Developments in BPS Wall-Crossing

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And, to appear…
Outline

1. Review BPS Wall-Crossing
2. The Kontsevich-Soibelman formula
3. N=2,D=4 Field Theory on $\mathbb{R}^3 \times S^1$
4. Twistor Space
5. One-particle corrections
7. Differential Equations
8. Summary & Concluding Remarks
Introduction

This talk is about the BPS spectrum of theories with d=4,N=2

Recently there has been some progress in understanding how the BPS spectrum depends on the vacuum.

These are called Wall-Crossing Formulae (WCF)

Last year: WCF derived with Frederik Denef

Subsequently, Kontsevich & Soibelman proposed a remarkable wall-crossing formula for generalized Donaldson-Thomas invariants of CY 3-folds

This talk will give a physical explanation & derivation of the KS formula
Review of BPS Wall-Crossing-I

Consider a theory on $\mathbb{R}^4$ with $\mathcal{N} = 2$ SUSY

For $u \in \mathcal{M}_v$, the moduli space of vacua,

let $\mathcal{H}_u$ be the one-particle Hilbert space.

Low energy theory: an unbroken rank $r$ abelian gauge theory

$\Gamma$: Symplectic lattice of electric \& magnetic charges $\gamma$

$$\mathcal{H}_u = \bigoplus_{\gamma \in \Gamma} \mathcal{H}_{\gamma,u}$$
BPS -II

$\mathcal{N} = 2$ central charge operator $\hat{Z} = Z_\gamma(u)$ on $\mathcal{H}_{\gamma,u}$

$\mathcal{H}_u^{BPS}$: Subspace of $\mathcal{H}_{\gamma,u}$ with $E = |Z_\gamma(u)|$

$$\Omega(\gamma; u) := -\frac{1}{2} \text{Tr}_{\mathcal{H}_{\gamma,u}^{BPS}} (2J_3)^2 (-1)^{2J_3}$$

Some BPS states are boundstates of other BPS states.

$\psi \in \mathcal{H}_{\gamma,u}^{BPS}$, a boundstate of BPS states: $\gamma = \gamma_1 + \gamma_2$,

$\psi$ will decay as $u$ crosses a wall of marginal stability where $Z_{\gamma_1}(u)$ and $Z_{\gamma_2}(u)$ are parallel.

(Cecotti, Fendley, Intriligator, Vafa; Seiberg & Witten)
Semi-Primitive Wall-Crossing

Marginal Stability Wall:

\[ MS(\gamma_1, \gamma_2) := \{ u \big| \frac{Z_{\gamma_1}(u)}{Z_{\gamma_2}(u)} \in \mathbb{R}_+ \} \]

Denef & Moore gave formulae for \( \Delta \Omega \)

for decays of the form \( \gamma \rightarrow \gamma_1 + N\gamma_2 \quad N \geq 1 \)

Based on Denef’s multi-centered solutions of sugra, and quiver quantum mechanics.

Do not easily generalize to \( \gamma \rightarrow N_1\gamma_1 + N_2\gamma_2 \quad N_i \geq 1 \)
BPS Rays

For each $\gamma \in \Gamma$ associate a ray in the $\zeta$ plane:

$$\ell_\gamma := Z_\gamma(u) \mathbb{R}_-$$

As $u$ crosses an MS wall some BPS rays will coalesce:
Symplectic transformations

\[ T := \Gamma^* \otimes \mathbb{C}^* \cong (\mathbb{C}^*)^{2r} \]

\( \mathcal{K}_\gamma \): Symplectic transformations of \( T \)

KS assign a group element \( S_\gamma \) to each BPS ray \( \ell_\gamma \)

\[ S_\gamma := \prod_{\gamma' \parallel \gamma} \mathcal{K}^{\Omega_\gamma(\gamma';u)}_{\gamma'} \]
KS WCF

For generic $u$, and convex cone $\mathcal{V}$ in the $\zeta$-plane

$$\mathcal{A}(\mathcal{V}) := \prod_{\ell, \gamma \in \mathcal{V}} S_\gamma = \prod_{-Z_\gamma(u) \in \mathcal{V}} \mathcal{K}_\gamma^{\Omega}(\gamma; u)$$

Main statement: The product is **INDEPENDENT OF** $u$

This is a wall-crossing formula !!
KS Transformations

\[ T = \Gamma^* \otimes \mathbb{C}^* \Rightarrow \text{Fourier modes: } X_\gamma : T \to \mathbb{C}^* \]

\[ \epsilon^{ij} = \langle \gamma^i, \gamma^j \rangle \Rightarrow \varpi^T := \frac{1}{2} \epsilon_{ij} \frac{dX^i}{X^i} \frac{dX^j}{X^j} \]

\[ \mathcal{K}_\gamma : X_{\gamma'} \to X_{\gamma'} e^{\langle \gamma, \gamma' \rangle} \log(1 - X_\gamma) \]

Example for \( r=1 \): \[ T = \mathbb{C}^* \times \mathbb{C}^* \quad \varpi^T = \frac{dx}{x} \frac{dy}{y} \]

\[ \mathcal{K}_{a,b} : (x, y) \to (x(1 - x^a y^b)^b, y(1 - x^a y^b)^{-a}) \]
Seiberg-Witten Theory

Consider a $d = 4, \mathcal{N} = 2$ field theory with a semisimple gauge group of rank $r$.

$$\mathcal{M}_v \cong \mathbb{C}^r \quad (u_2 = \langle \text{Tr}\Phi^2 \rangle, u_3 = \langle \text{Tr}\Phi^3 \rangle, \cdots)$$

$$\Gamma \to \mathcal{M}_v \quad \text{is now a local system}$$

Locally, we may choose a duality frame $\Gamma \cong \Gamma_{el} \oplus \Gamma_{mg}$

Special coordinates $\quad a^I = Z_{\alpha I}(u)$

$$Z_\gamma(u) = a \cdot \gamma_{el} + a_D \cdot \gamma_{mg}$$
Low Energy Theory on $\mathbb{R}^4$

Torus fibration: $\mathcal{J}_u := \Gamma_u^* \otimes (\mathbb{R}/2\pi \mathbb{Z})$

Choosing a duality frame:

$$
\mathcal{L} = -\frac{1}{4\pi} \text{Im} \tau_{IJ} (da^I \ast d\tilde{a}^J + F^I \ast F^J) + \frac{1}{4\pi} \text{Re} \tau_{IJ} F^I F^J + \cdots
$$
Example of G=SU(2)

\[ \Sigma_u : \quad y + \frac{\Lambda^4}{y} = x^2 - 2u \]

\[ \mathcal{K}_{2,-1} \mathcal{K}_{0,1} = \mathcal{K}_{0,1} \mathcal{K}_{2,1} \mathcal{K}_{4,1} \cdots \mathcal{K}_{2,0}^{-2} \cdots \mathcal{K}_{4,-1} \mathcal{K}_{2,-1} \]

It’s true!!!
Low Energy theory on $\mathbb{R}^3 \times S^1$

(Seiberg & Witten)

3D sigma model with target space $\mathcal{J}$

$\varphi^I_e = \int_{S^1} A^I_4 \, dx^4$

Periodic coordinates

$\varphi_{m,I} = \int_{S^1} (A_{D,4})_I \, dx^4$ for $\mathcal{J}_u$

Susy $\rightarrow$ $\mathcal{J}$ is hyperkähler
Semiflat Metric

\[ R = \text{radius of } S^1 \quad R \to \infty \]

KK reduce and dualize the 3D gauge field:

\[ g^{sf} = R \text{Im}\tau |da|^2 + (R \text{Im}\tau)^{-1} |dz|^2 \]

\[ dz_I = d\varphi_{m,I} - \tau_{IJ} d\varphi_e^J \]
The Main Idea

- $g_{\text{sf}}$ is quantum-corrected by BPS states (instanton = worldline of BPS particle on $S^1$)

- So, quantum corrections depend on the BPS spectrum

- The spectrum jumps, but the true metric $g$ must be smooth across MS walls.

- This implies a WCF!
Twistor Space

\[ Z := J \times \mathbb{C}P^1 \xrightarrow{p} \mathbb{C}P^1 \]

\( J_\zeta \) has complex structure \( \zeta \in \mathbb{C}P^1 \)

A HK metric \( g \) is equivalent to a fiberwise holomorphic symplectic form

\[ \varpi \in \Omega^2_{Z/\mathbb{C}P^1} \otimes \mathcal{O}(2) \]

\[ \varpi_\zeta = \zeta^{-1}\omega_+ + \omega_3 + \zeta\omega_- , \quad \zeta \in \mathbb{C}^* \]
Holomorphic Fourier Modes

\[ \mathcal{J}_\zeta \to \mathcal{M}_\nu \text{ is a torus fibration} \]

\[ \exists \text{ a basis of holomorphic functions } \mathcal{X}_\gamma(\zeta), \quad \gamma \in \Gamma \]

Restriction to torus \((\mathcal{J}_\zeta)_u \Rightarrow \mathcal{X}_\gamma(\zeta) = \exp [i\theta_\gamma + \cdots] \)

\[ \mathcal{J}_u = \Gamma_u^* \otimes (\mathbb{R}/2\pi \mathbb{Z}) \quad T_u = \Gamma_u^* \otimes \mathbb{C}^* \]

\[ \mathcal{X}(\zeta) : \mathcal{J}_\zeta \to T \quad \mathcal{X}_\gamma(\zeta) = \mathcal{X}(\zeta)^* (X_\gamma) \]

\[ \omega_\zeta = \mathcal{X}(\zeta)^* (\omega^T) \]
Semiflat holomorphic Fourier modes

\[ \chi^{\text{sf}} = \exp \left[ \pi R \zeta^{-1} Z_\gamma + i \theta_\gamma + \pi R \zeta \bar{Z}_\gamma \right] \]

(Neitzke, Pioline, & Vandoren)

**Strategy:** Compute quantum corrections to \( \chi^{\text{sf}} \)

\[ \chi_\gamma = \chi^{\text{sf}} \chi^{\text{inst}} \]

Recover the metric from

\[ \varpi_\zeta = \chi(\zeta)^*(\varpi^T) \]
One Particle Corrections

- Work near a point \( u^* \) where one HM, \( \mathcal{H} \) becomes massless
- Dominant QC’s from instantons of these BPS particles
- Choose a duality frame where \( \mathcal{H} \) has electric charge \( q>0 \)
- Do an effective Lagrangian computation
Periodic Taub-NUT

$r = 1$: Gibbons-Hawking metric on $U(1)$ fibration over $\mathbb{R}^2 \times S^1$

$$g^{\text{PTN}} = V^{-1}(d\varphi_m + A)^2 + V \left[|da|^2 + (d\varphi_e)^2\right]$$

$$V = V^{\text{sf}} + V^{\text{inst}}$$

$$V^{\text{sf}} = -q^2 R \left( \log \frac{a}{\Lambda} + \log \frac{\bar{a}}{\bar{\Lambda}} \right)$$

$$V^{\text{inst}} = q^2 R \sum_n e^{inq\varphi_e} K_0(R|nqa|)$$

(S&W, Ooguri & Vafa, Seiberg & Shenker)
Twistor coordinates for PTN

From the metric $g^{PTN}$ we compute $\omega^{PTN}$

$$\omega^{PTN} = \chi^*(\omega^T) = \frac{d\chi_e}{\chi_e} \frac{d\chi_m}{\chi_m}$$

$$\chi_e(\zeta) = \exp[i\varphi_e + \cdots]$$

$$\chi_m(\zeta) = \exp[i\varphi_m + \cdots]$$

Differential equation for twistor coordinates
Explicit PTN twistor coordinates

\[ \mathcal{X}_e(\zeta) = \mathcal{X}_e^{sf} = \exp[R\zeta^{-1}a + i\varphi_e + R\zeta\bar{a}] \]

\[ \mathcal{X}_m(\zeta) = \mathcal{X}_m^{sf} \mathcal{X}_m^{\text{inst}} \]

\[ \mathcal{X}_m^{sf} = \exp[R\zeta^{-1}a_D + i\varphi_m + R\zeta\bar{a}_D] \]

\[ \mathcal{X}_m^{\text{inst}}(\zeta) \sim \exp \left\{ q \int_{\gamma_e} [d\zeta'] \frac{\log[1 - \mathcal{X}_e(\zeta')^q]}{\zeta' - \zeta} \right\} \]
Key features of the coordinates

1. As a function of $\zeta$, $\mathcal{X}_m$ is discontinuous across the electric BPS ray $\ell_{\gamma_e}$.

The discontinuity is given by a KS transformation!

$$\begin{align*}
(\mathcal{X}_e, \mathcal{X}_m)^{cw} &= \mathcal{K}_{\gamma_e}(\mathcal{X}_e, \mathcal{X}_m)^{cw} \\
\mathcal{X}_m(\zeta) &\sim_{\zeta \to 0, \infty} \mathcal{X}_m^{sf}(\zeta)
\end{align*}$$

As befits instanton corrections.
Multi-Particle Contributions

- To take into account the instanton corrections from \textit{ALL} the BPS particles we cannot use an effective field theory computation.

- Mutually nonlocal fields in $L_{\text{eff}}$ are illegal!

- We propose to circumvent this problem by reformulating the instanton corrections as a Riemann-Hilbert problem in the $\zeta$ plane.
Riemann-Hilbert problem

\( \mathcal{X}(\zeta) : \mathcal{J}_{\zeta} \rightarrow T \)  
*Piecewise holomorphic family*

1. Across each BPS ray \( \ell_\gamma \)

\[ \mathcal{X}(\zeta)^{ccw} = S_\gamma \mathcal{X}(\zeta)^{cw} \]

2. \( \mathcal{X}(\zeta) \rightarrow \mathcal{X}^{sf}(\zeta) \)

*Exponentially fast for*  
\( \zeta \rightarrow 0, \infty \)
Solution to the RH problem

\[ X_\gamma(\zeta) = X^{{sf}}_\gamma(\zeta) \]

\[ \cdot \exp \left\{ \sum_{\gamma'} \Omega(\gamma'; u) \langle \gamma, \gamma' \rangle \int_{e_{\gamma'}} \left[ d\zeta' \right] \frac{\log(1 - X_{\gamma'}(\zeta'))}{\zeta' - \zeta} \right\} \]

Explicit instanton expansion as a sum over trees

\[ X_\gamma(\zeta) = X^{{sf}}_\gamma(\zeta) X^{{inst}}_\gamma(\zeta) \]
KS WCF = Continuity of the metric

As $u$ crosses a wall, BPS rays pile up

KS WCF $\iff$ Discontinuity from $\zeta_+$ to $\zeta_-$ is unchanged

$\Rightarrow$ $\ni$ and hence $g$ is continuous across a wall!
Differential Equations

The Riemann-Hilbert problem is equivalent to a flat system of differential equations:

\[
\zeta \partial_\zeta \mathcal{X} = \left( \zeta^{-1} A^-_\zeta + A^0_\zeta + \zeta A^+_\zeta \right) \mathcal{X} \quad \text{U}(1)_R \text{ symmetry}
\]

\[
R \partial_R \mathcal{X} = \left( \zeta^{-1} A^-_R + A^0_R + \zeta A^+_R \right) \mathcal{X} \quad \text{scale symmetry}
\]

\[
\partial_u \mathcal{X} = \left( \zeta^{-1} A^-_u + A^0_u \right) \mathcal{X} \quad \text{holomorphy}
\]

\[\rightarrow \] Stokes factors are independent of \( u, R \)

\[\rightarrow \] Compute at large \( R \): Stokes factors = KS factors \( S_\gamma \)
Summary

• We constructed the HK metric for circle compactification of SW theories.

• Quantum corrections to the dim. red. metric $g^{sf}$ encode the BPS spectrum.

• Continuity of the metric across walls of MS is equivalent to the KS WCF.

• Use the twistor transform to include quantum corrections of mutually nonlocal particles.
Other Things We Have Studied

• The $X_\gamma$ are Wilson-'tHooft-Maldacena loop operators, and generate the chiral ring of a 3D TFT

• Analogies to tt* geometry of Cecotti & Vafa

• Relations to Hitchin systems and D4/NS5 branes following Cherkis & Kapustin.
Open Problems

- Singularities at superconformal points
- Relations to integrable systems?
- Meaning of KS "motivic WCF formula"?
- Relation to the work of Joyce & Bridgeland/Toledano Laredo
- Generalization to SUGRA
- QC’s to hypermultiplet moduli spaces