Wall Crossing and an Entropy Enigma

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Work done with Frederik Denef

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and with Emanuel Diaconescu arXiv:0706.3193

Related References:

G. Lopes Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, "Black hole partition functions and duality," 0601108.

D. Gaiotto, A. Strominger and X. Yin, "From AdS(3)/CFT(2) to black holes / topological strings," 0602046;

P. Kraus and F. Larsen, "Partition functions and elliptic genera from supergravity," 0607138.

J. de Boer, M. C. N. Cheng, R. Dijkgraaf, J. Manschot and E. Verlinde, "A farey tail for attractor black holes," 0608059.

C. Beasley, D. Gaiotto, M. Guica, L. Huang, A. Strominger and X. Yin, "Why $Z_{\rm sugra} = |Z_{\rm top}|^2$," 0608021.

A. Dabholkar, D. Gaiotto and S. Nampuri, "Comments on the spectrum of CHL dyons," 0702150.

A. Sen, "Walls of Marginal Stability and Dyon Spectrum in N=4 Supersymmetric String Theories," 0702141, 0705.3874

M. x. Huang, A. Klemm, M. Marino and A. Tavanfar, "Black Holes and Large Order Quantum Geometry," 0704.2440.

M. C. N. Cheng and E. Verlinde, "Dying Dyons Don't Count," 0706.2363.

Outline

1. Setting the Scene
2. Wall-Crossing Formula
3. Mathematical Applications
4. D6-D2-D0 Partition Functions –
5. Derivation of OSV
6. The Entropy Enigma
7. A Nonperturbative Z_{top} ?
8. A Concluding Riddle

Setting the Scene

Setting: Type IIA strings on a compact Calabi-Yau 3-fold *X*.

BPS States in the d=4, N=2 SUGRA are

wrapped D6-D4-D2-D0 branes in X

They carry RR charge

$$\begin{aligned} \Gamma &= \quad p^0 \quad \oplus \quad P \quad \oplus \quad Q \quad \oplus \quad q_0 \\ &= H^0(X) \oplus H^2(X) \oplus H^4(X) \oplus H^6(X) \end{aligned}$$

But existence of states also depends on boundary conditions on VM scalars:

$$t_\infty := \lim_{\vec{x} \to \infty} t(\vec{x}) := \lim_{\vec{x} \to \infty} (B + iJ)(\vec{x})$$

Space of BPS States $\mathcal{H}(\Gamma; t_\infty)$

Is finite dimensional and depends on t_∞ (Seiberg & Witten)

These state spaces are interesting!

Accounts for black hole entropy (Strominger & Vafa; Maldacena, Strominger, Witten)

Plays a key role in the OSV conjecture

Plays a key role in quantum corrections to effective Sugra

Possibly carry algebraic structures generalizing GKM (Harvey & Moore)

Possibly categorify knot invariants (Gukov, Vafa, et. al.)

N=2 Central Charges

$\langle \Gamma_1, \Gamma_2 \rangle$: Symplectic product on $H^{\mathrm{even}}(X)$

Duality invariant product of electric+magnetic charges

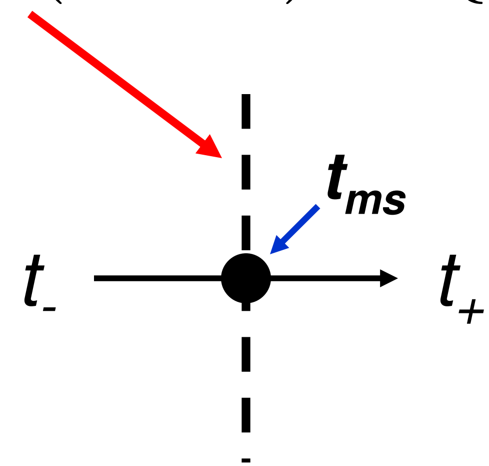
$$\text{Period vector: } \omega \sim e^t = e^{B+iJ}$$

$$Z_{\mathrm{holo}}(\Gamma; t) = \langle \Gamma, \omega \rangle \sim -\int_X e^{-t}\Gamma$$

$$Z_{\mathrm{normalized}}(\Gamma; t) \cong \frac{\frac{1}{6}p^0 t^3 - \frac{1}{2}Pt^2 + Qt - q_0}{\sqrt{(\mathrm{Im}t)^3}}$$

Wall-Crossing Formula

Marginal Stability Wall: $$MS(\Gamma_1, \Gamma_2) := \{t | \frac{Z(\Gamma_1;t)}{Z(\Gamma_2;t)} \in \mathbb{R}_+\}$$



$$\Delta\mathcal{H}(\Gamma_1, \Gamma_2; t_{ms}) = (J_{12}) \otimes \mathcal{H}(\Gamma_1; t_{ms}) \otimes \mathcal{H}(\Gamma_2; t_{ms})$$

$$J_{12} = \frac{1}{2}(|\langle\Gamma_1, \Gamma_2\rangle| - 1) := \frac{1}{2}(|I_{12}| - 1)$$

Remarks – I

- From $\langle\Gamma_1,\Gamma_2\rangle \mathrm{Im}(Z_1 Z_2^*) > 0$ To $\langle\Gamma_1,\Gamma_2\rangle \mathrm{Im}(Z_1 Z_2^*) < 0$
- $\natural$ Conjecture: A universal formula for N=2,d=4 BPS states
- Useful Corollary: Set $\Omega(\Gamma;t) := \mathrm{Tr}_{\mathcal{H}(\Gamma;t)}(-1)^F$

$$\Delta\Omega(\Gamma_1,\Gamma_2;t_{\rm ms}) = (-1)^{I_{12}-1}|I_{12}|\Omega(\Gamma_1;t_{\rm ms})\Omega(\Gamma_2;t_{\rm ms})$$

$$I_{12} := \langle\Gamma_1,\Gamma_2\rangle$$

Remarks - II

- Γ_1 and Γ_2 are primitive
- $t_{ms} \in MS(\Gamma_1, \Gamma_2)$ is generic
- We give three proofs

D-brane moduli space changes topology by blowing-down and blowing-up projective spaces.

Macroscopic 2-centered solutions

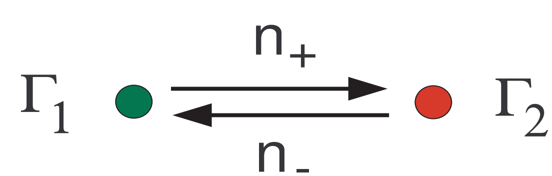
Microscopic Quiver Quantum Mechanics

Macro-proof: Denef Multicenter Solutions

$$R_{12} = \frac{1}{2}\langle\Gamma_1,\Gamma_2\rangle\frac{|Z_1+Z_2|_\infty}{\mathrm{Im}(Z_1 Z_2^*)_\infty}$$

$$\text{For } t_\infty \to t_{ms}\ ,\ \mathrm{Im}(Z_1 Z_2^*) \to 0, \Rightarrow R_{12} \to \infty$$

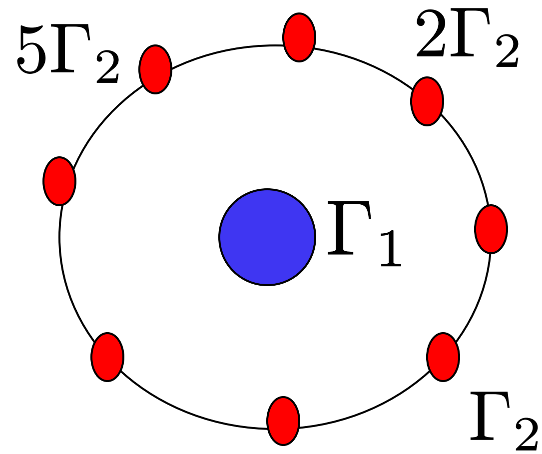
Micro-Proof : Quiver Quantum Mechanics (Denef, QQHHH):



Decay modelled by $N = 4$ $0+1$ SUSY QM with $U(1)$ VM (A_0, x^i, λ) + $n_\pm$ charge ± 1 CM's.
There are $|I_{12}| = |n_+ - n_-|$ Coulomb branch BPS states with support of the wavefunction at $|\vec{x}| = R_{12}$
They are in a spin J_{12} multiplet

Nonprimitive Wall-Crossing

$$MS(\Gamma_1, \Gamma_2) = MS(\Gamma_1, N\Gamma_2)$$



Denef's Halos:

FOCK SPACE $\oplus_{N>0} \Delta\mathcal{H}(\Gamma_1, N\Gamma_2; t)$

$|\langle \Gamma_1, k\Gamma_2 \rangle| \Omega(k\Gamma_2; t_{ms})$ CREATION/ANN. OPERATORS

$$\Omega_1 + \sum_{N>0} \Delta\Omega(\Gamma_1 + N\Gamma_2)|_{t_{\rm ms}} \, u^N =$$

$$= \Omega(\Gamma_1)|_{t_{\rm ms}} \prod_{k>0} \left(1 - (-1)^{k\langle \Gamma_1, \Gamma_2 \rangle} \, u^k \right)^{|\langle \Gamma_1, k\Gamma_2 \rangle| \, \Omega(k\Gamma_2)|_{t_{\rm ms}}}$$

Gives products such as McMahon, similar to DT, GV infinite products

Mathematical Tests/Applications

(with Emanuel Diaconescu)

D4 wraps rigid surface: $S \hookrightarrow X$ *with holomorphic bundle* $\mathcal{E} \to S,$

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$$

→ *Change of J will induce decay*

$$\Omega(\Gamma; t; x, y) := \mathrm{Tr}_{\mathcal{H}(\Gamma;t)} (x)^{J_3+R} (y)^{J_3-R}$$

Hodge Polynomial of $\mathcal{M}(\mathcal{E} \to S)$

$$\Omega_+ - \Omega_- = (-1)^{I_{12}-1} (xy)^{-\frac{1}{2}(I_{12}+1)} \frac{1-(xy)^{I_{12}}}{1-xy} \Omega_1 \Omega_2$$

Reproduces nontrivial results of Gottsche, Yoshioka

Makes new math predictions

Moduli of D4 branes is NOT the moduli of coherent sheaves!

D6-D2-D0 PartitionFunctions

$$\Gamma(\beta,n) := (1-\beta+ndV)$$

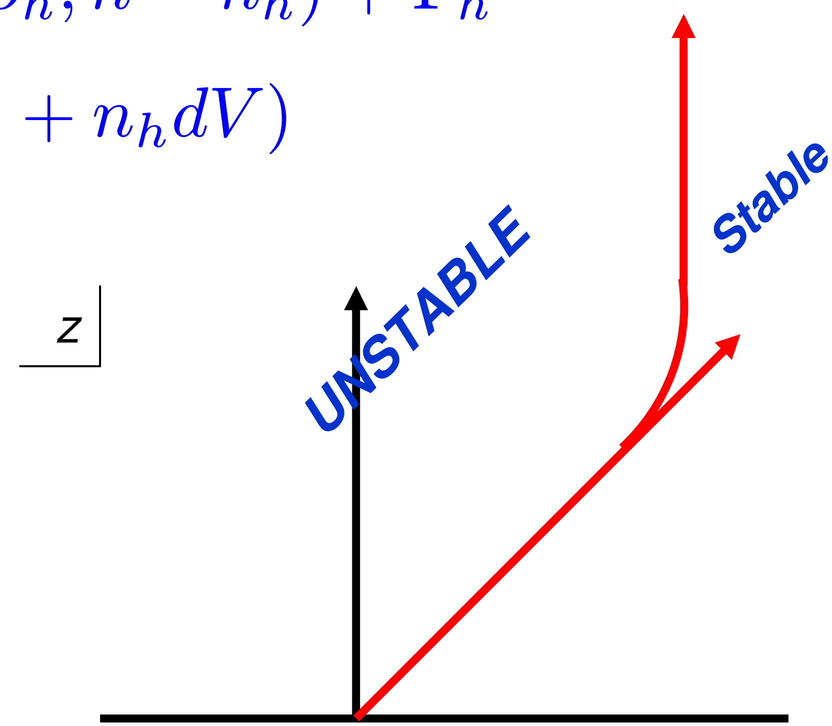
$$\beta = \text{PD of a holomorphic curve in } X,\ n\in\mathbb{Z}$$

$$\Gamma(\beta,n) \to \Gamma(\beta-\beta_h, n-n_h) + \Gamma_h$$

$$\Gamma_h := (-\beta_h + n_h dV)$$

Let $B+iJ = zP$, $P\in$ Kähler cone.

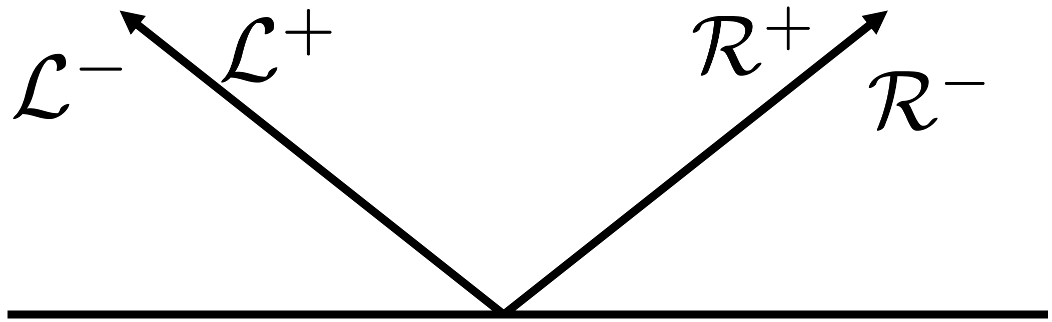
Marginal Stability Wall:



Relation to Donaldson-Thomas

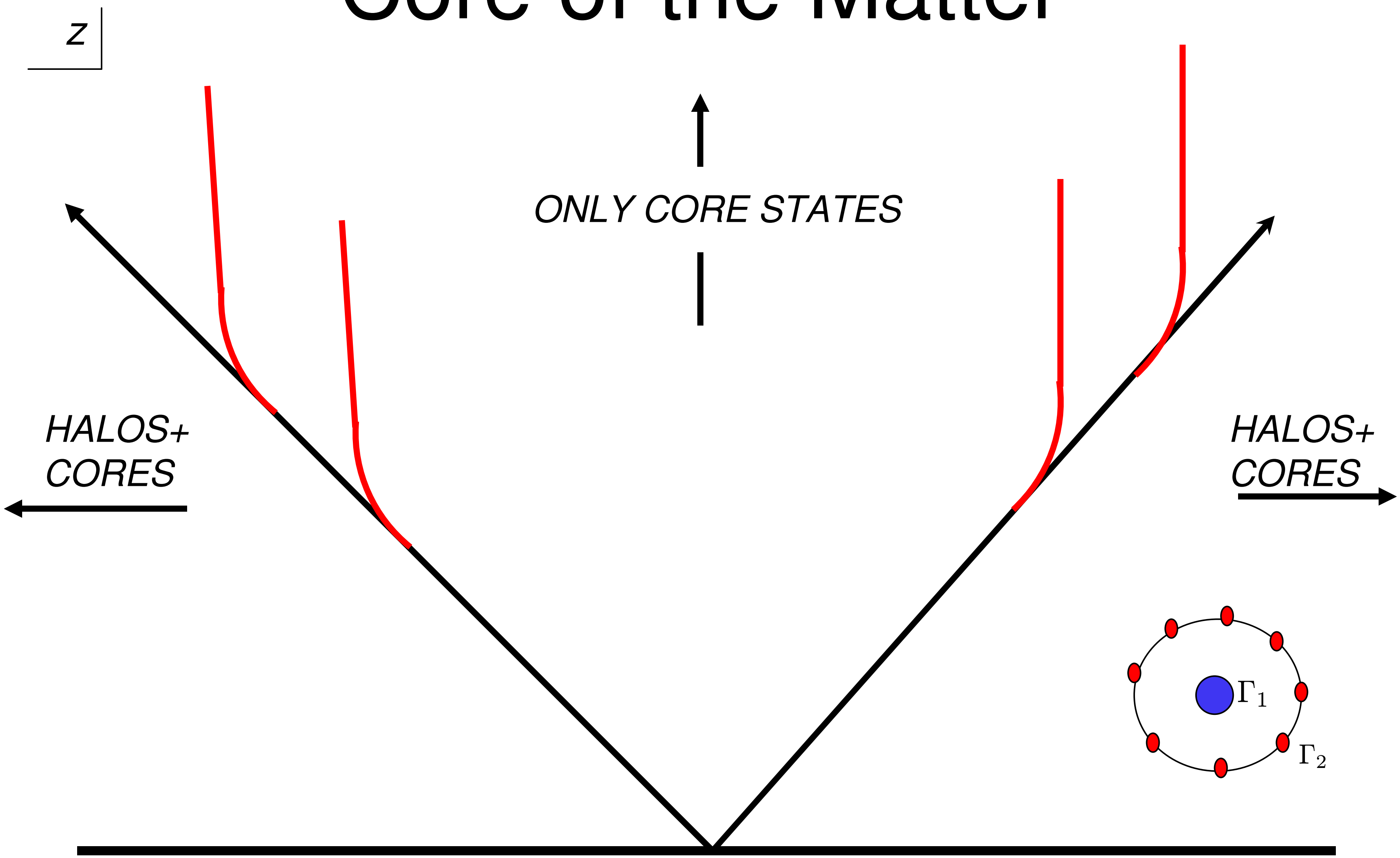
$$Z_{D6D2D0}(u,v;t) := \sum_{\beta,n} \Omega(\Gamma(\beta,n);t) u^n v^\beta$$

$$Z_{DT}(u,v) = \sum N_{DT}(\beta,n) u^n v^\beta$$



$$\begin{aligned}
\lim_{z\to\mathcal{L}^-} \mathcal{Z}_{D6-D2-D0}(u,v;zP) &= \mathcal{Z}_{DT}(u^{-1},v) \\
\lim_{z\to\mathcal{L}^+} \mathcal{Z}_{D6-D2-D0}(u,v;zP) &= \mathcal{Z}'_{DT}(u^{-1},v) \\
\lim_{z\to\mathcal{R}^-} \mathcal{Z}_{D6-D2-D0}(u,v;zP) &= \mathcal{Z}_{DT}(u,v) \\
\lim_{z\to\mathcal{R}^+} \mathcal{Z}_{D6-D2-D0}(u,v;zP) &= \mathcal{Z}'_{DT}(u,v)
\end{aligned} \quad (1)$$

Core of the Matter



OSV

$$\Omega(\Gamma) \sim \int d\phi |Z_{top}(g_{top}; t)|^2 e^{-2\pi q \cdot \phi}$$

$p^0 \neq 0$ → *Wall crossing in B* → *Problematic*

If $p^0 = 0$, i.e., if $\Gamma = P \oplus Q \oplus q_0$ and $P \in$ Kähler cone then:

$$\lim_{J \to \infty} \Omega(\Gamma; B + iJ)$$

Still has wall-crossing as a function of J !!

Diaconescu-Moore *Denef-Van den Bleeken*

A Refined OSV Formula

$$\Gamma = P \oplus Q \oplus q_0$$

$$\Omega(\Gamma)_\infty := \lim_{\lambda \to \infty} \Omega(\Gamma; B + i\lambda P)$$

Has no wall-crossing. In fact is B-independent

$$\Omega(\Gamma)_\infty = \int d\phi \mu(\phi) |Z^{\epsilon}_{top}(g_{top}; t)|^2 e^{-2\pi q \cdot \phi} + ET(\epsilon)$$

$$g_{\rm top} \equiv \frac{2\pi}{\phi^0}, \qquad t^A \equiv \frac{1}{\phi^0}(\phi^A + i\frac{P^A}{2})$$

$$\mu(\phi) = \phi^0 (\tfrac{P^3}{6} + \tfrac{c_2 P}{12}) + \mathcal{O}(e^{-|P|/\phi^0})$$

ET = Error term with precise estimates depending on cutoff ϵ

Polar States

For D4-D2-D0 Charge $$\Gamma = P \oplus Q \oplus q_0$$

Define: $$\hat{q}_0 = q_0 - \frac{1}{2}(D_{ABC}P^C)^{-1}Q_A Q_B$$

Then $$\Omega(\Gamma)_\infty = \Omega(P, \hat{q}_0)$$ *(slightly lying here)*

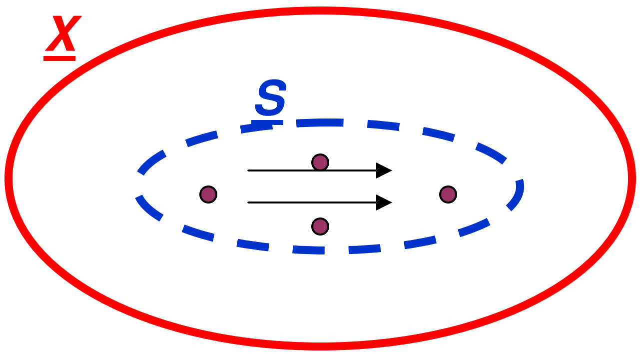
U-duality implies $$H(\tau) = \sum_{\hat{q}_0} \Omega(P, \hat{q}_0) e^{-2\pi i \tau \hat{q}_0}$$ *is modular*

$\Rightarrow$ Polar states: $\hat{q}_0 > 0$ determine all degeneracies.

$\hat{q}_0 < 0$ *Black hole degeneracies (Fareytail story)*

Microscopic Polar States

Single $D4$ wraps $S \in |P|$ with $U(1)$ flux F and N $\overline{DO}$ branes.



$$\hat{q}_0 = \frac{\chi(P)}{24} + \frac{1}{2}(F^-)^2 - N \Rightarrow$$

$$\hat{q}_0 \leq (\hat{q}_0)_{\max} := \frac{\chi(P)}{24} = \frac{P^3 + c_2 \cdot P}{24}$$

Large $P \Rightarrow$

$\Rightarrow$ a finite (but large) set of Polar States

$$0 < \hat{q}_0 \leq (\hat{q}_0)_{\max}$$

Macroscopic Polar States

Attractor formula

$$S(\Gamma) = 2\pi\sqrt{-\frac{1}{6}\hat{q}_0\chi(P)}$$

Polar states are realized as Denef's multi-centered solutions

Extreme *Polar States:* $\epsilon = \frac{(\hat{q}_0)_{\mathrm{max}} - \hat{q}_0}{(\hat{q}_0)_{\mathrm{max}}} \ll 1$

These are $1D6 + 1\overline{D6}$ boundstates with a "gas" of $D2D0$ particles:

Sketch of Derivation of OSV

- Fareytail

$$Z_{D4D2D0} = H(\tau) = \Sigma_{A \in SL(2,\mathbb{Z})} H^{\rm polar}(A\tau)$$

- (Extreme) Polar States Split as $D6\overline{D6}$:

$$H^{\rm polar} = Z^{Polar}_{D4D2D0} = Z^{\epsilon}_{D6\overline{D6}}(t_{ms}) + ET(\epsilon)$$

(for $\epsilon \ll 1$)

- Dilute Gas Approximation + "Swing state conjecture":

$$Z^{\epsilon}_{D6\overline{D6}}(t_{ms}) = Z^{\epsilon}_{D6D2D0}(t_{ms}) Z^{\epsilon}_{\overline{D6D2D0}}(t_{ms})$$

$$Z^{\epsilon}_{D6D2D0}(t_{ms}) = Z^{\epsilon}_{DT} \qquad \epsilon \sim |P|^{-1} \qquad P \to \infty$$

- DT =GW Conjecture: $Z_{DT} = Z_{GW} := Z_{top}$

Limitations on the Derivation

The derivation crucially depends on using only the extreme polar states

the derivation is only valid at LARGE coupling:

$$g_{top} \sim \sqrt{-\frac{\hat{q}_0}{P^3}}$$

$$\Gamma = P \oplus Q \oplus q_0 \to \lambda\Gamma \Rightarrow g_{top} \to g_{top}/\lambda$$

$$\text{Barely Polar States } 0 < \hat{q}_0 \sim \mathcal{O}(1)$$

<u>These states can have large entropy!</u>

Entropy Enigma

For suitable Q, q_0 $\exists$ splits

$$\lambda\Gamma = \Gamma_1^\lambda + \Gamma_2^\lambda$$

$$\Gamma_1^\lambda = r \oplus \frac{\lambda}{2} P \oplus Q_1(\lambda) \oplus \frac{\lambda}{2} q_0$$

$$\Gamma_2^\lambda = -r \oplus \frac{\lambda}{2} P \oplus Q_2(\lambda) \oplus \frac{\lambda}{2} q_0$$

$$S_{2-center} = S(\Gamma_1^\lambda) + S(\Gamma_2^\lambda) \sim (\lambda P)^3/r \sim \lambda^3$$

BUT $S(\lambda\Gamma) \sim \lambda^2$! ➡

2-Centered Solution Dominates the Entropy!

Magical Cancellations?

$$\log|\Omega(\lambda\Gamma)_\infty| \overset{?}{\sim} \lambda^3$$

Contradicts OSV…

… and even black hole dominance of the asymptotic degeneracy of states!

While we found contributions to $\Omega(\lambda\Gamma)_\infty$ of order e^{λ^3}, we cannot rigorously exclude cancellations, bringing it down to order e^{λ^2}.

Closely related question:

$$k := \lim_{\lambda\to\infty} \frac{\log\log|N_{DT}(\lambda^2\beta, \lambda^3 n)|}{\log\lambda}$$

- k=3 indicates the entropy enigma, k=2 suggests there are magical cancellations…
- Huang,Klemm,Marino,Tavanfar find tentative evidence for k=2, and not k=3!
- *The issue is open and important.*

Degeneracy Dichotomy

Either *there are no magical cancellations, and we have the entropy enigma,*

or, *there, are magical cancellations. In that case we must worry about*

$$\dim \mathcal{H}(\Gamma; t) \ \text{vs.} \ \Omega(\Gamma; t)$$

- Physically, the dimension determines the entropy.
- All successful microstate entropy computations have used the index.
- We expect in the full theory dimension=index.

If indeed $\log |\Omega(\lambda\Gamma)_\infty| \sim \lambda^2$ then we expect a spectrum of the form:

$$E = 0 : \qquad \sim \exp[\lambda^2] \ \text{states}$$
$$E \sim e^{-1/g_s} : \qquad \sim \exp[\lambda^3] \ \text{states}$$

``landscape of metastable states''

Remark on the nonperturbative topological string

•One interesting point of OSV was the promise of a nonperturbative definition of the topological string.

•The Dondaldson-Thomas product formula natrually splits as a spin zero and positive spin factor:

$$\mathcal{Z}'_{DT}(u,v) = \mathcal{Z}'^{,r=0}_{DT}(u,v)\,\mathcal{Z}'^{,r>0}_{DT}(u,v) \tag{1}$$

$$\mathcal{Z}'^{,r=0}_{DT}(u,v) = \prod_{Q>0,k>0}(1-(-u)^k v^Q)^{kn^0_Q} \tag{2}$$

$$\mathcal{Z}'^{,r>0}_{DT}(u,v) = \prod_{Q>0,r>0}\prod_{\ell=0}^{2r-2}\left(1-(-u)^{r-\ell-1}v^Q\right)^{(-1)^{r+\ell}\binom{2r-2}{\ell}n^r_Q}. \tag{3}$$

Physical Interpretation: $Z_{DT} = Z_{HALO} \times Z_{CORE}$

- Z_{halo} is convergent for sufficiently large Kähler classes.
- The product Z_{core} is never convergent (probably asymptotic)
- Nevertheless! If $k = 2$ then $Z_{DT} = \sum_{\beta,n} N_{DT}(\beta, n) u^n v^\beta$ might converge: If so – defines Z_{top} nonperturbatively.
- If $k > 2$ then Z_{DT} cannot converge.
- However, $Z^{\rm polar}_{D4D2D0}$ provides a nonperturbative version of $|Z_{\rm top}|^2$.
- This suggests: Choose a P and sum over $\Gamma(\beta, n)$ which "fit" into a D4D2D0 boundstate of charge P. This defines a finite $Z_{\rm top}(P)$ such that

$$\lim_{P\to\infty} Z_{\rm top}(P) = Z_{top}$$

Concluding Riddle:

Why did the BPS state cross the wall?

- We want to understand black hole entropy

  - We found lots of ``irrelevant'' stuff – Halos, Multi-Centered Core states, Swing States, *…*

    - Single-centered black holes always cross the wall.

      - Multi-centered solutions might or might not –but the multicentered ``scaling solutions'' which cross the wall have macroscopic entropy.)

    - So we really want to count the states which cross the wall…

- So we need an answer to our riddle at the microscopic level!!

Related References:

G. Lopes Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, "Black hole partition functions and duality," 0601108.

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