Marginal Stability and $\mathcal{N} = 4$ Dyon Spectrum

Collaborators:

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Our goal:

1. Find exact formula for the degeneracy of quarter BPS dyons in a class of $\mathcal{N} = 4$ supersymmetric string theories.

2. Compare the result with black hole entropy including higher derivative corrections.

By degeneracy we shall mean the index that counts:

No. of bosonic supermultiplets - No. of fermionic supermultiplets

CHL models based on $\mathbb{Z}_N$ orbifolds

Choudhury, Hockney, Lykken

1. Begin with heterotic string theory on

$$T^4 \times S^1 \times \widehat{S}^1$$

   T^4: A four torus

   S^1, $\widehat{S}^1$: two circles with period 2π

2. Take the orbifold by a $\mathbb{Z}_N$ group generated by $2\pi/N$ shift along S^1 + an order N internal symmetry preserving $\mathcal{N}=4$ supersymmetry.

Dual description

1. Begin with type IIB string theory on

$$K3 \times S^1 \times \widetilde{S}^1$$

2. Take the orbifold by a $\mathbb{Z}_N$ group generated by $2\pi/N$ shift along S^1 + an appropriate order N internal symmetry of type IIB string theory on $K3$.

The resulting theory is $\mathcal{N} = 4$ supersymmetric.

Special choices of N:

$$N = 1, 2, 3, 5, 7$$

$N = 1$: heterotic string theory on T^6.

For these theories the rank of the gauge group is

$$r = 2k + 8, \qquad k = \frac{24}{N+1} - 2$$

For $N = 1$ we have $k = 10$ and $r = 28$.

Although we shall focus on these theories, the analysis may be generalized for

1. other values of N,

2. $\mathcal{N}=4$ supersymmetric asymmetric $\mathbb{Z}_N$ orbifolds of type IIA on $T^4 \times S^1 \times \widehat{S}^1$ – with all supersymmetries coming from the right-moving sector

States of this theory are characterized by (r dimensional) electric charge vector Q and magnetic charge vector P.

$$Define: \qquad L = \begin{pmatrix} I_6 & \\ & -I_{r-6} \end{pmatrix}$$

Heterotic T-duality transformation is generated by a set of matrices Ω satisfying $\Omega L \Omega^T = L$ and preserving the charge lattice.

$$Q \to \Omega Q, \qquad P \to \Omega P$$

T-duality invariants

$$P^2 = P^T L P, \qquad Q^2 = Q^T L Q, \qquad P \cdot Q = P^T L Q$$

The heterotic S-duality group of this theory can be found by studying the T-duality group in type IIB description. Vafa, Witten

Result: It is $\Gamma_1(N)$ subgroup of SL(2,Z), consisting of matrices of the form:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 1$$

$$a, b, c, d \in \mathbb{Z}, \quad a, d = 1 \text{ mod } N, \quad c = 0 \text{ mod } N$$

$$\begin{pmatrix} Q \\ P \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix}$$

Our goal: To find the degeneracy $d(Q, P)$ of dyons of charge (Q, P)

For the toroidal compactification a formula for $d(Q, P)$ was proposed by Dijkgraaf, Verlinde, Verlinde.

Additional arguments for this formula were provided by Shih, Strominger, Yin and by Gaiotto.

Our goal will be to

• generalize the proposal to $\mathbb{Z}_N$ CHL string theories

• prove the proposal.

We shall describe the computation for a specific class of (Q, P).

Description in IIB on $K3 \times S^1 \times \widetilde{S}^1/\mathbb{Z}_N$:

1) Q_5 D5-brane wrapped on $K3 \times S^1$

2) Q_1 D1-branes wrapped on S^1

3) $-k/N$ units of momentum along S^1

4) J units of momentum along $\widetilde{S}^1$

5) One Kaluza-Klein monopole along $\widetilde{S}^1$

– BMPV black hole at the center of Taub-NUT

After translated to the heterotic description, this gives

$$P^2 = 2Q_5(Q_1 - Q_5), \quad Q^2 = 2k/N, \quad Q \cdot P = J$$

Using explicit computation / T-duality symmetry, the analysis can be generalized to more general charge vectors as long as Q corresponds to electric charge carried by a twisted sector state.

$d(Q, P)$, expressed in terms of P^2, Q^2 and $Q \cdot P$, remains the same.

In the weakly coupled type IIB description the low energy dynamics of the system is described by three weakly interacting pieces:

1) The closed string excitations around the Kaluza-Klein monopole

2) The dynamics of the D1-D5 center of mass coordinate in the Kaluza-Klein monopole background

3) The relative motion between the D1 and the D5-brane

Taking the coupling $\to 0$ limit we can make the three pieces non-interacting.

The generating function of the spectrum of BPS states is given by the product of the generating function of each of these three different systems.

Note: Individual pieces can be interacting.

e.g. D1-D5 system binds strongly to the Kaluza-Klein monopole.

Result for the generating function:

$$1/\Phi_k(\rho,\sigma,v)$$

Φ_k is a known function.

$-$ transforms as a modular form of weight k of a subgroup of the modular group of genus two Riemann surfaces.

Define $g(m,n,p)$ through

$$\frac{1}{\Phi_k(\rho,\sigma,v)} = \sum_{\substack{m,n,p \\ m\ge -1, n\ge -1/N}} e^{2\pi i(m\rho+n\sigma+pv)} g(m,n,p)\,.$$

Then

$$d(Q,P) = g\left(\frac{1}{2}P^2, \frac{1}{2}Q^2, Q\cdot P\right)$$

$$\frac{1}{\Phi_k(\rho,\sigma,v)} = \sum_{\substack{m,n,p \\ m\geq -1, n\geq -1/N}} e^{2\pi i(m\rho+n\sigma+pv)} g(m,n,p)\,.$$

$1/\Phi_k$ can be expanded in a power series in positive powers of $e^{2\pi iv}$ or negative powers of $e^{2\pi iv}$.

Which one gives the correct $g(m,n,p)$?

Will be discussed later.

Equivalent description

$$d(Q,P) = \frac{1}{N}\int_{\mathcal{C}} d\rho d\sigma dv \frac{1}{\Phi_k(\rho,\sigma,v)} \exp\left[-i\pi(\rho P^2 + \sigma Q^2 + 2vQ\cdot P)\right],$$

ρ, σ, v: complex parameters

$\mathcal{C}$: a three real dimensional subspace:

$0 \leq Re\,\rho \leq 1, \quad 0 \leq Re\,\sigma \leq N, \quad 0 \leq Re\,v \leq 1\,.$

$$Im\,\rho = M_1, \quad Im\,\sigma = M_2, \quad Im\,v = M_3,$$

$$M_1, M_2 >> |M_3| >> 0$$

Sign of M_3 is related to whether we expand $1/\Phi_k$ in positive or negative powers of $e^{2\pi iv}$.

Walls of marginal stability

Moduli space of $\mathcal{N} = 4$ supersymmetric string theory has walls of marginal stability.

– codimension 1 supspaces on which a quarter BPS dyon becomes marginally unstable against decay into a pair of **half BPS states**.

Across these lines of marginal stability the dyon spectrum can change.

Our results are valid for weakly coupled type IIB string theory.

— corresponds to some fixed region of the moduli space.

How does the degeneracy change as we move away from this region crossing the walls of marginal stability?

Results for walls of marginal stability

For a fixed charge vector (Q, P) there are many walls corresponding to the possiblity of decay into various pairs.

$$m(Q,P) = m(Q_1,P_1) + m(Q-Q_1, P-P_1)$$

$$Q_1 \parallel P_1, \qquad (Q-Q_1) \parallel (P-P_1)$$

– codimension 1 subspaces of asymptotic moduli space.

Let $a + iS$ denote the heterotic axion-dilaton modulus parametrizing $SL(2,R)/U(1)$.

1. For fixed values of the other moduli, the walls of marginal stability are **circles** or **straight lines** in the $a + iS$ plane.

2. These curves never intersect in the interior of upper half plane.

3. They can intersect on the real axis but only at **rational points**.

$\rightarrow$ different domains bounded by walls of marginal stability have vertices at rational points or $i\infty$.

Results for the degeneracy

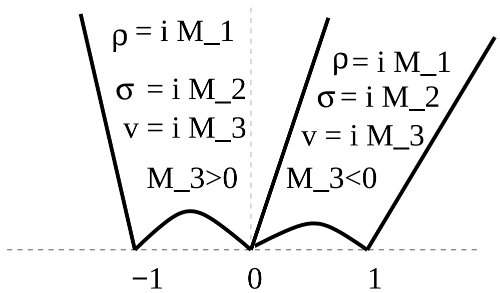
1. The degeneracies in different domains are given by the same integral formula, but with different integration contours $\mathcal{C}$.

2. As we deform one contour into another we pick up residues from the poles of $1/\Phi_k$.

– reflects change in the degeneracy across the walls of marginal stablity.

3. The change across a wall is proportional to the product of the degeneracies of a pair of half BPS dyons into which the original dyon could decay on the particular wall.

Example: $N = 1$ case, ı.e. heterotic on T^6



This jump is due to a jump in the spectrum in susy quantum mechanics describing D1-D5 centre of mass motion in KK monopole background Pope; Gauntlett, Kim, Park, Yi

In other domains we have different choices of the three dimensional integration contour $\mathcal{C}$.

Comparison with black hole entropy for large charges

$$d(Q,P) = \frac{1}{N}\int_{\mathcal{C}} d\rho d\sigma dv \frac{1}{\Phi_k(\rho,\sigma,v)} \exp\left[-i\pi(\rho P^2 + \sigma Q^2 + 2vQ\cdot P)\right] ,$$

a) Do the v integral by picking up residues from the poles of $1/\Phi_k$.

Result:

$$d(Q,P) = \int d\rho d\sigma e^{-F(\rho,\sigma)}$$

for some function $F(\rho,\sigma)$.

b) Do the ρ and σ integral using saddle point approximation.

For this we can treat

$$d(Q,P) = \int d\rho d\sigma e^{-F(\rho,\sigma)}$$

as we would treat a path integral and develop a Feynman diagram approach for evaluating the integral.

$\rightarrow$ 1PI effective action $-\Gamma(\rho,\sigma)$

ln $d(Q,P)$ is the value of $\Gamma(\rho,\sigma)$ at its extremum.

$\Gamma(\rho, \sigma)$ can be calculated using Feynman diagram expansion.

Loop expansion parameter: Inverse charge

Final statistical entropy to 'one loop' order, obtained by extremizing Γ, agrees with the black hole entropy after taking into account the effect of the Gauss-Bonnet term in the effective action.

Both sides involve complicated expressions involving Dedekind η function and the match is highly non-trivial.

$$\Gamma = \frac{\pi}{2}\Big[\left(\frac{Q^2}{S}+\frac{P^2}{S}(S^2+a^2)-2\,\frac{a}{S}\,Q\cdot P\right) + 128\,\pi\,\psi_k(a,S)\Big] + \mathcal{O}(Q^{-2},P^{-2})$$

$$\rho = i/(2NS), \qquad \sigma = iN(a^2+S^2)/(2S)$$

For $\mathbb{Z}_N$ orbifolds with $N = 1,2,3,5,7$

$$\psi_k(a,S) = -\frac{1}{64\pi^2}\Big((k+2)\ln S + \ln f^{(k)}(a+iS) + \ln f^{(k)}(-a+iS)\Big)$$

$$k = \frac{24}{N+1} - 2, \qquad f^{(k)}(\tau) = \eta(\tau)^{k+2}\,\eta(N\tau)^{k+2}$$

What is the role of the walls of marginal stability in this comparison?

In the large charge limit the change in the degeneracy across walls of marginal stability is exponentially suppressed compared to the leading term.

Thus we would expect that the asymptotic expansion of the black hole entropy should not change as we move across the walls of marginal stability.

Consistent with the generalized attractor mechanism.

However there is still an exponentially suppressed change in $d(Q,P)$ across walls of marginal stability.

Can we explain this on the black hole side?

It can be explained by taking into account the contribution from two centered small black holes.

Typically as we cross a wall of marginal stability, a particular 2-centered black hole (dis)-appear, causing a change in entropy.

Denef; Denef, Moore

This change is precisely in accordance with the prediction of the exact formula for $d(Q,P)$.

For the future:

Given the exact degeneracy formula, we can now expand the answer for the statistical entropy to include higher powers of inverse charges as well as exponentially suppressed terms.

Question: How does it compare with the black hole entropy?

$$\Phi_k(\rho,\sigma,v) = \exp\left(2\pi i\left(\frac{1}{N}\sigma+\rho+v\right)\right)$$

$$\prod_{r=0}^{N-1}\prod_{\substack{l,b\in Z, k'\in Z+\frac{r}{N}\\ k',l,b>0}}\left\{1-\exp(2\pi i(k'\sigma+l\rho+bv))\right\}^{e(k',l,b)}$$

$$e(k',l,b) = \sum_{s=0}^{N-1} e^{-2\pi i l s/N}\, c^{(r,s)}(4lk'-b^2)$$

$k',l,b > 0$: $(k' > 0, l \geq 0, b \in \mathbb{Z})$ or $(k' = 0, l > 0, b \in \mathbb{Z})$ or $(k' = 0, l = 0, b < 0)$

$c^{r,s}(n)$: known coefficients, given in terms of jacobi ϑ-functions and Dedekind η-functions.

Definition of $c^{(r,s)}(n)$:

$$A = \left[\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2} + \frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2} + \frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2}\right]$$

$$B = \eta(\tau)^{-6}\vartheta_1(\tau,z)^2$$

$$E_N(\tau) = \frac{12i}{\pi(N-1)}\,\partial_\tau\left[\ln\eta(\tau) - \ln\eta(N\tau)\right]$$

$$F^{(0,0)}(\tau,z) = \frac{8}{N}A\,,$$

$$F^{(0,s)}(\tau,z) = \frac{8}{N(N+1)}\,A - \frac{2}{N+1}\,B\,E_N(\tau)\,,$$

$$F^{(r,rk)}(\tau,z) = \frac{8}{N(N+1)}\,A + \frac{2}{N(N+1)}\,E_N\left(\frac{\tau+k}{N}\right)\,B\,,$$

for $1 \le s \le (N-1)$, $1 \le r \le (N-1)$, $0 \le k \le (N-1)$,

$$F^{(r,s)}(\tau,z) = \sum_{b\in\mathbf{Z},n} c^{(r,s)}(4n-b^2)\,q^n\,e^{2\pi izb}$$