and $\bar Q = \bar\lambda^{\dot\alpha} \bar d_{\dot\alpha}$ are mapped into

$$Q = Z^A{}_J \left(G^{-1}\tfrac{\partial}{\partial\tau}G\right)^J{}_A \quad \text{and} \quad \bar Q = \bar Z^J{}_A \left(G^{-1}\tfrac{\partial}{\partial\tau}G\right)^A{}_J .$$

Can easily define b and $\bar b$ satisfying $\{Q,b\} = \{\bar Q,\bar b\} = H$ as

$$b = \bar Y^J{}_A \left(G^{-1}\tfrac{\partial}{\partial\tau}G\right)^A{}_J \quad \text{and} \quad \bar b = Y^A{}_J \left(G^{-1}\tfrac{\partial}{\partial\tau}G\right)^J{}_A .$$

Worldline supersymmetry can be made manifest by combining

$(\theta^A{}_J, Z^A{}_J, Y^A{}_J)$ and $(\bar\theta^J{}_A, \bar Z^J{}_A, \bar Y^J{}_A)$ into chiral and antichiral superfields

$$\Theta^A{}_J(\tau^+,\kappa,\kappa') = \theta^A{}_J(\tau^+) + \kappa Z^A{}_J(\tau^+) + \kappa' Y^A{}_J(\tau^+) + \kappa\kappa' F^A{}_J(\tau^+)$$

$$\bar\Theta^J{}_A(\tau^-,\bar\kappa,\bar\kappa') = \bar\theta^J{}_A(\tau^-) + \bar\kappa \bar Z^J{}_A(\tau^-) + \bar\kappa' \bar Y^J{}_A(\tau^-) + \bar\kappa\bar\kappa' \bar F^J{}_A(\tau^-)$$

auxiliary fields

$$\tau^\pm = \tau \pm \kappa\bar\kappa' \pm \kappa'\bar\kappa$$

Topological A-model:

Pure spinor version of $N=2$ superparticle action on $AdS_5 \times S^5$ can be written in terms of Θ^A_J and $\bar{\Theta}^J_A$ as

$$S = R^2 \int d\tau \int d^4\kappa \; Tr\left[\log(1+\Theta\bar{\Theta})\right].$$

$R = AdS_5$ radius

Worldline action is $t \to \infty$ limit of the topological A-model

$$S = t \int d^2z \int d^4\kappa \; Tr\left[\log(1+\Theta\bar{\Theta})\right]$$

where $\Theta(z,\bar{z},\kappa,\kappa')$ is worldsheet superfield,

which can also be expressed as a gauged linear sigma model

$$S = \int d^2z \int d^4\kappa \; Tr\left[\Phi e^V \bar{\Phi} + \Theta e^V \bar{\Theta} + tV\right]$$

(M. Roček, private comm.)

which is based on the fermionic coset $\dfrac{U(2,2|4)}{U(2,2)\times U(4)}$.

V is $U(4)$ gauge superfield and $(\Phi^J_K, \bar{\Phi}^J_K)$ are superfields which can be gauge-fixed to $\Phi^J_K = \bar{\Phi}^J_K = \delta^J_K$.

Relation of S to superstring action on $AdS_5 \times S^5$ is <u>unclear</u> because of Wess-Zumino term in superstring action.

Speculation: Gauged linear sigma model may be useful for proving Maldacena conjecture using methods of Gopakumar-Ooguri-Vafa for proving open-closed duality in conifold transition.

Near $t=0$, both Higgs and Coulomb phases can coexist in gauged sigma model.

Closed string amplitudes can be reinterpreted as gauge theory Feynman diagrams where "holes" in Coulomb phase become "faces" in diagram.

Conifold / Chern-Simons duality: Coulomb phase describes D-branes with boundary condition $X^j = \delta^{jk} \bar{X}_k$ which is Witten's open topological A-model for Chern-Simons on S^3.

$AdS_5 \times S^5$ / super-Yang-Mills duality: Coulomb phase describes D-branes with boundary condition $\Theta^A_J = \epsilon^{AB} \delta_{JK} \bar{\Theta}^K_B$ which is conjectured to be an open topological A-model for $N=4$ $d=4$ super-Yang-Mills on AdS_4.

Open topological A-model is constructed from fermionic coset $\dfrac{OSp(4|4)}{Sp(4)\times SO(4)}$.

Summary

Pure spinor formalism is useful for computing multiloop superstring amplitudes and studying superstring in $AdS_5 \times S^5$ background.

When $1 < g \leq 6$, low-energy contribution to 4-point g-loop superstring amplitude starts at $\partial^{2g} R^4$.
Suggests that N=8 d=4 supergravity is finite up to 8 loops, but not beyond 8 loops.

Superparticle on $AdS_5 \times S^5$ background can be described as $t \to \infty$ limit of topological A-model based on fermionic coset $\frac{U(2,2|4)}{U(2,2) \times U(4)}$.
May be useful for generalizing Gopakumar-Ooguri-Vafa proof of open-closed conifold duality to Maldacena conjecture.
But many crucial points are <u>not</u> <u>yet</u> <u>understood</u>.