Evidence for Ultraviolet Finiteness of $N = 8$ Supergravity

Strings, Madrid
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Zvi Bern, UCLA

Z. Bern, N.E.J. Bjerrum-Bohr, D. Dunbar, hep-th/0501137
Z. Bern, L. J. Dixon, R. Roiban, hep-th/0611086
Why $N = 8$ Supergravity?

- UV finiteness of $N = 8$ supergravity would imply a new symmetry or non-trivial dynamical mechanism.
- The discovery of either would have a fundamental impact on our understanding of gravity.
- High degree of supersymmetry makes this the most promising theory to investigate.
- By $N = 8$ we mean ungauged Cremmer-Julia supergravity.

No known superspace or supersymmetry argument prevents divergences from appearing at some loop order.

$\frac{1}{\epsilon} D^n R^4$  

Potential counterterm predicted by susy power counting

Range of opinions on where this can happen — from 3 to 9 loops, depending on assumptions.
Reasons to Reexamine This

1) The number of *established* counterterms in *any* supergravity theory is zero.

2) Discovery of remarkable cancellations at 1 loop – the “no-triangle hypothesis”. ZB, Dixon, Perelstein, Rozowsky.
   ZB, Bjerrum-Bohr and Dunbar; Bjerrum-Bohr, Dunbar, Ita, Perkins and Risager.

3) *Every* explicit loop calculation to date finds $N = 8$ supergravity has identical power counting as in $N = 4$ super-Yang-Mills theory, which is UV finite. Green, Schwarz and Brink; ZB, Dixon, Dunbar, Perelstein, Rozowsky; Bjerrum-Bohr, Dunbar, Ita, PerkinsRisager; ZB, Carrasco, Dixon, Johanson, Kosower, Roiban.

4) Very interesting hint from string dualities. Chalmers; Green, Vanhove, Russo.
   – Dualities restrict form of effective action. May prevent divergences from appearing in $D = 4$ supergravity.
   – Difficulties with decoupling of towers of massive states. See Russo’s talk for latest status.

5) Berkovits’ string non-renormalization theorems suggest $N = 8$ supergravity may be finite through 8 loops. See Berkovits’ talk.
   No argument beyond this. Green, Vanhove, Russo.
Gravity Feynman Rules

\[ \mathcal{L} = \frac{2}{k^2} \sqrt{g} R, \quad g_{\mu \nu} = \eta_{\mu \nu} + \kappa h_{\mu \nu} \]

Propagator in de Donder gauge:

\[ P_{\mu \nu; \alpha \beta}(k) = \frac{1}{2} \left[ \eta_{\mu \nu} \eta_{\alpha \beta} + \eta_{\mu \beta} \eta_{\nu \alpha} - \frac{2}{D-2} \eta_{\mu \alpha} \eta_{\nu \beta} \right] \frac{i}{k^2 + i\epsilon} \]

Three vertex has about 100 terms:

\[ G_{3\mu, \nu, \sigma}(k_1, k_2, k_3) = \]

\[
\begin{align*}
&\text{sym}[ -\frac{1}{2} P_3(k_1 \cdot k_2 \eta_{\mu \alpha} \eta_{\nu \beta} \eta_{\sigma \gamma}) - \frac{1}{2} P_6(k_1 k_2 \eta_{\mu \alpha} \eta_{\sigma \gamma}) + \frac{1}{2} P_3(k_1 \cdot k_2 \eta_{\mu \nu} \eta_{\alpha \beta} \eta_{\sigma \gamma}) \\
&+ P_6(k_1 \cdot k_2 \eta_{\mu \alpha} \eta_{\nu \sigma} \eta_{\beta \gamma}) + 2 P_3(k_1 k_2 \eta_{\mu \alpha} \eta_{\beta \gamma}) - P_3(k_1 k_2 \eta_{\mu \nu} \eta_{\alpha \sigma} \eta_{\beta \gamma}) \\
&+ P_3(k_1 k_2 \eta_{\mu \nu} \eta_{\alpha \beta} \eta_{\gamma \sigma}) + P_6(k_1 k_2 \eta_{\mu \nu} \eta_{\alpha \sigma} \eta_{\beta \gamma}) + 2 P_6(k_1 k_2 \eta_{\beta \gamma} \eta_{\alpha \sigma} \eta_{\mu \nu}) \\
&+ 2 P_3(k_1 k_2 \eta_{\mu \nu} \eta_{\beta \gamma} \eta_{\alpha \sigma}) - 2 P_3(k_1 \cdot k_2 \eta_{\alpha \sigma} \eta_{\gamma \mu} \eta_{\beta \gamma}) ]
\end{align*}
\]

An infinite number of other messy vertices

\[ \sim 10^{20} \text{ terms} \]

It is “impossible” to calculate
Why are Feynman diagrams clumsy for loop or high-multiplicity processes?

• Vertices and propagators involve gauge-dependent off-shell states. Origin of the complexity.

\[ \begin{align*}
\gamma + \gamma + \gamma + \cdots \\
\quad \quad \quad \quad \text{\( p^2 \neq 0 \)}
\end{align*} \]

• To get at root cause of the trouble we must rewrite perturbative quantum gravity.

\[ \begin{align*}
\quad \quad \quad \quad \text{\( p^2 \neq 0 \)}
\end{align*} \]

• All steps should be in terms of gauge invariant on-shell states. \( p^2 = 0 \)

• Need on-shell formalism.
• Kawai-Lewellen-Tye relations: sum of products of gauge theory tree amplitudes gives gravity tree amplitudes.

• Unitarity method: efficient formalism for perturbatively quantizing gauge and gravity theories. Loop amplitudes from tree amplitudes.

Key features of this approach:

• Gravity calculations mapped into much simpler gauge theory calculations.

• Only on-shell states appear.
At tree level Kawai, Lewellen and Tye presented a relationship between closed and open string amplitudes. In field theory limit, relationship is between gravity and gauge theory where we have stripped all coupling constants:

\[
M_4^{\text{tree}}(1, 2, 3, 4) = s_{12}A_4^{\text{tree}}(1, 2, 3, 4) A_4^{\text{tree}}(1, 2, 4, 3),
\]
\[
M_5^{\text{tree}}(1, 2, 3, 4, 5) = s_{12}s_{34}A_5^{\text{tree}}(1, 2, 3, 4, 5) A_5^{\text{tree}}(2, 1, 4, 3, 5)
+ s_{13}s_{24}A_5^{\text{tree}}(1, 3, 2, 4, 5) A_5^{\text{tree}}(3, 1, 4, 2, 5)
\]

Holds for any external states. See review: gr-qc/0206071

Progress in gauge theory can be imported into gravity theories.
The unitarity method gives us a means for directly going between on-shell tree amplitudes and loop amplitudes

- Lagrangian not needed.
- No Feynman diagrams.
- No gauge fixing required.
- No unphysical off-shell states.
- KLT relations can be used to determine tree amplitudes.

A number of recent improvements to method, which I won’t discuss here
All-loop Resummation in $\mathcal{N} = 4$ Super-YM Theory

Obtained using unitarity method

- Conjecture that planar scattering amplitudes iterate to all loop orders and may be resummable.
- Explicit form of conjecture determined for MHV amplitudes.
- Four-loop cusp anomalous dimension.

$\mathcal{A}_n = \mathcal{A}_n^{\text{tree}} \mathcal{A}_n^{\text{divergent}} \exp \left[ \frac{1}{4} \gamma_K F_n^{1\text{-loop}} + C \right]$

- all-loop resummed amplitude
- IR divergences
- cusp anomalous dimension
- finite part of one-loop amplitude

Gives a definite prediction for all values of coupling given the Beisert, Eden, Staudacher integral equation for the cusp anomalous dimension. See Beisert’s talk

In a beautiful paper Alday and Maldacena confirmed this conjecture at strong coupling from AdS string computation. See Maldacena’s talk

Anastasiou, ZB, Dixon, Kosower; ZB, Dixon, Smirnov

ZB, Czakon, Dixon, Kosower, Smirnov
**N = 8 Power Counting To All Loop Orders**

From ’98 paper:

- Assumed iterated 2 particle cuts give the generic UV behavior.
- Assumed no cancellations with other uncalculated terms.

No evidence was found that more than 12 powers of loop momenta come out of the integrals.

Result from ’98 paper

Elementary power counting gave finiteness condition:

\[ D < \frac{10}{L} + 2 \quad (L > 1) \]

In \( D = 4 \) diverges for \( L \geq 5 \).

\( L \) is number of loops.

\( D^4 R^4 \) counterterm was expected in \( D = 4 \), for \( L = 5 \)
Additional Cancellations at One Loop

Crucial hint of additional cancellation comes from one loop.

Surprising cancellations not explained by any known susy mechanism are found beyond four points.

Two derivative coupling means $N = 8$ should have a worse power counting relative to $N = 4$ super-Yang-Mills theory.

- Cancellations observed in MHV amplitudes. (ZB, Dixon, Perelstein Rozowsky (1999))
- “No-triangle hypothesis” — cancellations in all other amplitudes. (ZB, Bjerrum-Bohr and Dunbar (2006))
- Confirmed by explicit calculations at 6,7 points. (Bjerrum-Bohr, Dunbar, Ita, Perkins, Risager (2006))
One-loop $D = 4$ theorem: Any one loop massless amplitude is a linear combination of scalar box, triangle and bubble integrals with rational coefficients:

$$A_{n}^{1-\text{loop}} = \sum_{i} d_{i} I_{4}^{(i)} + \sum_{i} c_{i} I_{3}^{(i)} + \sum_{i} b_{i} I_{2}^{(i)}$$

- In $N = 4$ Yang-Mills only box integrals appear. No triangle integrals and no bubble integrals.
- The “no-triangle hypothesis” is the statement that same holds in $N = 8$ supergravity.
**L-Loop Observation**

From 2 particle cut:

\[
(k_1 + k_2)^2 [2(L-2)]
\]

numerator factor

From \(L\)-particle cut:

\[
(l + k_4)^2 [2(L-2)]
\]

numerator factor

Above numerator violates no-triangle hypothesis. Too many powers of loop momentum.

There must be additional cancellation with other contributions!

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ZB, Dixon, Roiban

Using generalized unitarity and no-triangle hypothesis all one-loop subamplitudes should have power counting of \(N = 4\) Yang-Mills
Complete Three Loop Calculation

Besides iterated two-particle cuts need following cuts:

For first cut have:

\[
\sum_{N=8 \text{ states}} M_4^{\text{tree}}(1, 2, l_3, l_1) \times M_5^{\text{tree}}(-l_1, -l_3, q_3, q_2, q_1) \times M_5^{\text{tree}}(3, 4, -q_1, -q_2, -q_3)
\]

Use KLT

\[
M_4^{\text{tree}}(1, 2, l_3, l_1) = -is_{12} A_4^{\text{tree}}(1, 2, l_3, l_1) A_4^{\text{tree}}(2, 1, l_3, l_1)
\]

\[
M_5^{\text{tree}}(-l_1, -l_3, q_3, q_2, q_1) = is_{l_1 q_1} s_{l_3 q_3} A_5^{\text{tree}}(-l_1, -l_3, q_3, q_2, q_1) A_5^{\text{tree}}(-l_1, q_1, q_3, -l_3, q_2) + \{l_1 \leftrightarrow l_3\}
\]

\[N = 8\] supergravity cuts are sums of products of \[N = 4\] super-Yang-Mills cuts

ZB, Carrasco, Dixon, Johansson, Kosower, Roiban
Complete three loop result

All obtainable from iterated two-particle cuts, except (h), (i), which are new.

\[ l_{i,j}^2 = (l_i + l_j)^2 \]
\[ s = (k_1 + k_2)^2 \]
\[ t = (k_1 + k_4)^2 \]

<table>
<thead>
<tr>
<th>Integral ( I^{(x)} )</th>
<th>( \mathcal{N} = 4 ) Super-Yang-Mills</th>
<th>( \mathcal{N} = 8 ) Supergravity</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)–(d) ( s^2 )</td>
<td>( s (l_1 + k_4)^2 )</td>
<td>( [s^2]^2 )</td>
</tr>
<tr>
<td>(e)–(g) ( s(l_1 + k_4)^2 )</td>
<td>( s(l_1 + k_4)^2 )</td>
<td>( [s^2]^2 )</td>
</tr>
<tr>
<td>(h) ( sl_{1,2}^2 + tl_{3,4}^2 - sl_5^2 - tl_6^2 - st )</td>
<td>( (sl_{1,2}^2 + tl_{3,4}^2 - st)^2 - s^2(2(l_{1,2}^2 - t) + l_5^2)t_5^2 - t^2(2(l_{3,4}^2 - s) + l_6^2)t_6^2 )</td>
<td>( -s^2(2l_{1,2}^2 t_5^2 + 2l_{3,4}^2 t_7^2 + l_1^2 l_5^2 + l_3^2 l_5^2) - t^2(2l_{3,4}^2 t_10 + 2l_{4,5}^2 t_2 + l_3^2 t_5 + l_3^2 l_5 + l_4^2 l_5 + 2s l_5^2 t_6) )</td>
</tr>
<tr>
<td>(i) ( sl_{1,2}^2 - tl_{3,4}^2 - \frac{1}{3}(s - t)l_5^2 )</td>
<td>( (sl_{1,2}^2 - tl_{3,4}^2)^2 - (s^2 l_{1,2}^2 + t^2 l_{3,4}^2 + \frac{1}{3} st u) t_5^2 )</td>
<td>( \sum_{S_3} \left[ I^{(a)} + I^{(b)} + \frac{1}{2} I^{(c)} + \frac{1}{4} I^{(d)} + 2I^{(e)} + 2I^{(f)} + 4I^{(g)} + \frac{1}{2} I^{(h)} + 2I^{(i)} \right] )</td>
</tr>
</tbody>
</table>
To check leading UV behavior we can expand in external momenta keeping only leading term.

Get vacuum type diagrams:

\[(l_1 + k_4)^4\]

After combining contributions:

The leading UV behavior cancels!!
Finiteness Conditions

Through $L = 3$ loops the correct finiteness condition is ($L > 1$):

```
\text{“superfinite”}
\text{in $D = 4$}
\begin{equation}
D < \frac{6}{L} + 4
\end{equation}
```

\text{not the weaker result from iterated two-particle cuts:}

```
\text{finite}
\text{in $D = 4$ for $L = 3, 4$}
\begin{equation}
D < \frac{10}{L} + 2
\end{equation}
\text{('98 prediction)}
```

Beyond $L = 3$, as already explained, from special cuts we have good reason to believe that the cancellations continue.

All one-loop subamplitudes should have same UV power-counting as $N = 4$ super-Yang-Mills theory.
There does not appear to be a supersymmetry explanation for observed cancellations, especially if they hold to all loop orders, as we have argued.

If it is *not* supersymmetry what might it be?
Tree Cancellations in Pure Gravity

Unitarity method implies all loop cancellations come from tree amplitudes. Can we find tree cancellations?

You don’t need to look far: proof of BCFW tree-level on-shell recursion relations in gravity relies on the existence such cancellations!

Susy not required

Consider the shifted tree amplitude:

\[ k_1^{\mu} \rightarrow k_1^{\mu} + \frac{z}{2} \langle k_1^- | \gamma^\mu | k_2^- \rangle, \quad k_2^{\mu} \rightarrow k_2^{\mu} - \frac{z}{2} \langle k_1^- | \gamma^\mu | k_2^- \rangle, \]

How does \( M(z) \) behave as \( z \to \infty \)?

Proof of BCFW recursion requires \( M(z) \to 0 \)
Tree Cancellations in Pure Gravity

Summing over all Feynman diagrams, correct gravity scaling is:

\[ M_n^{\text{tree}}(z) \sim \frac{1}{z^2} \]

Remarkable tree-level cancellations!

Bedford, Brandhuber, Spence, Travaglini;
Cachazo and Svrcek;
Benincasa, Boucher-Veronneau, Cachazo
Proposal: This continues to higher loops, so that most of the observed $N = 8$ multi-loop cancellations are \textit{not} due to susy but in fact are generic to gravity theories!
What needs to be done?

• $N = 8$ four-loop computation. Can we demonstrate that four-loop $N = 8$ amplitude has the same UV power counting as $N = 4$ super-Yang-Mills? Certainly feasible (but non-trivial).

• Can we construct a proof of perturbative UV finiteness of $N = 8$? Perhaps possible using unitarity method – formalism is recursive.

• Investigate higher-loop pure gravity power counting to study cancellations. (It does diverge.) Goroff and Sagnotti; van de Ven

• Link to a twistor string description of $N = 8$? Abou-Zeid, Hull, Mason

• Can we find other examples with less susy that may be finite? Guess: $N = 6$ supergravity theories will be perturbatively finite.
Summary

- Unitarity method gives us a powerful means for studying ultraviolet properties of quantum gravity.
- At four points through three loops, established $N = 8$ supergravity has same power counting as $N = 4$ Yang-Mills.
- One-loop $N = 8$ “no-triangle hypothesis” – one-loop cancellations.
- No-triangle hypothesis implies cancellations in a class of terms to all loop orders. No known superspace argument gives such cancellations.
- Proposed that most of the observed $N = 8$ cancellations are present in generic gravity theories, with susy cancellations on top of these.

$N = 8$ supergravity may be the first example of a point-like perturbatively UV finite theory of quantum gravity in $D = 4$. Proof is an open challenge.