Maximal Supersymmetry and Duality

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Interplay of maximal supersymmetry (N=8 in four dimensions) and duality symmetries of M/String theory leads to powerful constraints.

How powerful?

Outline:

1) **Maximal supersymmetry and low energy expansion**
   - Higher derivative interactions – important at high curvature.
   - e.g. IIB in d=10. Differential equations on moduli space;
   - Exact non-perturbative coefficients.

2) **Four-graviton scattering amplitude**
   - Duality with multi-loop Feynman diagrams of
   - eleven-dimensional supergravity compactified on two-torus.

3) **Connections with maximal supergravity**
   - UV divergence properties.
1) **Maximal supersymmetry and low energy expansion**

- **Closed Superstring/M-Theory** reduces at low energy to maximal supergravity - IIA or IIB in d=10 and N=8 in d=4 - plus higher derivative terms.

- **Moduli-dependent coefficients** encode exact behaviour of higher-dimension terms.
  Perturbative + non-perturbative dependence on couplings

  e.g. 4-graviton amplitude - effective action

  \[
  \frac{1}{\alpha'^4} \int d^{10}x \sqrt{G} \left( e^{-2\phi} R + \alpha'^3 \mathcal{F}(\phi, \ldots) R^4 + \ldots \right)
  \]

  Moduli-dependent coefficients

String pert. expansion \[ \mathcal{F}(\phi, \ldots) = \sum_{h=0}^{\infty} e^{2(h-1)\phi} f_h + \text{nonpert.} \]

\(\alpha'\) series important for high curvatures
**Type IIB supergravity**

**Fields:**  
**coset scalars** \( \Omega = \Omega_1 + i\Omega_2 \)  \( \Omega_2 \equiv e^{-\phi} = g^{-1}_B \)  
\[ \partial_\mu \Omega/\Omega_2 \]  \[ \Omega_2^{-\frac{3}{2}} (F_{\mu \nu \rho} + i\Omega_2 H_{\mu \nu \rho}) \]  
\( P_\mu, \lambda, \ G_{\mu \nu \rho}, \ \psi_\mu, \ F_5, \ g_{\mu \nu} \)

**Dilaton, dilatino, 3-form, gravitino, 5-form, metric**

\( u_\Phi : \ -2, \ -3/2, \ -1, \ -1/2, \ 0, \ 0 \)

**U(1) charges** in \( \text{SL}(2,\mathbb{R})/\text{U}(1) \)

**Coset becomes** \( \text{SL}(2,\mathbb{Z})/\text{SL}(2,\mathbb{R})/\text{U}(1) \) in string theory

\[ \implies \text{Pattern of} \ u \text{ non-conserving higher-order interactions.} \]
Higher-derivative terms in IIB:

Consider composite operator $\mathcal{P}^{(u)}_{2n+2}$: U(1) charge $u$, dimension $\Delta = 2n + 2$

e.g. $\mathcal{R}^4 \quad u = 0, \quad \Delta = 8$

$\lambda^{16} \quad u = -24, \quad \Delta = 8$

$(G\bar{G})^p \mathcal{R}^4 \quad u = 0, \quad \Delta = 2p + 8$

$SL(2,\mathbb{Z})$ - invariant action:

$$S^{(n)} = \alpha'^{n-4} \sum_{u} \int d^{10}x \mathcal{F}^{(u)}_n (\Omega, \bar{\Omega}) \mathcal{P}^{(-u)}_{2n+2}$$

Index $i$ labels degenerate terms

$\mathcal{F}^{(u)}_n$ has holomorphic and antiholomorphic weights $\mp u/2$.

$$\mathcal{F}^{(u)}_n (\Omega, \bar{\Omega}) \rightarrow \left( \frac{c\bar{\Omega} + d}{c\Omega + d} \right)^{u/2} \mathcal{F}^{(u)}_n (\Omega, \bar{\Omega}) \quad \Omega \rightarrow \frac{a\Omega + b}{c\Omega + d}$$

How is $\mathcal{F}^{(u)}_n$ constrained by supersymmetry??
Consequences of supersymmetry

Invariance of action

\[
\sum_{m=0}^{\infty} \delta^{(m)} \sum_{n=0}^{\infty} S^{(n)} = 0
\]

i.e,

\[
(\delta^{(0)} + \alpha'^3 \delta^{(3)} + \ldots)(S^{(0)} + \alpha'^3 S^{(3)} + \ldots) = 0
\]

On-shell algebra

\[
[\delta, \delta] \Phi = [\delta^{(0)} + \alpha'^3 \delta^{(3)} + \ldots, \delta^{(0)} + \alpha'^3 \delta^{(3)} + \ldots] \Phi
\]

\[
= a \cdot P \Phi + \Phi \text{ eqn. of motion} + \delta_{gauge} \Phi
\]

Strongly constrains the form of \( F^{(u)}_n \), \( \delta^{(m)} \)

Difficult to implement in detail in absence of off-shell superspace formalism.  **Modified torsion constraints.**

Consider general form of component supersymmetry.
Classical IIB supersymmetry transformations

\[ \delta^{(0)} \Omega = 2 \lambda \epsilon \Omega_2 \]

\[ \delta^{(0)} \Phi(u) = \hat{\delta}^{(0)} \Phi(u) + \tilde{\delta}_u^{(0)} \Phi(u) \]

where \( \Phi(u) \) is any field with U(1) charge \( u \) and

\[ \tilde{\delta}_u^{(0)} \Phi(u) = u (\lambda \epsilon - \lambda^* \epsilon^*) \Phi(u) \]

Classical supersymmetry:

\[ \delta^{(0)} S^{(n)} = \alpha'^{n-4} \int d^{10}x \sum_u \left( \mathcal{F}_n(u)^i \hat{\delta}^{(0)} \left( \mathcal{P}_{2n+2}^{(-u)} i \right) \right) \]

\[ -2i \mathcal{D} \mathcal{F}_n(u)^i \lambda \epsilon \mathcal{P}_{2n+2}^{(-u)} i + 2i \mathcal{D} \mathcal{F}_n(u)^i \lambda^* \epsilon^* \mathcal{P}_{2n+2}^{(-u)} i \]

where \( \mathcal{D} = i \Omega_2 \frac{\partial}{\partial \Omega} - \frac{u}{4} \) is modular covariant derivative on charge \( u \).

\[ \mathcal{D} f^{(u)} = f^{(u+1)} \]
Add $\sum_{m} \delta^{(m)} S^{(n-m)}$ terms, and require closure of superalgebra, $[\delta^{(m)}, \delta^{(n)}] \Phi \approx 0$, leads to expression of general form (suppressing superscripts and coefficients)

$$DF_n = F_n + F_{m_1} F_{n-m_1} + F_{m_1} F_{m_2} F_{n-m_1-m_2} + \cdots + F_{m_1} F_{m_1+m_2} \cdots F_{n-m_1-\cdots-m_{n-1}} + \cdots$$

Detailed coefficients need a more complete analysis.

Apply $\bar{D}$ to above equation:

$$\Rightarrow \quad \text{Inhomogeneous Laplace (Poisson) equation}$$

$$\bar{D} DF_n = \bar{D} F_n + \bar{D} (\ldots \ldots \ldots)$$
$$= F_n + F_{m_1} F_{n-m} + \ldots$$
Simple cases can be analyzed in detail:

(a) Simple nondegenerate examples: \((\text{index } i \text{ on } \mathcal{F}^{(u)}_n)\) is redundant

\[
\mathcal{D} \mathcal{F}^{(u)}_n = c_u \mathcal{F}^{(u+2)}_n \quad \bar{\mathcal{D}} \mathcal{F}^{(u+2)}_n = \bar{c}_{u+2} \mathcal{F}^{(u)}_n
\]

i.e. Laplace eigenvalue equation

\[
\bar{\mathcal{D}} \mathcal{D} \mathcal{F}^{(u)}_n = c_u \bar{c}_{u+2} \mathcal{F}^{(u)}_n
\]

e.g. i) U(1) preserving

\[
u = 0, \quad c_0 \bar{c}_2 = s(s - 1) \quad \text{where} \quad n = 2s = \frac{1}{2} \Delta - 1
\]

\[
\nabla^2_{\Omega} = \Omega^2_2 \partial_\Omega \partial_\Omega \quad \Rightarrow \quad \nabla^2_{\Omega} \mathcal{F}^{(0)}_n = s(s - 1) \mathcal{F}^{(0)}_n
\]

Solution is nonholomorphic Eisenstein series
\[ \mathcal{F}_n^{(0)} = E_s = \sum_{(\hat{m}, \hat{n}) \neq (0,0)} \frac{\Omega_2^s}{|\hat{m} + \hat{n}\Omega|^{2s}} \]

Natural SL(2,\mathbb{Z}) generalization of Riemann Zeta Values

\[ \sim 2\zeta(2s)\Omega_2^s + (\ldots)\zeta(2s - 1)\Omega_2^{1-s} + \sum_{k \neq 0} \mu(k, s) \left( e^{2\pi i k \Omega} + c.c. \right) (1 + O(\Omega_2^{-1})) \]

* TREE-level terms *
* GENUS-(s - \frac{1}{2}) term *
* D-INSTANTON terms with pert. corrections *

non-renormalization at higher loops

examples:
\[ E_{\frac{3}{2}} \mathcal{R}^4, \quad E_{\frac{5}{2}} D^4 \mathcal{R}^4 \]

ii) U(1) -violating processes at order n=3:

\[ \mathcal{F}_3^{(u)} = \mathcal{D}^u \mathcal{F}_3^{(0)} = \mathcal{D}^u E_{\frac{3}{2}} \]

examples:
\[ \mathcal{F}_3^{(8)} \mathcal{G}^8, \quad \mathcal{F}_3^{(24)} \lambda^{16} \]
iii) Higher order: \( \mathcal{F}_6^{(0)} \mathcal{D}^6 \mathcal{R}^4 \) \( (u = 0, \ n = 6) \)

\[
(\nabla^2_{\Omega} - 12) \mathcal{F}_6^{(0)} = -6E_{\frac{3}{2}} E_{\frac{3}{2}}
\]

Not (yet) derived purely from supersymmetry but motivated by four-graviton scattering amplitude.

[Other examples in very recent paper by Basu + Sethi.]

(b) Degenerate cases:

In general Laplace eigenvalue equation generalizes to inhomogeneous simultaneous equations:

\[
(\delta_{ij} \bar{\mathcal{D}} \mathcal{D} - \lambda^{(u)}_{n;ij}) \mathcal{F}^{(u)}_n j = \sum_{j,k,m,v} f_{ijk} \mathcal{F}^{(v)}_m j \mathcal{F}^{(u-v)}_n m + \ldots
\]

Lower order source coefficients

Interesting \( SL(2,\mathbb{Z}) \) generalization of Multiple Zeta Values. (Zagier)

Illustrated by four-graviton amplitude.
2) **Four - graviton scattering amplitude**

Tiny subsector of complete theory type II theory derivatives of curvature (zero fluxes, fixed dilaton).

\[ \mathcal{R}^4, \partial^4 \mathcal{R}^4, \ldots, \partial^{2k} \mathcal{R}^4, \ldots \]

linearized Weyl curvatures contracted with familiar sixteen-index tensor

Low-energy expansion of string perturbation theory:

**TREE-LEVEL** (Virasoro-Shapiro Model) - all-orders expansion.

**ONE-LOOP** (Genus-one world-sheet) - recent results in \(d=9, 10\).

**TWO-LOOP** (Genus-two world-sheet) - little explicitly known.

Boundary “data” for non-perturbative structure.
Genus-one amplitude:

\[ I = \int_{F} \frac{d^2 \tau}{\tau_2} F(\tau, \bar{\tau}; s, t, u) \]

Integral of modular function

\[ A_4^{h=1} = \mathcal{R}^4 I(s, t, u) \]

Expansion in powers of

\[ \sigma_2 = s^2 + t^2 + u^2 \]

and

\[ \sigma_3 = s^3 + t^3 + u^3 = 3stu \]

Analytic part - subtract threshold cuts

\[ I^{an} = \frac{\pi \alpha'}{3} + 0\sigma_2 + \frac{\pi \alpha'^6}{3} \zeta(3) \sigma_3 + 0\sigma_2^2 + \alpha'^6 \frac{97}{1080} \zeta(5) \sigma_2 \sigma_3 \]

\[ -\alpha'^{12} \left( \frac{1}{30} \zeta(3)^2 \sigma_2^3 + \frac{61}{1080} \zeta(3)^2 \sigma_3^2 \right) + \ldots \]

- no \( S^2 \) or \( S^4 \) terms

MBG, Russo, Vanhove  arXiv:0801.0322
Compactify on circle radius $r$ \hfill (d=9)

\[ A_{4}^{h=1}(r; s, t) = \frac{\pi}{3} \left[ r + r^{-1} + \sigma_{2} \left( \frac{\zeta(3)}{15} r^{3} + \frac{\zeta(3)}{15} r^{-3} \right) \right. \]
\[ + \sigma_{3} \left( \frac{\zeta(5)}{63} r^{5} + \frac{\zeta(3)}{3} r + \frac{\zeta(3)}{3} r^{-1} + \frac{\zeta(5)}{63} r^{-5} \right) \]
\[ + \sigma_{2}^{2} \left( \frac{\zeta(7)}{315} r^{7} + \frac{2\zeta(3)}{15} r \log(r^2 \lambda_{4}) + \frac{\zeta(5)}{36} r^{-3} + \frac{\zeta(3)^{2}}{315} r^{-5} + \frac{\zeta(7)}{1050} r^{-7} \right) \]
\[ + \sigma_{2} \sigma_{3} \left( \frac{7\zeta(9)}{2970} r^{9} + \frac{\zeta(3)^{2}}{21} r^{3} + \frac{97\zeta(5)}{1080} r + \frac{29\zeta(5)}{135} r^{-1} + O(r^{-3}) \right) \]
\[ + \sigma_{2}^{3} \left( \frac{3\zeta(11)}{8008} r^{11} + \frac{2\zeta(3)\zeta(5)}{525} r^{5} + \frac{11\zeta(5)}{210} r \log(r^2 \lambda_{6}) + \frac{\zeta(3)^{2}}{30} r + \frac{\zeta(3)^{2}}{30} r^{-1} + O(r^{-3}) \right) \]
\[ + \sigma_{3}^{2} \left( \frac{109\zeta(11)}{225225} r^{11} + \frac{8\zeta(3)\zeta(5)}{1575} r^{5} + \frac{\zeta(5)}{15} r \log(r^2 \lambda_{6}) + \frac{61\zeta(3)^{2}}{1080} r + \frac{61\zeta(3)^{2}}{6144} r^{-1} \right. \]
\[ + O(r^{-3}) \bigg] + O(e^{-r}) \]

Intriguing pattern of coefficients
- rational numbers \times products of zeta values.

Relevant to M-theory compactified on $T^2$
What is non-perturbative completion??

\( \text{SL}(2, \mathbb{Z}) - \text{invariant effective IIB action} \quad \text{(string frame)} \)

\[
\alpha' S = \int d^{10}x \sqrt{g} \left( e^{-2\phi} R + \alpha'^3 e^{-\phi/2} E_2^3 R^4 + \alpha'^5 e^{\phi/2} E_5^5 D^4 R^4 \right)
\]

[What is the complete list of \( O(1/\alpha') \) interactions??

- absence of superspace formalism makes things difficult
- exact dependence on \( F_5 \):
  \[
  R^4 \rightarrow \frac{1}{\alpha'} (R + DF_5 + F_5^2)^4
  \]
gives info concerning AdS/CFT plasma viscosity
  (Buchel, Myers, Paulos, Sinha)
- Stretched horizon of stringy black holes]
Higher derivative interactions ??

Clues from M-theory/String Theory duality -

Connections with eleven-dimensional supergravity

Recall: CLASSICALLY:

Eleven-dimensional M-theory on $T^2$ is dual to type II on a circle of radius $r_A = r_B^{-1}$

Torus volume: \[ V = \exp \left( \frac{1}{3} \phi^B \right) r_B^{-\frac{4}{3}} \]

Complex structure: \[ \Omega = \Omega_1 + i \Omega_2 = \text{Complex IIB coupling} \]
\[ \Omega_1 = C^{(0)} = C_9^{(1)}, \quad \Omega_2 = \exp (-\phi^B) = r_A \exp (-\phi^A). \]

Type IIB in d=10: \[ V \to 0 \quad \implies \quad r_B \to \infty \]

Type IIA in d=10: \[ R_{10} \to \infty \quad \implies \quad r_A \to \infty \]
What about quantum effects??

Feynman diagrams \( L \) loops - UV divergent (in 11 dimensions on two-torus)

Regulate, e.g., momentum scale \( \Lambda \).

Naive degree of divergence \( \Lambda^{(d-2)L+2} \).

Actual degree of divergence much less due to overall factor of \( \mathcal{R}^4 \) (eight powers of momentum).

Further powers of \( S, T, U \) as \( L \) increases - lower degree of divergence (see last part of talk).

Subtract divergences with counterterms - unknown coefficients encoding short distance features of \( M \)-theory. Some of these (how many?) are determined by requiring consistency with string perturbation theory.
L=1  One loop in 11 dimensions on $T^2$

Sum of all Feynman diagrams:

Sum over windings of loop around cycles of $T^2$
Winding numbers $\hat{m}, \hat{n}$

$\Lambda^3 \mathcal{V}$  divergence in zero winding number sector $\hat{m} = \hat{n} = 0$
suppressed in limit $\mathcal{V} \to 0$

$\mathcal{V}^{-\frac{1}{2}}$ from non-zero windings
\( L=1 \) One loop in 11 dimensions on \( T^2 \)

Sum of all Feynman diagrams:

\[ \mathcal{R} \sim k k h \]

\[ \mathcal{R}^4 \]

Box diagram of \textbf{SCALAR} field theory

(i) \textbf{LOW ENERGY}  \( S, T, U \rightarrow 0 \)  \( 11\text{-dim. Mandelstam variables} \)
\[ A = \sum_{(\hat{m}, \hat{n}) \neq (0,0)} \frac{\Omega_{2}^{\frac{3}{2}}}{|\hat{m} + \hat{n}\Omega|^3} \nu^{-\frac{1}{2}} \mathcal{R}^4 = E_{\frac{3}{2}} \nu^{-\frac{1}{2}} \mathcal{R}^4 \]
TEN-DIMENSIONAL IIB limit:

\[
\nu \to 0 \quad \frac{1}{\alpha'} e^{-\frac{1}{2} \phi^B} E_{\frac{3}{2}} (\Omega, \bar{\Omega}) \mathcal{R}^4
\]

\(E_{\frac{3}{2}} (\Omega, \bar{\Omega})\) contains TREE - LEVEL and GENUS-ONE string perturbative terms together with non-perturbative D-instantons

(ii) HIGHER ORDERS in \(S, T, U\):

Infinite series of terms in IIA limit: \((r^A \to \infty)\)

\[c_h e^{2(h-1)\phi^A} s^h \mathcal{R}^4\]

finite coefficients - no contributions from higher loops!
Higher-loop 11-dim. sugra on $T^2$

$L=2 \cdot$ TWO LOOPS - factor out overall $S^2 \mathcal{R}^4$

resulting in scalar field theory diagrams

(Bern, Dixon, Dunbar, Perelstein, Rozowsky 1998)

$S^2 \mathcal{R}^4$ \hspace{1cm} $S^2 \mathcal{R}^4$

+ $T$ and $U$ diagrams

Sum over windings of both loops around cycles of $T^2$

Winding numbers $\hat{m}_1, \hat{m}_2, \hat{n}_1, \hat{n}_2$ \hspace{1cm} (MBG, Vanhove 1999)

Use one-loop counterterm for sub-divergences
Evaluation of two-loop integrals

Redefinition of three Schwinger parameters

\[ L_1, L_2, L_3 \rightarrow V, \tau_1, \tau_2 \] (and four vertex positions)

where

\[ \tau_1 = \frac{L_2}{L_1 + L_2}, \quad \tau_2 = \frac{\sqrt{\Delta}}{L_1 + L_2}, \quad V = \sqrt{\Delta} \]

\[ \Delta = L_1L_2 + L_2L_3 + L_3L_1 \]

\[ \tau = \tau_1 + i\tau_2 \] (c.f. genus-one world-sheet)

Integration Domain (a)
= 3 copies of \( SL(2,\mathbb{Z}) \) fund. domain in (b)
Inhomogeneous Laplace equations

\[ A = I(S, T, U) \mathcal{R}^4 = \sum_{(p,q)} \sigma_2^p \sigma_3^q I_{(p,q)} \mathcal{R}^4 \]

General terms: \[ I_{(p,q)} = \sum_i h_{(p,q)}^i \quad \text{where} \]

i.e., \textit{SUM} of modular functions

\[
(\nabla^2_\Omega - i(i + 1)) h_{(p,q)}^i = \sum_{r,s} c_{(p,q)}^{rs} E_r E_s
\]

Some examples:
i) Limit \( S, T, U \to 0 \) gives \( \sigma_2 \mathcal{R}^4 \sim D^4 \mathcal{R}^4 \)

Ten-dimensional IIB

\[
\to \alpha' e^{\frac{1}{2} \phi^B} E_{\frac{5}{2}} (\Omega, \bar{\Omega}) \sigma_2 \mathcal{R}^4
\]

\( \nu \to 0 \)

(recall \( (\nabla^2_{\Omega} - \frac{15}{4}) E_{\frac{5}{2}} = 0 \) so source term is zero)

\( E_{\frac{5}{2}} (\Omega, \bar{\Omega}) \) contains TREE-LEVEL and GENUS-TWO terms together with non-perturbative D-instantons coinciding with tree-level and genus-two string results
ii) Next order in $S, T, U$ gives $\sigma_3 \mathcal{R}^4 \sim D^6 \mathcal{R}^4$

**Ten-dimensional IIB**

$$S_{D^6 \mathcal{R}^4} = \alpha' e^{\phi^B} \mathcal{E}_{(0,1)}(\Omega) D^6 \mathcal{R}^4$$

$s \to 0$

$$\nabla_\Omega^2 \mathcal{E}_{(0,1)} - 12 \mathcal{E}_{(0,1)} = -6 E_{\frac{3}{2}} E_{\frac{3}{2}}$$

New effect:

Mixing of $\delta^{(3)} S^{(3)}$ with $\delta^{(0)} S^{(6)}$ leads to source term in Poisson eqn. for coefficient of $S^3 \mathcal{R}^4$ term.

$\mathcal{E}_{(0,1)}$ contains genus: 0, 1, 2, 3: Agrees with string pert. theory as far as can be checked.

i.e. genus 0, 1, 3
iii) Expand Two-Loop Supergravity to higher orders in $S,T,U$:

(GBG, Russo, Vanhove arXiv:0807.0389)

Leads to $d=9$ modular invariant interactions of form:

$$\frac{1}{r_B^m} \alpha'^{2p+3q} \mathcal{E}^{(m+1)}_{(p,q)}(\Omega, \bar{\Omega}) \sigma_2^p \sigma_3^q \mathcal{R}^4 \quad m = 4p + 6q - 7$$

$$\frac{1}{r_B} \alpha'^4 \mathcal{E}^{(2)}_{(2,0)}(\Omega, \bar{\Omega}) \sigma_2^2 \mathcal{R}^4$$

$$\frac{1}{r_B^3} \alpha'^5 \mathcal{E}^{(4)}_{(1,1)}(\Omega, \bar{\Omega}) \sigma_2 \sigma_3 \mathcal{R}^4$$

$$\frac{1}{r_B^5} \alpha'^6 \mathcal{E}^{(6)}_{(3,0)}(\Omega, \bar{\Omega}) \sigma_2^3 \mathcal{R}^4$$

$$\frac{1}{r_B^5} \alpha'^6 \mathcal{E}^{(6)}_{(0,2)}(\Omega, \bar{\Omega}) \sigma_3^2 \mathcal{R}^4$$
New feature: Each modular function is the sum of solutions of Poisson equations,

\[ \mathcal{E}^{(4)}_{(1,1)} = \sum_{r=0}^{5} \mathcal{E}^{(4)}_{(1,1)} \]

where

\[ (\Delta \Omega - j(j + 1)) \mathcal{E}^{(4)}_{(1,1)} = -2v_j E_{\frac{3}{2}} E_{\frac{3}{2}} - 24w_j \zeta(2) E_{\frac{1}{2}} E_{\frac{1}{2}} \]

\( v_j, w_j \) are constants, \( E_{\frac{1}{2}} \sim \Omega_2 \log \Omega_2 + \ldots \)

- Solve for perturbative coefficients:
  Many agreements with string pert. theory and unitarity.

- Higher supergravity loops \((L>2)\) will reproduce further string theory terms.
3) \textit{Connections with maximal supergravity}

Higher-loop supergravity?? \textbf{All L}

\textbf{THREE LOOPS} - extra power of $S$
- anticipated from successes of one and two loops
- explicit construction (Bern, Carrasco, Johansson, Dixon, Kosower, Roiban)

\textbf{HIGHER LOOPS} - are there further powers of $S$???

Little known of details beyond three loops - but, \textbf{subject to important assumptions}, duality with string theory points to possibly important constraints.
Simple dimensional argument:

$L$ - LOOP Maximal Supergravity in $d$ dimensions

Naive divergence
(count vertices and propagators)

$A_L \sim \Lambda^{(d-2)L+2}$

External momentum factors reduce divergence of sum of all Feynman diagrams

$A_L \sim S^{\beta_L} R^4 \Lambda^{(d-2)L-6-2\beta_L}$

QUESTION - what is the value of $\beta_L$?
Direct multi-loop calculations 1982 - 2006

\[ L = 1 \quad \beta_L = 0, \quad A_1 \sim \mathcal{R}^4 \Lambda^{d-8} \]

\[ L = 2 \quad \beta_L = 2, \quad A_2 \sim S^2 \mathcal{R}^4 \Lambda^{2d-14} \]

i.e \( \mathcal{R}^4 \) not renormalized beyond 1 loop – hence NO 3-LOOP \( \mathcal{R}^4 \) counterterm in d=4.

\[ L = 3 \quad \beta_L = 3, \quad A_2 \sim S^3 \mathcal{R}^4 \Lambda^{3d-18} \]

i.e \( S^2 \mathcal{R}^4 \) not renormalized beyond 2 loops – NO 5-LOOP \( S^2 \mathcal{R}^4 \) counterterm in d=4

\[ L = 4 \quad \beta_L = 4, \quad A_2 \sim S^4 \mathcal{R}^4 \Lambda^{4d-22} \]

i.e \( S^3 \mathcal{R}^4 \) not renormalized beyond 2 loops – NO 6-LOOP \( S^3 \mathcal{R}^4 \) counterterm in d=4
Fermionic zero mode argument (Berkovits, 2006)

\[ \beta_L = L \quad L \leq 5 , \quad \beta_L \geq 6 \quad L \geq 6 \]

Based on pure spinor string theory – builds in full supersymmetry

No UV divergence up to \textbf{9 LOOPS} in d=4 (MBG, Russo, Vanhove 2006a)

– manifestly \( S^6 \mathcal{R}^4 \) duality-invariant counterterm.

**IF** \( \beta_L = L \) for all \( L \) \( \Rightarrow A_L \sim S^L \mathcal{R}^4 \Lambda^{(d-4)L-6} \)

Motivated by duality of eleven-dim. supergravity and string theory. (MBG, J. Russo, P. Vanhove 2006b)

Ultraviolet finite when: \( d < 4 + \frac{6}{L} \)

i.e., finite for all \( L \) when \( d=4 \) – (as in maximal Yang-Mills).
Consider L-loop 11-dim. SUGRA on circle (radius $R_{11}$)

Compactified expression (arbitrary $\beta_L$) contains powers of $(S R_{11}^2)^{\nu}$ and $(R_{11} \Lambda)^{-w}$ \hspace{1cm} (w > 0 for subdvergence)

Transforming to IIA parameters $R_{11}^3 \rightarrow g_A^2$, $S \rightarrow s R_{11}$

$$A_L \sim s^{\beta_L+\nu} g_A^{2(\nu+\frac{1}{3}(\beta_L-w))} \mathcal{R}^4 \Lambda^{9L-6-2\beta_L-w}$$

i) L=1: \hspace{0.5cm} $\beta_1 = 0$, $w = 3$ \hspace{1cm} $s^h g_A^{2(h-1)} \mathcal{R}^4 \hspace{1cm} (h = \nu > 1)$

ii) L > 1: $\beta_L \geq 2$ \hspace{0.5cm} contributes to lower powers of $g_A$

$$s^h g_A^{2(h-1)} g_A^{-\frac{2}{3}(2\beta_L+w-3)} \mathcal{R}^4 \Lambda^{9L-6-2\beta_L-w}$$

Hence leading behaviour at genus $h$: $s^h \mathcal{R}^4 \hspace{1cm} (\beta_h = h)$
UV
FINITE ???

Ceci n'est pas une pipe.\text{\underline{preuve}}
Comment on relation to superstring:

**CANNOT** decouple maximal supergravity quantum field theory from closed string (in $d > 3$ dimensions). (MBG, Ooguri Schwarz)

Suggests that maximal supergravity probably does not make sense in isolation from string theory for $d > 3$. [i.e., String theory is crucial UV completion.]

**VIZ:** Open string theory reduces to N=4 maximally supersymmetric Yang–Mills theory. **CAN** be decoupled from string theory in $d=4$ (as in AdS/CFT limit of D3-branes). N=4 Yang–Mills is UV finite and is a sensible decoupled local quantum field theory.
From $L=1$ to $L=2$  
$L=2$ to $L=3$  
$L=3$ to $L=4$  

...............  

Converges  

$L = \infty$  

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