BPS Black holes and topological strings: a review

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Introduction I

- Explaining the microscopic origin of Bekenstein-Hawking entropy of black holes is a pass/fail test for any theory of quantum gravity. String theory has passed this test with celebrated success for a class of BPS or near BPS black holes in the limit $Q = \infty$.

  Strominger Vafa; ...

- For BPS BH preserving 4 supercharges in $D = 4, \mathcal{N} = 4$ or $\mathcal{N} = 8$, a beautiful picture has emerged: exact microscopic degeneracies at finite Q are encoded as Fourier coefficients of certain modular forms. Derivations exist at least for certain duality orbits.

  Dijkgraaf, Verlinde, Verlinde; ...

Introduction II

- In this lecture, I will review some recent progress in trying to achieve the same level of accuracy for BPS black holes in $D = 4, \mathcal{N} = 2$ string theories.
- While this may sound academic, asking such detailed questions is bound to uncover many fruitful connections with mathematics, and perhaps some general lessons about QG.

Outline

Outline

Set-up I

- Consider type IIA string theory compactified on a Calabi-Yau three-fold X. The LEEA is $\mathcal{N} = 2$, $D = 4$ (ungauged) supergravity, with $n_V = h^{1,1}(X)$ vector multiplets and $n_H = h^{2,1}(X) + 1$ hypermultiplets.
- At two-derivative level, the moduli space splits into a product of $\mathcal{M}_V \times \mathcal{M}_H$. The first factor describes the Kähler structure and B-field on X, the second describes the complex structure of X, RR scalars and axiodilaton. *Forget $\mathcal{M}_H$ for now.*
- $\mathcal{M}_V$ is a special Kähler manifold. Its geometry is encoded in a holomorphic prepotential $F(X^I)$, $I = 0, \ldots, n_v$, homogeneous of degree 2: $t^A = X^A/X^0 = \int_{\gamma^A}(B + iJ)$ are the complexified Kähler moduli.
- X^I is the lowest component of a chiral superfield $\Phi^I = X^I + F^I_{\mu\nu}\theta\sigma^{\mu\nu}\theta + \ldots$, so $F(X^I)$ also controls the kinetic terms and theta angles of the $n_V + 1$ gauge fields $F^I_{\mu\nu}$.

Set-up II

- A special class of higher-derivative F-term corrections can be incorporated by letting $F = \sum_g F_g(X^I) W^{2g}$ depend on the (square of the) Weyl multiplet $W_{\mu\nu} = T_{\mu\nu} + R_{\mu\nu\rho\sigma}\theta\sigma^{\mu\nu}\theta + \dots$, where T is (related to) the graviphoton:

$$\int d^4\theta d^4x\, F(\Phi^I, W^2) + cc = S_{kin} + \sum_{g=1}^{\infty} F_g(X^I)(g\, R^2\, T^{2g-2} + \dots)$$

- F_g is given by the A-twisted topological string amplitude at genus g on X. In the large volume limit,

$$F = -C_{ABC}\frac{X^A X^B X^C}{6X^0} - \frac{W^2}{64}\frac{c_{2A}X^A}{24X^0} - \frac{(X^0)^2}{(2\pi i)^3}\sum_{g,\beta} N_{g,\beta}\, q^\beta \left(\frac{\pi W}{4X^0}\right)^{2g}$$

where $C_{ABC} = \int_X J_A J_B J_C$, $c_{2A} = \int_X J_A c_2(TX)$, $q^\beta = e^{2\pi i \beta_A X^A / X^0}$, and $N_{g,\beta}$ are the Gromov-Witten invariants.

Bershadsky Cecotti Ooguri Vafa; Antoniadis Gava Narain Taylor

Set-up III

- Type IIA string theory on X= M-theory on $X \times S^1$. $V/l_M^6 = 1/g_4^2$ is a hyper, while $V/l_s^6 = R_{11}^3 V/l_M^9 \sim (\mathrm{Im} t)^3$ is a vector. The overall scaling $\mathrm{Im} t^A \to \infty$, keeping g_4 fixed, leads to 5D SUGRA, with moduli space given by the cubic hypersurface $C_{ABC} r^A r^B r^C = 1$.
- The GW instanton series follows from one-loop contributions of M2-branes wrapped on $\beta \in H_2(X, \mathbb{Z})$ with spin $J_3^R = g/2$, in a self-dual graviphoton background:

$$\sum_h \sum_m n_\beta^g \int \frac{ds}{s} \left(2 \sin \frac{s}{2}\right)^{2h-2} e^{-\frac{2\pi s}{\lambda}(\beta_A t^A + im)} = \sum_g N_{g,\beta}\, q^\beta \lambda^{2g-2}$$

The BPS invariants n_β^g count (with signs) complex curves in class β with Lefschetz spin g.

Gopakumar Vafa

Spectrum of BPS states I

- BPS states, preserving 4 SUSY, are labelled by their conserved electric charges q_I, magnetic charges p^I and angular momentum J^2, J_3. Their mass in 4D Planck units is given by the modulus of the central charge Z

$$\mathcal{M} = |Z(p, q, t^i, \bar{t}^{\bar{i}})| \,, \qquad Z(p, q, t^i, \bar{t}^{\bar{i}}) = e^{K/2}(q_I X^I - p^I F_I)$$

- We are interested in the degeneracies of BPS states in the sector $\mathcal{H}(p, q, J; t)$, for a given value of the moduli t at infinity. For computability, consider instead the second helicity supertrace

$$\Omega(p, q; t) = -\frac{1}{2}\mathrm{Tr}_{\mathcal{H}(p,q;t)}[(-1)^{2J_3} J_3^2]$$

While $\Omega(p, q; t)$ is locally constant, it may still jump on lines of marginal stability.

Static, spherically symmetric solutions I

- Consider the ansatz $ds_4^2 = -e^{2U(r)}dt^2 + e^{-2U(r)}(dr^2 + r^2 d\Omega_2^2)$. The radial evolution of the scalars U and t^i is governed by

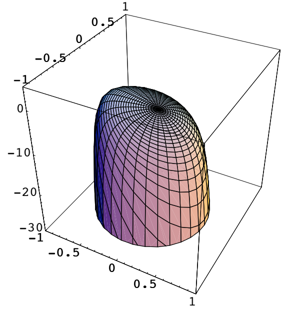
$$H = \dot{U}^2 + g_{i\bar{j}}\dot{t}^i\dot{\bar{t}}^{\bar{j}} + V_{BH} \equiv 0 \,, \quad V_{BH} = -e^{2U}\left(|Z|^2 + 4\partial_i|Z|g^{i\bar{j}}\partial_{\bar{j}}|Z|\right)$$

  Note that the potential is unbounded from below.

Ferrara Gibbons Kallosh

- Extremal (non-BPS) BH solutions with $AdS_2 \times S^2$ throat require fine-tuning the gradients of U and t^i at infinity, so as to reach extremum of V_{BH} with zero velocity.

Khuri Ortin; Dhar Mandal; ...



Static, spherically symmetric solutions II

- H is the bosonic part of a supersymmetric quantum mechanics with 4 real supercharges. One way to uncover it to lift the motion on $\mathcal{M}_V$ to geodesic motion on the 3D vector moduli space.

  Gunaydin Neitzke BP Waldron; Neitzke BP Vandoren

- BPS black holes are special solutions given by 1st order "attractor flow" eqs,

$$\frac{dU}{d\tau} = -e^U |Z| \,, \quad \frac{dz^i}{d\tau} = -2e^U g_{i\bar{j}} \partial_{\bar{j}} |Z|$$

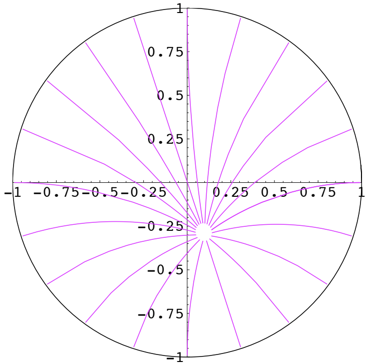
  where $\tau = 1/r$ is the inverse radial distance.

- The ODE can be integrated into

$$2e^{-U+K/2} \mathrm{Im} \left[e^{-i\alpha} \begin{pmatrix} X^I \\ F_I \end{pmatrix} \right] = \begin{pmatrix} p^I \tau + c^I \\ q_I \tau + d^I \end{pmatrix}$$

Static, spherically symmetric solutions III

- The "attractor flow" is a gradient flow for $\log|Z|$ on $\mathcal{M}_V$. $|Z|$ decreases monotonically from spatial infinity towards the origin.



- Moduli are attracted to $t = t_*(p, q)$ which minimizes $|Z|$ locally.
- t_* is locally independent of the moduli at infinity, but different basins of attraction are possible.

Static, spherically symmetric solutions IV

- If $|Z_*| > 0$, one obtains an smooth BH with $AdS_2 \times S^2$ throat, with moduli at the horizon, given by the "stabilization eqs"

$$\mathrm{Im}\begin{pmatrix} X^I \\ F_I \end{pmatrix} = \begin{pmatrix} p^I \\ q_I \end{pmatrix} \ , \quad S_{BH} = \pi |Z_*|^2 = \frac{i\pi}{4}(X^I \bar{F}_I - \bar{X}^I F_I)$$

Ferrara Kallosh Strominger

- If $|Z_*| = 0$ and t_* is a regular point in $\mathcal{M}_V$, the solution is nakedly singular and must be dismissed. *Does that mean that BPS states with such charges don't exist ?*

Moore; Denef

- In the latter case, higher derivative corrections must be included. Moreover, spherically symmetric solutions are just the tip of the iceberg...

Higher derivative corrections I

- In the presence of higher-derivative F-term corrections, the full solution can be obtained only numerically. It exhibits oscillatory fluctuations due to non-physical modes, which can in principle be absorbed by field redefinitions.

Cardoso de Wit Käppeli Mohaupt;Sen; Hubeny Maloney Rangamani

- The NH geometry can be obtained explicitly, using SUSY enhancement. The stabilization eqs still hold, with $F(X^I)$ replaced with $F(X^I, W^2 = 2^8)$. The macroscopic entropy is now given by the Bekenstein-Hawking-Wald formula.
- Alternatively, one may use the "entropy function formalism", valid for any extremal BH with $AdS_2 \times S^2$ throat, $F_{\theta\phi} \sim p^I, \ , F_{rt} \sim \phi^I$,

$$S_{BHW}(p^I, q_I) = \langle \Sigma(p^I, \phi^I) - q_I \phi^I \rangle_{\phi^I} \ , \quad \Sigma = \int d^4x \sqrt{-g_4}\, \mathcal{L}$$

Sen; Kraus Larsen

Higher derivative corrections II

- For $N = 2$ BPS BH in the presence of higher derivative F-terms, the entropy function is given by the "topological free energy",

$$\Sigma(p^I, \phi^I) = \mathrm{Im}[F(p^I + i\phi^I, 2^8)]$$

Cardoso de Wit Mohaupt; Ooguri Strominger Vafa; Sahoo Sen

- This observation has prompted the famous OSV conjecture

$$\boxed{\Omega(p^I, q_I) \sim \int d\phi^I \, |\Psi_{\rm top}(p^I + i\phi^I)|^2 \, e^{-\phi^I q_I}}$$

to all orders in inverse charges. $\Psi_{\rm top}(X^I) = e^{iF(X, W^2 = 2^8)}$ is the topological wave function. The equality $\log \Omega(p, q) \sim S_{BHW}$ follows automatically in the saddle point approximation at large charges.

Ooguri Strominger Vafa

Higher derivative corrections III

- Much of the recent activity in this area has been prompted by trying to answer some of the many questions raised by OSV:

  - To what extent are the two sides really well-defined ?
  - In what regime of moduli space and charges should it hold ?

  Denef Moore

  - How can it be consistent with electric-magnetic duality ?
  - How about holomorphic anomalies ?

  Cardoso de Wit Mahapatra

  - Is there a quantum mechanical interpretation ?

  Ooguri Vafa Verlinde; ...

  - Can one control non-perturbative corrections ?
  - etc.

Outline

Multi-centered solutions I

- Relaxing staticity assumption, multi-centered BPS solutions can be obtained by superposing N single centered solutions,

$$2e^{-U+K/2}\mathrm{Im}\left[e^{-i\alpha}\begin{pmatrix} X^I \\ F_I \end{pmatrix}\right] = \begin{pmatrix} \sum p^I_a/|\vec{x}-\vec{x}_a| + c^I \\ \sum q_{Ia}/|\vec{x}-\vec{x}_a| + d_I \end{pmatrix}$$

  provided the centers $\vec{x}_a$ satisfy $N-1$ constraints

$$\sum_b \langle \Gamma_a, \Gamma_b \rangle / r_{ab} = \langle \Gamma_a, \Gamma_\infty \rangle$$

  where $\Gamma_a = (p^I_a, q_{Ia}), \Gamma_\infty = (c^I, d_I), \langle \Gamma, \Gamma' \rangle = p^I q'_I - p'^I q_I$.

  Sabra; Behrndt Luest Sabra; Denef

- The solution carries angular momentum

$$\vec{J} = \sum_{a<b} \frac{1}{2} \langle \Gamma_a, \Gamma_b \rangle \frac{\vec{x}_a - \vec{x}_b}{r_{ab}} + \vec{J}_Q$$

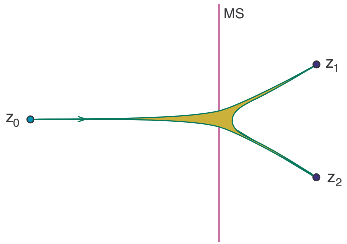
Multi-centered solutions II

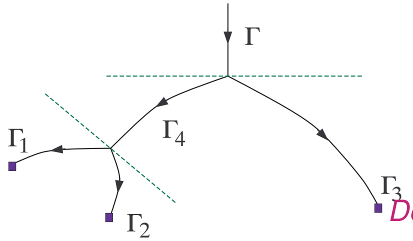
- In addition, one should require that $U(\vec{x})$ and $t^i(\vec{x})$ stay in their domains of definition.
- For 2 centers with mutually non-local charges, these conditions are easily checked: the distance is fixed to

$$r_{12} = \frac{1}{2}\langle \Gamma_1, \Gamma_2 \rangle \left[|Z_1 + Z_2| / \mathrm{Im}(Z_1 \bar{Z}_2) \right] |_\infty ,$$

  Note that r_{12} diverges on the line of marginal stability (LMS): $\arg Z_1 = \arg Z_2 [2\pi]$, where the decay $\Gamma \to \Gamma_1 + \Gamma_2$ is possible.
- The 2-centered solution only exists on the side of LMS where $\langle \Gamma_1, \Gamma_2 \rangle \mathrm{Im}(Z_1 \bar{Z}_2)|_\infty > 0$.

Multi-centered solutions III





Denef; Denef Moore

- As $\vec{x}$ varies in $\mathbb{R}^3$, $t^i(\vec{x})$ maps a domain in $\mathcal{M}_V$, well approximated by a "split attractor flow" which forks on the LMS.
- Given a total charge vector Γ and asymptotic moduli Γ_∞, there may exist many different attractor trees.
- According to the "Split Attractor Conjecture", a multi-centered solution exists iff the corresponding attractor tree exists, and the number of such trees for fixed (Γ_∞, Γ) is finite.

Multi-centered solutions IV

- As the line of marginal stability is crossed, one expects to lose the BPS states corresponding to the 2-centered configurations. When both Γ_1 and Γ_2 are primitive,

$$\Delta\Omega(\Gamma, t) = (-1)^{\langle\Gamma_1,\Gamma_2\rangle} \langle\Gamma_1, \Gamma_2\rangle \, \Omega(\Gamma_1, t) \, \Omega(\Gamma_2, t)$$

Denef Moore

- The factor $\langle\Gamma_1, \Gamma_2\rangle$ comes from quantizing the orbital degrees of freedom of the 2-centered system, with $J_Q = 1/2$. This can be extended to 3 and more centers.

de Boer El Showk Messamah Van de Bleeken

- Kontsevich and Soibelman have proposed a far-reaching generalization of this formula, more on this later.

Outline

The MSW (0,4) SCFT I

- For vanishing D6-brane charge, a useful realization of a D4-D2-D0 black hole is as M5-brane wrapping $S^1 \times P$, where $P = p^A \gamma_A$ is a divisor inside X, with self-dual flux $H \in H_2(P)$ projecting to $q_A \gamma^A \in H_2(X)$, and with left-moving momentum q_0 around S^1.

Maldacena Strominger Witten

- The reduction of the M5 (0,2) worldvolume theory on P leads to a 2D $(0,4)$ SCFT. It can be described as a non-linear $(0,4)$ sigma model on the HK manifold $\mathbb{R}^3 \times S^1 \times T_*(\mathcal{M}_P)/\Gamma$, coupled to a hyperholomorphic torus bundle of rank $b_2^- - b_2^+$. Here $\mathcal{M}_P$ is the moduli space of P inside X, and the torus directions correspond to the self-dual H-field on M5.

Minasian Moore Tsimpis

- When P is ample (i.e. lies inside the Kähler cone), $\mathcal{M}_P = \mathbb{C}P^N$ with $N = C_{ABC} p^A p^B p^C + \frac{1}{12} c_{2A} p^A$.

The MSW (0,4) SCFT II

- The Cardy formula, valid when $(-\hat{q}_0) \gg C(p)$,

$$S = 2\pi\sqrt{\frac{c_L}{6}(-\hat{q}_0)}\ , \quad \hat{q}_0 = q_0 - \frac{1}{12} D^{AB} q_A q_B < 0$$

  with

$$c_L = 4N + 4 + b_2^- - b_2^+ = 6 C_{ABC} p^A p^B p^C + c_{2A} p^A$$

  precisely reproduces the BHW entropy, including the one-loop R^2 correction proportional to $c_{2A} p^A$!

  Maldacena Strominger Witten

- The sigma model picture is valid only in a small neighborhood away from the discriminant locus where P becomes singular, and membrane instanton effects take place.

The MSW (0,4) SCFT III

- To go further, one may consider the (0,4) elliptic genus

$$\chi_P(\tau,\bar\tau,z) = \mathrm{Tr}_R \left[\frac{F^2}{2} (-1)^F e^{i\pi p_A q^A} \, e^{2\pi i \tau (L_0 - \frac{c_L}{24}) - 2\pi i \bar\tau (\bar L_0 - \frac{c_R}{24}) + 2\pi i z^A q_A} \right]$$

- Invariance under spectral flow (shifts of M2-flux) and dualities imply

$$\chi_P(\tau,\bar\tau,z) = \sum_{\mu \in L_X^*/L_X} H_\mu(\tau)\, \theta_\mu(\tau,\bar\tau,z)$$

  where θ_μ are non-hol. Siegel-Narain theta functions for the signature $(1, h-1)$ lattice $L_X = H_2(X,\mathbb{Z})$, h_μ is a vector of hol. modular forms with weight $-1-\frac{1}{2}h^{1,1}(X)$, and μ run over $\det(6C_{AB})$ possible "glue vectors" in $H_2(P,\mathbb{Z})/L_X \oplus L_X^\perp$.

Gaiotto Strominger Yin; de Boer Cheng Dijkgraaf Manschot Verlinde

The MSW (0,4) SCFT IV

- The knowledge of the polar terms with $N - \Delta_\mu < 0$ in $H_\mu(\tau) = \sum_N H_\mu(N) q^{N-\Delta_\mu}$ is sufficient to determine χ_P completely, via the "Farey tail expansion", very schematically

$$\chi_P(\tau, z) = \sum_{\gamma \in Sl(2,\mathbb{Z})/\Gamma_\infty} h_\mu^- \left(\frac{a\tau + b}{c\tau + d} \right) \theta_\mu \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right)$$

  This can be interpreted as a sum over Euclidean asymptotically AdS_3 geometries, corresponding to all possible fillings of the torus T^2 at infinity.

  Dijkgraaf Maldacena Moore Verlinde; Manschot Moore

- Polar terms correspond to states with $0 < \hat{q}_0 \leq \frac{c_L}{24}$. The associated single centered black hole entropy would be imaginary. States closest to the unitarity (upper) bound dominate the asymptotic density of states.

The MSW (0,4) SCFT V

- Inspired by the OSV conjecture, two (equivalent) descriptions have been proposed for the polar states:

  • In the $AdS^3 \times S^2 \times X_*$ attractor geometry: M2 and $\overline{\mathrm{M2}}$ branes which wrap 2-cycles in X and tile Landau levels around the north and south poles of S^2

  Gaiotto Strominger Yin

  • In the original type II/X picture: as two-centered solutions $1\,D6 - D4 - D2 - D0$ and $1\,\overline{D6} - D4 - D2 - D0$ with no net $D6$ brane charge.

  Denef Moore

  We follow the second approach.

Outline

single D6-D2-D0 I

- Bound states of a single D6 with $Q_A \gamma^A$ D2 and n D0, in the limit where $B + iJ$ is scaled to infinity, are described by ideal sheaves on X, i.e. roughly $U(1)$ instantons in a non-commutative 6D gauge theory on X with $c_1 = 0, c_2 = Q_A J^A, c_3 = n$. The (indexed) number of such objects is the Donaldson-Thomas invariant $N_{DT}(Q_A, n)$.
- Theorem: DT invariants are related to GW invariants via

$$Z_{DT} \equiv \sum_{Q_A, J} (-1)^n N_{DT}(Q_A, n)\, e^{-\lambda n + 2\pi i Q_A t^A} = [M(e^{-\lambda})]^{\chi/2} e^{F_{\rm hol} - F_{\rm pol}}$$

where $M(q) = \prod (1 - q^n)^{-n}$ is the Mac-Mahon function.

Maulik Nekrasov Okounkov Pandharipande

single D6-D2-D0 II

- Using the relation between GW and BPS invariants n_Q^g,

$$Z_{DT} = [M(e^{-\lambda})]^{\chi} \prod_{Q_A>0,k>0} \left(1 - e^{-k\lambda+2\pi i Q_A t^A}\right)^{k n_Q^0}$$

$$\times \prod_{Q_A>0,g>0} \prod_{k=-(g-1)}^{g-1} \left(1 - e^{-k\lambda+2\pi i Q_A t^A}\right)^{(-1)^{k+1} \binom{2g-2}{g-1-k} n_Q^g}$$

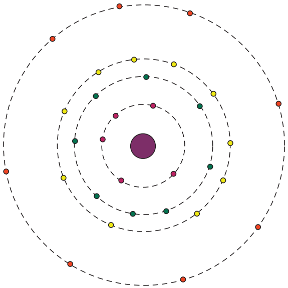
Gopakumar Vafa

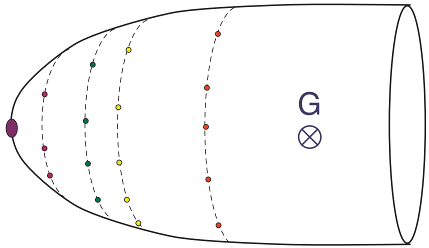
- In particular, D6-D0 bound states are counted by M^{χ}. Terms in 1st line correspond to "halos" of D2-D0 around the D6. They only exist for large enough B/J.

Denef Moore

single D6-D2-D0 III

- In contrast, the 2nd line describes "core" D6-D2-D0 bound states, stable for any B at large enough J. The black hole looks like an onion:





Denef Moore

- D4 charge can be introduced by spectral flow, i.e. tensoring by a line bundle.

single D6-D2-D0 IV

- In the decompactification to 5D, only "core" states remain. For primitive charges, using the 4D/5D connection, one obtains agreement with M2-brane counting,

$$\Omega_{5D}(Q_A, J) = \sum_{g>0} (-1)^{2J} \binom{2g-2}{g-1-2J} n_Q^g$$

Katz Klemm Vafa; Dijkgraaf Vafa Verlinde

- There is numerical evidence, up to genus ~ 50, that $\Omega_5(\lambda^2 Q, \lambda^3 J)$ grows like e^{λ^3}, in agreement with 5D BH entropy $S \sim \sqrt{Q^3 - J^2}$.
- On the other hand $N_{DT}(\lambda^2 Q, \lambda^3 J)$ appears to grow like e^{λ^2} only. Such "miraculous" cancellations would be needed if the OSV formula is to hold at weak topological coupling.

Huang Klemm Marino Tavanfar

Outline

An improved OSV formula I

Putting all of this together, Denef and Moore manage to (im)prove the OSV conjecture for zero D6 charge, subject to several key assumptions:

- split attractor flow conjecture: multi-centered sols are 1-1 with attractor trees
- extreme polar state conjecture: polar states close to unitarity bound are $1\,D6 - 1\,\overline{D6}$ with sufficiently small D2,D0
- the effect of "swing states" (states which jump between infinity and the LMS) can be neglected, as quantified by some exponent ξ.

An improved OSV formula II

Under these favorable circumstances, OSV follows,

- for zero total D6 brane charge, in the strict limit $t = i\infty$,
- with suitable cut-off on the DT partition function,
- with extra measure factor $(P^3 + \frac{1}{2}c_2 P)\phi^0$,
- with corrections of order $e^{-\alpha|P|^{3-\xi}/\phi^0}$, smaller than $e^{-\beta_A p^A/\phi^0}$
- in the regime where $\lambda = 1/\phi^0 \gg \mathcal{O}(|P|^{\kappa-3})$.

There is evidence that $\xi = 1, \kappa = 3$, which validates OSV at $\mathcal{O}(1)$ topological coupling. The entropy enigma (entropic dominance of multi-centered sols over single centered) suggests that OSV breaks down at weak coupling, barring "miraculous" cancellations.

Denef Moore

Outline

4D Black holes and 3D Instantons I

- To lift some of the ambiguities of the OSV formula, it would be desirable to patch up the local degeneracies $\Omega(p, q; t)$ into a globally well defined object, continuous across the lines of marginal stability. *cf. chamber dependence of threshold corrections*.
- One natural candidate is the 3D moduli space metric after reduction on a circle of radius $R = e^U$: it factorizes into the product of two QK spaces, exchanged under T-duality along S_1,

$$\mathcal{M} = \mathcal{M}_V^A \times \mathcal{M}_H^A = \mathcal{M}_H^B \times \mathcal{M}_V^B$$

4D Black holes and 3D Instantons II

- $\mathcal{M}_H^A$ is identical to the hypermultiplet metric in 4D, while at large radius, $\mathcal{M}_V^A$ is the *c*-map of the vector multiplet $\mathcal{M}_V$ in 4D:

$$\mathcal{M}_V^A = \mathbb{R}_U \times \mathcal{M}_V \times \tilde{T}^{2h+3}_{\zeta^I,\tilde{\zeta}_I,\sigma}$$

Cecotti Ferrara Girardello; Ferrara Sabharwal

- At finite radius, $\mathcal{M}_V^A$ receives instanton corrections from 4D BPS black holes winding around the loop. There are also extra $e^{ik\sigma}$ contributions, with non-zero NUT charge k around S^1.
- By T-duality, black hole corrections to $\mathcal{M}_V^A$ are mapped to D-instanton corrections to the hypermultiplet moduli space $\mathcal{M}_H^B$ in type IIB/X.

Seiberg Shenker

4D Black holes and 3D Instantons III

- QK metrics are difficult to describe, since they cannot be encoded in a simple holomorphic function. By the superconformal quotient construction, they are equivalent to HK cones. Using twistor methods they can be described in terms of a holomorphic symplectic space with a real structure. Indeed on a HK manifold $\Omega = \omega_1 + i\omega_2$ is holomorphic wrt to J_3.

Salamon; Swann; de Wit Rocek Vandoren

- Locally, one can choose holomorphic Darboux coordinates $\eta^I_{[i]}(\zeta)$ and $\mu_I^{[i]}(\zeta)$ such that $\Omega = d\mu_I^{[i]} \wedge d\eta^I_{[i]}$. On the overlap of two patches, they are related by a symplectomorphism,

$$\mu_I^{[i]} = \partial_{\eta^I_{[i]}} S\,, \quad \eta^I_{[j]} = \partial_{\mu_I^{[j]}} S\,, \quad S = S(\eta^I_{[i]}, \mu_I^{[j]}, \zeta)$$

Hitchin Karlhede Lindström Roček; Alexandrov BP Saueressig Vandoren; Lindström Roček

4D Black holes and 3D Instantons IV

- If $S = \eta^I_{[i]}\mu_I^{[i]} + H(\eta^I_{[i]}, \zeta)$, one recovers the standard Legendre transform construction of HK metrics with tri-holomorphic isometries, with generalized prepotential $H(\eta^I, \zeta)$. Superconformal invariance requires H to be quasi-homogeneous of degree 1 and ζ-independent.
- In the absence of instanton corrections, the metric on $\mathcal{M}_H$ is given by

$$H_{\text{pert}} = -\frac{i}{2}\frac{F(\eta^\Lambda)}{\eta^\flat} - \frac{i}{24\pi}\chi\,\eta^\flat \log \eta^\flat$$

where $\eta^\flat$ is the "superconformal compensator".

Roček Vafa Vandoren; Robles Llana Saueressig Vandoren

4D Black holes and 3D Instantons V

- Covariantizing the GW instanton sum under $Sl(2,\mathbb{Z})$ S-duality of type IIB, one obtains the exact contribution of all D0 and D2 branes. Restoring symplectic invariance, the form of general D-brane instantons, to linear (one-instanton) order, is given by

$$S = \eta^I \mu_I + H_{\rm pert} + \eta^\flat \sum_{p,q} n_{p^\Lambda,q_\Lambda} \sum_n \frac{1}{n^2} e^{2\pi i n (q_\Lambda \frac{\eta^\Lambda}{\eta^\flat} - p^\Lambda \mu_\Lambda)} + \dots$$

where n_{p^Λ,q_Λ} are a priori unknown, except when $p^\Lambda = 0$.

Robles-Llana Roček Saueressig Theis Vandoren; Alexandrov BP Saueressig Vandoren

- Note that $(\eta^\Lambda/\eta^\flat, \mu^\Lambda)$ parametrize an algebraic torus $\mathbb{C}^{\times(2n_v+2)}$
- This result is very reminiscent of the KS wall-crossing formula, which we now review.

4D Black holes and 3D Instantons VI

- Kontsevich and Soibelman show that across a LMS, the infinite non-commutative products

$$\prod_{\arg(Z_{p,q})\nearrow} U_{p,q}^{\Omega_+(p,q)} = \prod_{\arg(Z_{p,q})\searrow} U_{p,q}^{\Omega_-(p,q)} ,$$

  where $\Omega_\pm$ are "motivic GW invariants", $U_{p,q}$ are formal group elements

$$U_{p,q} = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n^2} e_{np^\Lambda, nq_\Lambda}\right)$$

  and $e_{p,q}$ satisfy the Lie algebra

$$\left[e_{p,q}, e_{p',q'}\right] = (-1)^{p^\Lambda q'_\Lambda - p'^\Lambda q_\Lambda} \left(p^\Lambda q'_\Lambda - p'^\Lambda q_\Lambda\right) e_{p+p',q+q'} .$$

- Up to subtle sign, $U_{p,q}$ may be interpreted as a symplecto-morphism of a complex torus $\mathbb{C}^{\times 2n_V}$.

4D Black holes and 3D Instantons VII

- This matches the hypermultiplet instanton corrections provided

$$n_{p,q} \equiv \Omega(p,q) \ , \quad e_{p,q} = i(q_\Lambda \frac{\eta^\Lambda}{\eta^\flat} - p^\Lambda \mu_\Lambda) \ , \quad [*,*] = \{*,*\}_{PB}$$

- Indeed, in the context of 4D/3D $\mathcal{N} = 2$ gauge theories the KS formula guarantees that the full instanton-corrected metric on the 3D moduli space is well defined and continuous across the LMS.

Gaiotto Neitzke Moore

- Generalizing SYM $\to$ SUGRA is challenging, due to exponential growth of Ω. Moreover, the instanton measure $n_{p,q}$ could differ from BH degeneracy. *cf. D(-1) measure vs D0 index in 10D*

Yi; Sethi Stern;Green Gutperle

- When the NS5-brane charge is non-zero, electric and magnetic translations no longer commute: Landau-type wave functions, non-Abelian Fourier coefficients.

Conclusion and open problems I

- Thanks to key physical insights (multi-centered solutions, 4D/5D connection, lines of marginal decay, dualities) and profound mathematical concepts (symplectic invariants, coherent sheaves, Rademacher expansions, ...), much progress towards precision counting of $\mathcal{N} = 2$ BPS black holes has been achieved. Yet our understanding is far from complete.
- Counting 4D black holes by computing instanton corrections in 3D seems very promising. If so, 3D U-dualities can act as spectrum generating symmetries for 4D black holes ! For $\mathcal{N} = 4, 8$, this suggests new relations between Siegel modular forms and automorphic forms of $SO(8, n_V + 2, \mathbb{Z})$ and $E_{8(8)}(\mathbb{Z})$.

Gunaydin Neitzke BP Waldron

Conclusion and open problems II

- For $\mathcal{N} = 2$, we are back to the problem of computing the exact metric on the hypermultiplet moduli space in 4D ! The utility of twistor techniques is just beginning to be appreciated. One may also contemplate a "triholomorphic" generalized topological string wave function, relevant for higher derivative corrections to the hypers.

Antoniadis Gava Narain Taylor; Gunaydin Neitzke BP

- The microscopic counting of 5D black holes and 5D black rings is still unsatisfactory. The reason why only F-terms contribute to the index remains mysterious. Can one count micro-states of extremal non-BPS BH reliably ? How about BH in $AdS_4 \times X$ vacua of gauged SUGRA ?

Conclusion and open problems III

Congratulations to Hirosi, Andy and Cumrun !

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Conclusion and open problems IV