Motivations

We will be interested in gluon scattering amplitudes of planar $N_l c$ super Yang-Mills. This can give non trivial information about more realistic theories but is more tractable. Perturbative computations are easier with higher loop computations possible. A proposal for all loops is possible. The strong coupling regime can be studied by means of the gauge-string duality through a weakly coupled string sigma model.
Motivations

We will be interested in gluon scattering amplitudes of planar $\mathcal{N} = 4$ super Yang-Mills.

Motivation: It can give non trivial information about more realistic theories but is more tractable.

- Perturbative computations are easier. Higher loop computations are possible $\rightarrow$ proposal for all loops (MHV) $n$-point amplitudes.
- The strong coupling regime can be studied, by means of the gauge/string duality, through a weakly coupled string sigma model.
Aim of these project

Prescription for computing scattering amplitudes of planar $\mathcal{N} = 4$ super Yang-Mills at strong coupling by using the $AdS/CFT$ correspondence.
Gauge theory amplitudes, Dixon’s talk

\[ A_n^{L,Full} \sim \sum_\rho Tr(T^{a_\rho(1)}...T^{a_\rho(n)})A_n^{(L)}(\rho(1), ..., \rho(2)) \]

- Leading $N_c$ color ordered $n$–points amplitude at $L$ loops: $A_n^{(L)}$
- The amplitudes are IR divergent.
- Dimensional regularization $D = 4 - 2\epsilon \rightarrow A_n^{(L)}(\epsilon) = 1/\epsilon^{2L} + ...$
- Focus on MHV amplitudes and scale out the tree amplitude $M_n^{(L)}(\epsilon) = A_n^{(L)}/A_n^{(0)}$.

BDS proposal for all loops MHV amplitudes (Bern, Dixon, Smirnov)

\[ \log \mathcal{M}_n = \sum_{i=1}^{n} \left( -\frac{1}{8\epsilon^2} f^{(-2)} \left( \frac{\lambda \mu^{2\epsilon}}{s_{i,i+1}^\epsilon} \right) - \frac{1}{4\epsilon} g^{(-1)} \left( \frac{\lambda \mu^{2\epsilon}}{s_{i,i+1}^\epsilon} \right) \right) + f(\lambda) \text{Fin}_n^{(1)}(k). \]

\[ f(\lambda), g(\lambda) \rightarrow \text{cusp/collinear anomalous dimension.} \]
We will study scattering amplitudes at strong coupling by using the \textit{AdS/CFT} duality.

- Set up the computation: Use a \textit{D-brane} as IR cut-off.
- Actual computations: Dimensional regularization.
String theory set up

\[ ds^2 = R^2 \frac{dx_{3+1}^2 + dz^2}{z^2} \]

- Place a D-brane extended along \( x_{3+1} \) and located at \( z_{IR} \gg R \).
  - The asymptotic states are open strings ending on the D-brane.
  - Consider the scattering of these open strings.
  - \( k_{pr} = k \frac{z_{IR}}{R} \) is very large: fixed angle and very high momentum.
  - Intuition from flat space (Gross and Mende): Amplitude is dominated by a saddle point.
World-sheet with the topology of a disk with vertex operator insertions (corresponding to external states)

- Near each vertex operator, the momentum of the external state fixes the form of the solution.
- The boundary of the world-sheet sits at $z = z_{IR}$.
- T-duality in $x^\mu$ directions followed by a change of coordinates $r = R^2/z \rightarrow$ we end up again with $AdS_5$!

$$ds^2 = R^2 \frac{dx^2_{3+1} + dz^2}{z^2} \quad \rightarrow \quad ds^2 = R^2 \frac{dy^2_{3+1} + dr^2}{r^2}$$
The world-sheet boundary is located at \( r = R^2/z_{IR} \) and is a particular line constructed as follows...

- For each particle with momentum \( k^\mu \) draw a segment joining two points separated by \( \Delta y^\mu = 2\pi k^\mu \)
- Concatenate the segments according to the ordering of the insertions on the disk (particular color ordering).
  - As \( z_{IR} \rightarrow \infty \) the boundary of the world-sheet moves to \( r = 0 \).
  - Exactly same computation as when computing the expectation value of a Wilson-Loop given by a sequence of light-like segments!
Prescription

- $A_n$: Leading exponential behavior of the $n$–point scattering amplitude.
- $A_{\text{min}}(k_1^\mu, k_2^\mu, \ldots, k_n^\mu)$: Area of a minimal surface that ends on a sequence of light-like segments on the boundary.

$$A_n \sim e^{-\frac{\sqrt{\lambda}}{2\pi} A_{\text{min}}}$$

- Prefactors are subleading in $1/\sqrt{\lambda}$, and we don’t compute them.
- In particular our computation is blind to helicity (and hence works also for non MHV)
Consider $k_1 + k_3 \rightarrow k_2 + k_4$

- The simplest case $s = t$.

Need to find the minimal surface ending on such sequence of light-like segments

$$r(y_1, y_2) = \sqrt{(1 - y_1^2)(1 - y_2^2)}$$

$$y_0 = y_1 y_2$$

- The "dual" AdS space possesses isometries $SO(2, 4)$.
- This dual conformal symmetry takes this solution to the most general one!
Let’s compute the area...

- Small problem: The area diverges!
- Dimensional reduction scheme: Start with $\mathcal{N} = 1$ in $D=10$ and go down to $D = 4 - 2\epsilon$.
- For integer $D$ this is exactly the low energy theory living on $Dp$–branes ($p = D - 1$)

**Gravity dual**

\[
ds^2 = h^{-1/2} dx^2_D + h^{1/2} \left( dr^2 + r^2 d\Omega^2_{9-D} \right), \quad h = \frac{c_D \lambda_D}{r^{8-D}} \\
\lambda_D = \frac{\lambda \mu^{2\epsilon}}{(4\pi e^{-\gamma})^\epsilon} \quad c_D = 2^{4\epsilon} \pi^{3\epsilon} \Gamma(2 + \epsilon)
\]
T-dual coordinates

\[ ds^2 = \sqrt{\lambda_D c_D} \left( \frac{dy_D^2 + dr^2}{r^2 + \epsilon} \right) \rightarrow S_\epsilon = \frac{\sqrt{\lambda_D c_D}}{2\pi} \int \frac{L_{\epsilon=0}}{r^\epsilon} \]

- Presence of \( \epsilon \) will make the integrals convergent.
- The eoms will depend on \( \epsilon \) but if we plug the original solution into the new action, the answer is accurate enough.
- Plugging everything into the action...

\[ S \approx -\frac{\sqrt{\lambda_D c_D}}{2\pi a^\epsilon} \quad _2F_1 \left( \frac{1}{2}, -\frac{\epsilon}{2}, \frac{1 - \epsilon}{2}; b^2 \right) + O(\epsilon) \]

- Just expand in powers of \( \epsilon \)...

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Scattering Amplitudes via AdS/CFT
Final answer

\[ \mathcal{A} = e^{iS} = \exp \left[ i S_{\text{div}} + \frac{\sqrt{\lambda}}{8\pi} \left( \log \frac{s}{t} \right)^2 + \tilde{C} \right] \]

\[ S_{\text{div}} = 2S_{\text{div},s} + 2S_{\text{div},t} \]

\[ S_{\text{div},s} = -\frac{1}{\epsilon^2} \frac{1}{2\pi} \sqrt{\frac{\lambda \mu^{2\epsilon}}{(-s)^\epsilon}} - \frac{1}{\epsilon} \frac{1}{4\pi} (1 - \log 2) \sqrt{\frac{\lambda \mu^{2\epsilon}}{(-s)^\epsilon}} \]

- Should be compared to the field theory answer

\[ \mathcal{A} \sim (A_{\text{div},s})^2 (A_{\text{div},t})^2 \exp \left\{ \frac{f(\lambda)}{8} (\ln s/t)^2 + \text{const} \right\} \]

\[ A_{\text{div},s} = \exp \left\{ -\frac{1}{8\epsilon^2} f(-2) \left( \frac{\lambda \mu^{2\epsilon}}{s^\epsilon} \right) - \frac{1}{4\epsilon} g(-1) \left( \frac{\lambda \mu^{2\epsilon}}{s^\epsilon} \right) \right\} \]
• $SO(2, 4)$ transformations fixed somehow the kinematical dependence of the finite piece.

• This dual conformal symmetry constrains the form of the amplitude

**Dual Ward identity**

$$\mathcal{O}_k A = 0 \quad \rightarrow \quad \mathcal{O}_k \text{Fin} = -\mathcal{O}_k \text{Div}$$

For $n = 4, 5$ the solution is unique and agrees with BDS! for $n = 6$ there is some freedom.

• Also perturbatively and even dual super-conformal symmetry on NMHV (Sokatchev’s talk).

• Dual super-conformal symmetry present for all values of the coupling! (Berkovits’s talk)
What about the BDS ansatz?

- Symmetries "protect" BDS from corrections, we need to consider \( n > 5 \). What about \( n = \infty \)?

We choose a zig-zag configuration that approximates the rectangular Wilson loop.

\[
\log < W_{\text{rect}}^{\text{weak}} > = \frac{\lambda}{8\pi} \frac{T}{L}
\]

\[
\log < W_{\text{rect}}^{\text{strong}} > = \frac{\sqrt{\lambda}}{4} \frac{4\pi^2}{\Gamma(\frac{1}{4})^4} \frac{T}{L}
\]

- While BDS \( \rightarrow \) \( \log < W_{\text{rect}}^{\text{strong}} > = \frac{\sqrt{\lambda}}{4} \frac{T}{L} \). Impressive explicit computations showed that indeed BDS fails for 6 gluons at two loops! (Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich)
This computation shows a relation between Wilson loops and scattering amplitudes.

**Scattering amplitudes vs. WL**

\[ \mathcal{M}(k_1, ..., k_n) \approx \langle W(x_1, ..., x_n) \rangle, \quad k_i = x_{i+1} - x_i \]

This relation holds also at weak coupling! (for MHV amplitudes)

- Four legs at one loop (Drummond, Korchemsky, Sokatchev)
- \( n \) legs at one loop (Brandhuber, Heslop, Travaglini)
- Up to six legs at two loops (Drummond, Henn, Korchemsky, Sokatchev)
Other processes (something like $\text{meson/photon} \rightarrow q + \bar{q}$)

- Brane $I$ extending along $(x^0, x^1)$ at $Z = Z_{IR}$.
- Brane $II$ extending along $(x^0, x^1, z)$.
- $\text{meson} \rightarrow (II, II)$, quarks $\rightarrow (I, II)$.

- $(0, \kappa) \rightarrow (\kappa/2, \kappa/2) + (-\kappa/2, \kappa/2)$
- After T-duality, a triangle in the $(y^0, y^1)$ plane with boundary conditions for $r$
- $r = 0$ in the red lines (quarks), $r = \infty$ in the blue line (meson).
Also possible to consider \( \textit{meson} \rightarrow q + \bar{q} + \textit{gluons} \).

Other processes like singlets into gluons.

We could not find the solutions but the singular behavior can be understood

\[
g_{\text{quark}}(\lambda) = \frac{\sqrt{\lambda}}{4\pi} (1 - 3 \log 2)
\]
Consider a massless gauge theory

**High spin, twist two operators**

\[ \mathcal{O}_S = \text{Tr}(\phi D^S \phi), \quad S \gg 1 \quad \Rightarrow \quad \Delta \mathcal{O}_S = f(\lambda) \log S - B(\lambda) + ... \]

**Scattering amplitudes**

\[ \log A = \frac{f(\lambda)}{\epsilon^2} + \frac{g(\lambda)}{\epsilon} + ... \]

- Is there any relation between \( B(\lambda) \) and \( g(\lambda) \)?
• They are not equal but $g_R - B_R = C_R X$, where $X$ is a universal function (Dixon, Magnea, Sterman).

• All this quantities can be computed at strong coupling in planar $\mathcal{N} = 4$ SYM for gluons (adjoint) and quarks (fundamental).

$$B_{gg} = \frac{\sqrt{\lambda}}{\pi} \left( \log \left( \frac{\sqrt{\lambda}}{2\pi} \right) + 1 - 2 \log 2 \right)$$

$$B_{qq} = \frac{\sqrt{\lambda}}{2\pi} \left( \log \left( \frac{\sqrt{\lambda}}{2\pi} \right) + 1 - 3 \log 2 \right)$$

• Universality seems to hold at strong coupling!
What needs to be done?

- Try to make explicit computations for $n > 4$, e.g. $n = 6$ is a good one.
- Subleading corrections in $1/\sqrt{\lambda}$? Information about helicity of the particles, etc.
- Gross and Mende computed higher genus amplitudes (in flat space) using similar ideas, can we do the same?
- Can we repeat the computation in other backgrounds?
- Deeper relation between Wilson loops and scattering amplitudes?
- Some powerful alternative to BDS?
Conclusions and Outlook

- A lot of structure (some discovered and hopefully more waiting to be discovered) behind scattering amplitudes of planar MSYM.

- For $n = 4, 5$, we think we know them to all values of the coupling!

- We haven’t assume/use at all the machinery of integrability.

Talk by someone at strings 2009:

- Expression for all planar MSYM amplitudes at all values of the coupling.