

Integrability of the AdS/CFT System

Matthias Staudacher



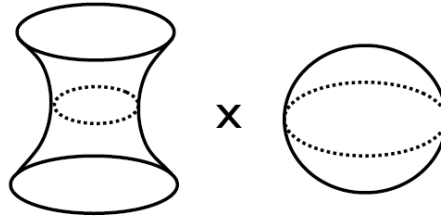
Max-Planck-Institut für Gravitationsphysik
AEI Potsdam

Strings 2008

CERN, GENEVA, 21.8.2008

The AdS/CFT Correspondence

IIB Superstrings on $AdS_5 \times S^5$



string tension: $\frac{1}{\alpha'}$

string coupling: g_s

$\mathcal{N} = 4$ $SU(N)$ supersymmetric gauge theory

't Hooft coupling: $\lambda = Ng_{\text{YM}}^2$ Inverse color number: $\frac{1}{N}$

Conjecture: exact duality between these two theories

[Maldacena '97; Gubser, Klebanov, Polyakov '98; Witten '98]

Fascinating Links

- Between quantum field theories without gravity, and string theories with (both classical and quantized) gravity
- Between exactly solvable two-dimensional quantum field theory and exactly solvable four-dimensional quantum field theory
- Between gauge/string theories and solved as well as unsolved problems of theoretical solid state physics
- Between gauge/string theories and mathematics (representation theory, quantum groups and Hopf algebras, complex analysis, integral equations, quantum geometry, ...)

$\mathcal{N} = 4$ Supersymmetric Gauge Theory, I

[Brink, Schwarz, Scherk '77; Gliozzi, Scherk, Olive '77]

Fields: All fields are in the adjoint representation, they are $N \times N$ matrices.

- gauge field \mathcal{A}_μ with $\mu = 0, 1, 2, 3$ of dimension $\Delta = 1$
- field strength $\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - i [\mathcal{A}_\mu, \mathcal{A}_\nu]$, $\Delta = 2$
- 6 real scalars Φ_m , with $m = 1, \dots, 6$, $\Delta = 1$
- 4×4 real fermions $\Psi_{\alpha a}, \dot{\Psi}_{\dot{\alpha}}^a$ mit $\alpha, \dot{\alpha} = 1, 2$, $a = 1, 2, 3, 4$, $\Delta = \frac{3}{2}$
- covariant derivatives: $\mathcal{D}_\mu = \partial_\mu - i \mathcal{A}_\mu$, $\Delta = 1$

$\mathcal{N} = 4$ Supersymmetric Gauge Theory, II

Action:

$$S = \frac{N}{\lambda} \int \frac{d^4x}{4\pi^2} \text{Tr} \left(\frac{1}{4} \mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu} + \frac{1}{2} \mathcal{D}^\mu \Phi^m \mathcal{D}_\mu \Phi_m - \frac{1}{4} [\Phi^m, \Phi^n] [\Phi_m, \Phi_n] \right. \\ \left. + \dot{\Psi}_{\dot{\alpha}}^a \sigma_{\mu}^{\dot{\alpha}\beta} \mathcal{D}^\mu \Psi_{\beta a} - \frac{i}{2} \Psi_{\alpha a} \sigma_m^{ab} \epsilon^{\alpha\beta} [\Phi^m, \Psi_{\beta b}] - \frac{i}{2} \dot{\Psi}_{\dot{\alpha}}^a \sigma_{ab}^m \epsilon^{\dot{\alpha}\beta} [\Phi_m, \dot{\Psi}_{\dot{\beta}}^b] \right)$$

Free parameters: $\lambda = N g_{\text{YM}}^2$ und N .

The “most beautiful” four-dimensional gauge theory. λ ist dimensionless.

Superconformal quantum field theory.

[Avdeev, Tarasov, Vladimirov '80; Grisaru, Rocek, Siegel '80]

[Sohnius, West '81; Caswell, Zanon '81; Brink, Lindgren, Nilsson '83; Mandelstam '83; Howe, Stelle, Townsend '84]

Global Symmetry: $\text{PSU}(2, 2|4)$.

IIB Superstring on $AdS_5 \times S^5$

Two-dimensional worldsheet with coordinates σ, τ :



Embedded into the coset space (Fermions act like “staples”)

$$\frac{\widetilde{\text{PSU}}(2, 2|4)}{\text{SO}(4, 1) \times \text{SO}(5)} = \overbrace{AdS_5 \times S^5}.$$

The IIB Superstring σ -Model on $AdS_5 \times S^5$

Action:

[Metsaev, Tseytlin '98]

$$S = \frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \left(\partial_a Z^M \partial^a Z_M + \partial_a Y_N \partial^a Y_N \right) + \text{Fermions}.$$

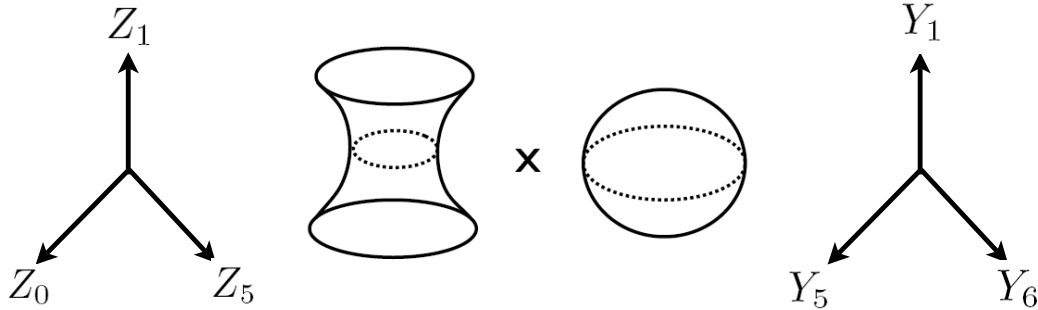
with

$$\begin{aligned} AdS_5 : \quad & -Z_0^2 + Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 - Z_5^2 = -R^2 \\ S^5 : \quad & Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 + Y_5^2 + Y_6^2 = R^2 \end{aligned}$$

The quantization of this model has not yet been understood.

However, see below ...

The Spectral Problem of IIB Superstrings on $AdS_5 \times S^5$



$Z_0 + i Z_5 = \rho_3 e^{i t}$, $Z_1 + i Z_2 = \rho_1 e^{i \alpha_1}$, $Z_3 + i Z_4 = \rho_2 e^{i \alpha_2}$:
 3 angles $t, \alpha_1, \alpha_2 \longrightarrow$ 3 conserved quantities E, S_1, S_2 . E is the energy.

$Y_1 + i Y_2 = r_1 e^{i \phi_1}$, $Y_3 + i Y_4 = r_2 e^{i \phi_2}$, $Y_5 + i Y_6 = r_3 e^{i \phi}$:
 3 angles $\phi_1, \phi_2, \phi \longrightarrow$ 3 conserved angular momenta J_1, J_2, J_3 .

The PSU(2, 2|4) Symmetry of the AdS/CFT System

32 bosonic generators and **32** fermionic generators $\mathfrak{Q}, \bar{\mathfrak{Q}}, \mathfrak{S}, \bar{\mathfrak{S}}$. $\mathfrak{su}(2, 2)$: conformal algebra, $\mathfrak{su}(4)$: R-symmetry. $\mathfrak{u}(2, 2|4)$ is reducible.

$\mathfrak{u}(2, 2)$				\mathfrak{Q}	\mathfrak{Q}	\mathfrak{Q}	\mathfrak{Q}
				\mathfrak{Q}	\mathfrak{Q}	\mathfrak{Q}	\mathfrak{Q}
				$\bar{\mathfrak{S}}$	$\bar{\mathfrak{S}}$	$\bar{\mathfrak{S}}$	$\bar{\mathfrak{S}}$
				$\bar{\mathfrak{S}}$	$\bar{\mathfrak{S}}$	$\bar{\mathfrak{S}}$	$\bar{\mathfrak{S}}$
\mathfrak{S}	\mathfrak{S}	$\bar{\mathfrak{Q}}$	$\bar{\mathfrak{Q}}$	$\mathfrak{u}(4)$			
\mathfrak{S}	\mathfrak{S}	$\bar{\mathfrak{Q}}$	$\bar{\mathfrak{Q}}$				
\mathfrak{S}	\mathfrak{S}	$\bar{\mathfrak{Q}}$	$\bar{\mathfrak{Q}}$				
\mathfrak{S}	\mathfrak{S}	$\bar{\mathfrak{Q}}$	$\bar{\mathfrak{Q}}$				

Instead of **8**, only **3 + 3** $\mathfrak{u}(1)$ Cartan charges: $(E, S_1, S_2 | J_1, J_2, J_3)$

Conformal Energy/Dilatation weight are a part of the symmetry!

The Spectral Problem of $\mathcal{N} = 4$ SYM

Conformal invariance restricts the structure of two-point functions:

$$\langle \mathcal{O}_n(x) \mathcal{O}_m(0) \rangle = \frac{\delta_{nm}}{x^{2\Delta_n}}.$$

Δ_n is the anomalous scaling dimension of the composite operator \mathcal{O}_n .

This leads to the mixing problem of $\mathcal{N} = 4$:

$$\mathcal{O} = \text{Tr}(\mathcal{X}\mathcal{Y}\mathcal{Z}\mathcal{F}_{\mu\nu}\Psi_{\alpha}^A(\mathcal{D}_{\mu}\mathcal{Z})\dots)\text{Tr}(\dots)\dots + \dots$$

The partons carry additive, protected Lorentz and R-symmetry charges S_1, S_2, J_1, J_2, J_3 . Here Δ_n is related to the dilatation generator \mathfrak{D} :

$$[\mathfrak{D}, \mathcal{O}_n(0)] = i \Delta_n \mathcal{O}_n(0).$$

$\Delta_n(\lambda)$ is not protected, it generically depends on the 't Hooft coupling λ .

The Spectral Problem of AdS/CFT and Integrability

A key prediction of AdS/CFT:

$$\begin{array}{ccc} \text{string energy} & \leftrightarrow & \text{scaling dimension} \\ E(\lambda) & = & \Delta(\lambda) \end{array}$$

- Solid Fact I: The $AdS_5 \times S^5$ string σ -model is classically integrable.

[Bena, Polchinski, Roiban '03]

It has been completely solved in terms of an algebraic curve.

[Kazakov, Marshakov, Minahan, Zarembo '04, Beisert, Kazakov, Sakai, Zarembo '05]

- Solid Fact II: The full one-loop dilatation operator of $\mathcal{N} = 4$ SYM can be mapped to a quantum integrable spin chain. It has been completely diagonalized by means of the Bethe ansatz.

[Minahan, Zarembo '02, Beisert, MS '03]

Mixing Problem in $\mathcal{N} = 4$ SYM and Spin Chains

Consider twist operators:

$$\mathcal{O} = \text{Tr} \left(\mathcal{D}^{S_1} \mathcal{Z}^{J_3} \right) + \dots$$

$\mathcal{D} = \mathcal{D}_1 + i \mathcal{D}_2$ mit $\mathcal{D}_\mu = \partial_\mu + i A_\mu$ is a covariant lightcone derivative.

The dilatation operator is regarded as the Hamiltonian of a spin chain.

The spin chain is

$$\text{Tr} \left((\mathcal{D}^{s_1} \mathcal{Z})(\mathcal{D}^{s_2} \mathcal{Z}) \dots (\mathcal{D}^{s_{J_3-1}} \mathcal{Z})(\mathcal{D}^{s_{J_3}} \mathcal{Z}) \right),$$

where $S_1 = s_1 + s_2 + \dots + s_{J_3-1} + s_{J_3} := M =$ Magnon number.

The Asymptotic Bethe Ansatz

[Sutherland '78; MS '04]

The excitations of the **integrable** gauge theory spin chain scatter according to matrix Bethe equations, where the p_k are **lattice** momenta:

$$e^{ip_k L} |\Psi\rangle = \left(\prod_{\substack{j=1 \\ j \neq k}}^M S(p_k, p_j) \right) \cdot |\Psi\rangle, \quad E = \sum_{k=1}^M q_2(p_k).$$

The (asymptotic) **S-matrix** is assumed to be **factorized**.

So far, factorization was only proved in **special cases** (at one loop for all, and up to four loops for some operators).

However, for finite size chains we are not allowed to assume exactness of the S-matrix, as it rests on **long-range** interactions: \rightarrow **wrapping problem!**

The AdS/CFT (internal) S-Matrix

[Arutyunov, Frolov, MS '04; MS '04; Beisert, MS '05; Beisert '05 + '06; Janik '06; Beisert, Hernandez, Lopez '06; Beisert, Eden, MS '06]

Die S-matrix should be unitary, and satisfy the Yang-Baxter-equation:

$$S_{12} S_{21} = 1, \quad S_{12} S_{13} S_{23} = S_{23} S_{13} S_{12}.$$

It was (ad-hoc) conjectured to also possess crossing symmetry: [Janik '06]

$$S_{12} S_{\bar{1}\bar{2}} = f_{12}^2.$$

The S-matrix for AdS/CFT has the following symmetry structure [Beisert '05]

$$S_{12} = \left(S_{12}^{\mathfrak{su}(2|2)_L} \otimes S_{12}^{\mathfrak{su}(2|2)_R} \right) \sigma_{12}^2,$$

It was first motivated from the gauge theory spin chain, and subsequently also using string theory arguments. [Arutyunov, Frolov, Plefka, Zamaklar '06]

The Asymptotic All-Loop Bethe Equations

[Beisert, MS '05]

$$1 = \prod_{\substack{j=1 \\ j \neq k}}^{K_2} \frac{u_{2,k} - u_{2,j} - i}{u_{2,k} - u_{2,j} + i} \prod_{j=1}^{K_3} \frac{u_{2,k} - u_{3,j} + \frac{i}{2}}{u_{2,k} - u_{3,j} - \frac{i}{2}},$$

$$1 = \prod_{j=1}^{K_2} \frac{u_{3,k} - u_{2,j} + \frac{i}{2}}{u_{3,k} - u_{2,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{3,k} - x_{4,j}^+}{x_{3,k} - x_{4,j}},$$

$$1 = \left(\frac{x_{4,k}^-}{x_{4,k}^+} \right)^L \prod_{\substack{j=1 \\ j \neq k}}^{K_4} \left(\frac{u_{4,k} - u_{4,j} + i}{u_{4,k} - u_{4,j} - i} \sigma^2(x_{4,k}, x_{4,j}) \right) \prod_{j=1}^{K_3} \frac{x_{4,k}^- - x_{3,j}}{x_{4,k}^+ - x_{3,j}} \prod_{j=1}^{K_5} \frac{x_{4,k}^- - x_{5,j}}{x_{4,k}^+ - x_{5,j}},$$

$$1 = \prod_{j=1}^{K_6} \frac{u_{5,k} - u_{6,j} + \frac{i}{2}}{u_{5,k} - u_{6,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{5,k} - x_{4,j}^+}{x_{5,k} - x_{4,j}},$$

$$1 = \prod_{\substack{j=1 \\ j \neq k}}^{K_6} \frac{u_{6,k} - u_{6,j} - i}{u_{6,k} - u_{6,j} + i} \prod_{j=1}^{K_5} \frac{u_{6,k} - u_{5,j} + \frac{i}{2}}{u_{6,k} - u_{5,j} - \frac{i}{2}},$$

$$E(g) = 2 \sum_{j=1}^{K_4} \left(\frac{i}{x_{4,j}^+} - \frac{i}{x_{4,j}^-} \right) = \frac{1}{g^2} \sum_{j=1}^{K_4} \left(\sqrt{1 + 16 g^2 \sin^2 \frac{p_j}{2}} - 1 \right), \quad \Delta = \Delta_0 + g^2 E(g), \quad K_4 = M.$$

$$1 = \prod_{j=1}^{K_4} \left(\frac{x_{4,j}^+}{x_{4,j}^-} \right) = \prod_{j=1}^{K_4} e^{i p_j}, \quad u_k = x_k + \frac{g^2}{x_k}, \quad u_k \pm \frac{i}{2} = x_k^\pm + \frac{g^2}{x_k^\pm}.$$

The Hubbard Connection

The AdS/CFT system is mysteriously related to the Hubbard model:

- Identical asymptotic dispersion law. Hubbard Hamiltonian is identical to the “rational part” of the $\mathfrak{su}(2)$ sector of the dilatation generator.

[Rej, Serban, Staudacher '05]

- S-matrix factors into two of Shastry's Hubbard R-matrices

[MS conjecture (unpublished) '05; Beisert '06]

- Dressing phase constants look like commuting charge expectation values in a half-filled “bosonic” Hubbard model

[Beisert, Eden, Staudacher '06]

The Interpolating Scaling Function

The scaling dimension of operators of low twist J_3 behaves in a very interesting logarithmic way at large spin $S_1 \rightarrow \infty$:

$$\Delta - S_1 - J_3 = f(g) \log S_1 + O(S_1^0) .$$

$f(g)$ is the universal scaling function, where $g^2 = \lambda/16 \pi^2$.

Also appears in the structure of MHV-amplitudes und in lightcone segmented Wilson loops \mathcal{W} ! Gluon 4-point function in $4 - 2\epsilon$ dimensions:

[Bern, Dixon, Smirnov]

$$\mathcal{M}_4^{\text{All-Loop}} \simeq \exp \left[f(g) \mathcal{M}_4^{\text{One-Loop}} \right] , \quad \mathcal{M}_4^{\text{All-Loop}} \simeq \langle \mathcal{W} \rangle .$$

The Interpolating Integral Equation

The non-linear asymptotic Bethe equations reduce in the limit $S_1 \rightarrow \infty$, where $L \rightarrow \infty$ with $L \ll \log S_1$, to a linear integral equation for the density $\hat{\sigma}$ of Bethe roots. These describe the one-dimensional “motion” of the covariant derivatives of the twist operators:

[Beisert, Eden, MS '06]

$$\hat{\sigma}(t) = \frac{t}{e^t - 1} \left[\hat{K}(2gt, 0) - 4g^2 \int_0^\infty dt' \hat{K}(2gt, 2gt') \hat{\sigma}(t') \right].$$

The universal scaling function $f(g)$ is then given by

$$f(g) = 16g^2 \hat{\sigma}(0).$$

The kernel \hat{K} is of a rather involved structure, it will not be written here.

Gauge Theory Meets String Theory

This equation is **analytic** at small g , and therefore valid for **arbitrary values** of the coupling constant g !

At **weak coupling** the equation was (numerically) tested up to **four loop order** in gauge theory: [Bern, Czakon, Dixon, Kosower, Smirnov, '06]:

$$f(g) = 8g^2 - \frac{8}{3}\pi^2 g^4 + \frac{88}{45}\pi^4 g^6 - 16\left(\frac{73}{630}\pi^6 + 4\zeta(3)^2\right)g^8 \pm \dots$$

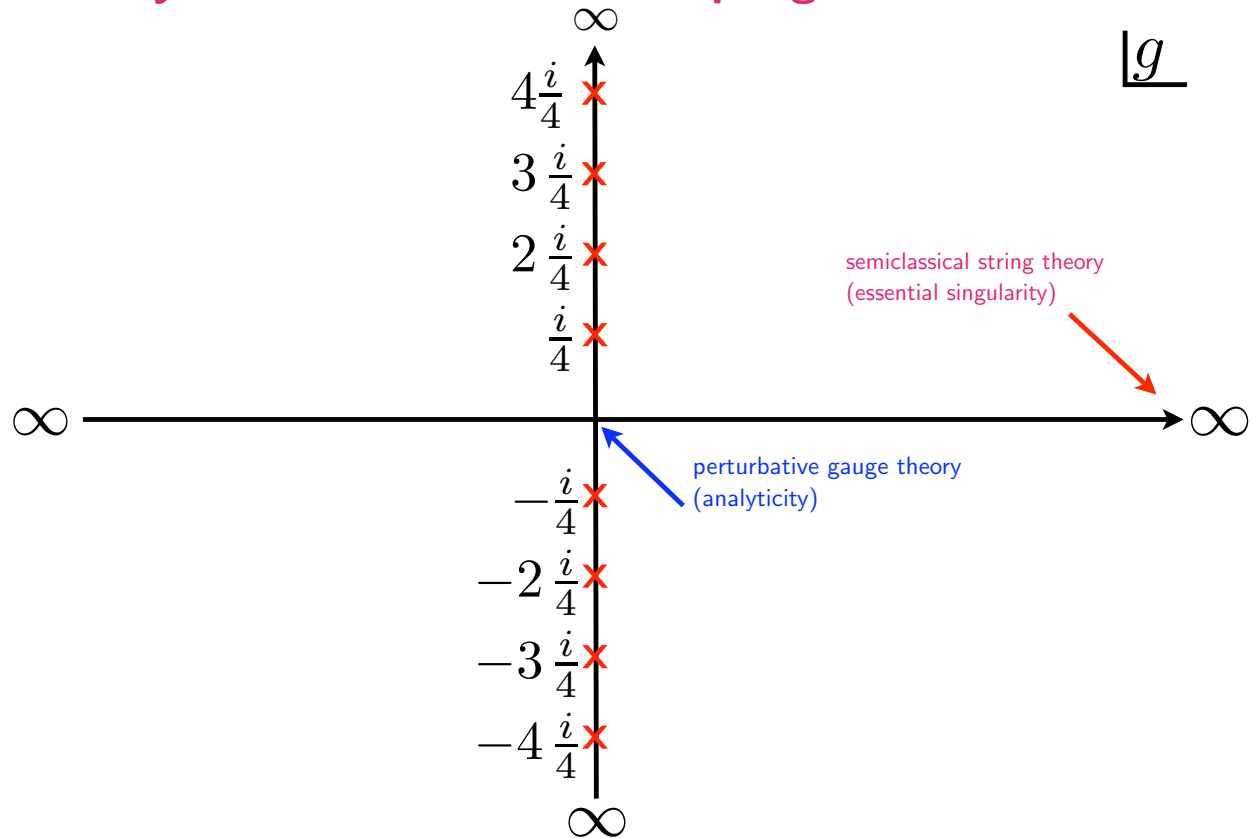
Improved to **0.001%** by [Cachazo, Spradlin, Volovich '06].

At **strong coupling** the scaling function agrees with string theory to the **three known orders** [Gubser, Klebanov, Polyakov '02], [Frolov, Tseytlin '02], [Roiban, Tirziu, Tseytlin '07; Roiban, Tseytlin '07] as was recently shown **analytically** from the equation [Basso, Korchemsky, Kotański '07] (see also [Kostov, Serban, Volin '08]):

$$f(g) = 4g - \frac{3\log 2}{\pi} - \frac{K}{4\pi^2} \frac{1}{g} - \dots$$

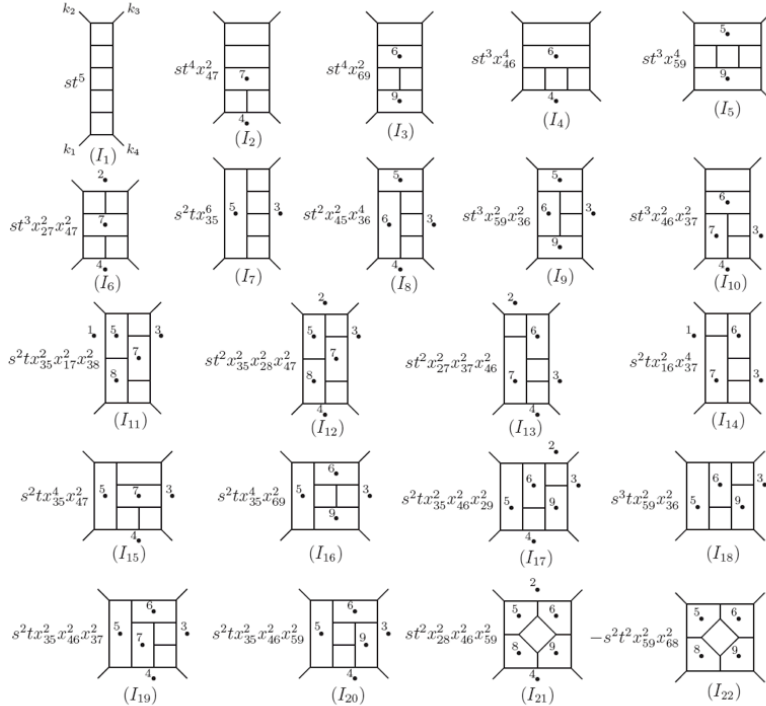
→ The AdS/CFT correspondence is **exactly true** !

Analytic Structure in the Coupling Constant Plane



Challenge I: Compute 5-Loop diagrams ...

[Z. Bern, J. J. M. Carrasco, H. Johansson and D. A. Kosower '07]



... and ...

[Z. Bern, J. J. M. Carrasco, H. Johansson and D. A. Kosower '07]

$$-st^3x_{45}^2 \quad (I_{23})$$

$$-s^2t^2x_{35}^2 \quad (I_{24})$$

$$-st^3 \quad (I_{25})$$

$$-st^2x_{45}^2 \quad (I_{26})$$

$$-s^2tx_{59}^2 \quad (I_{27})$$

$$-st^2x_{36}^2 \quad (I_{28})$$

$$-s^2tx_{37}^2 \quad (I_{29})$$

$$-st^2x_{35}^2x_{27}^2 \quad (I_{30})$$

$$-st^2x_{46}^2x_{29}^2 \quad (I_{31})$$

$$s^2t^2 \quad (I_{32})$$

$$-s^2tx_{16}^2 \quad (I_{33})$$

$$st^2 \quad (I_{34})$$

Challenge II: Compute 3-Loop String Corrections ...

... and compare to the 3-loop prediction

[Beisert, Eden, MS '06, Basso, Korchemsky, Kotański '07]

$$f\left(g + \frac{3 \log 2}{4 \pi}\right) = 4 g - \frac{K}{4 \pi^2} \frac{1}{g} - \frac{27 \zeta(3)}{2^9 \pi^3} \frac{1}{g^2} - \dots$$

Tough ... are there other ways to test the capacity of the asymptotic Bethe ansatz to interpolate between gauge and string theory?

... and compare to the 5-Loop Prediction

[Beisert, Eden, MS '06]

$$\begin{aligned} f(g) = & 8g^2 - \frac{8}{3}\pi^2g^4 + \frac{88}{45}\pi^4g^6 - 16\left(\frac{73}{630}\pi^6 + 4\zeta(3)^2\right)g^8 \\ & + 32\left(\frac{887}{14175}\pi^8 + \frac{4}{3}\pi^2\zeta(3)^2 + 40\zeta(3)\zeta(5)\right)g^{10} \mp \dots \end{aligned}$$

Challenge II: Compute 3-Loop String Corrections ...

... and compare to the 3-loop prediction

[Beisert, Eden, MS '06, Basso, Korchemsky, Kotarński '07]

$$f\left(g + \frac{3 \log 2}{4 \pi}\right) = 4 g - \frac{K}{4 \pi^2} \frac{1}{g} - \frac{27 \zeta(3)}{2^9 \pi^3} \frac{1}{g^2} - \dots$$

Tough ... are there other ways to test the capacity of the asymptotic Bethe ansatz to interpolate between gauge and string theory?

Bethe Ansatz: Hidden “Hole” Rapidities

[Belitsky, Gorsky, Korchemsky '06]

There are $S_1 = s_1 + \dots + s_J$ \mathcal{D} -particles in the background of the J_3 \mathcal{Z} 's:

$$\mathrm{Tr} \left((\mathcal{D}^{s_1} \mathcal{Z})(\mathcal{D}^{s_2} \mathcal{Z}) \dots (\mathcal{D}^{s_{J-1}} \mathcal{Z})(\mathcal{D}^{s_J} \mathcal{Z}) \right),$$

whose motion is described by S_1 rapidities u_k . Drop the index k :

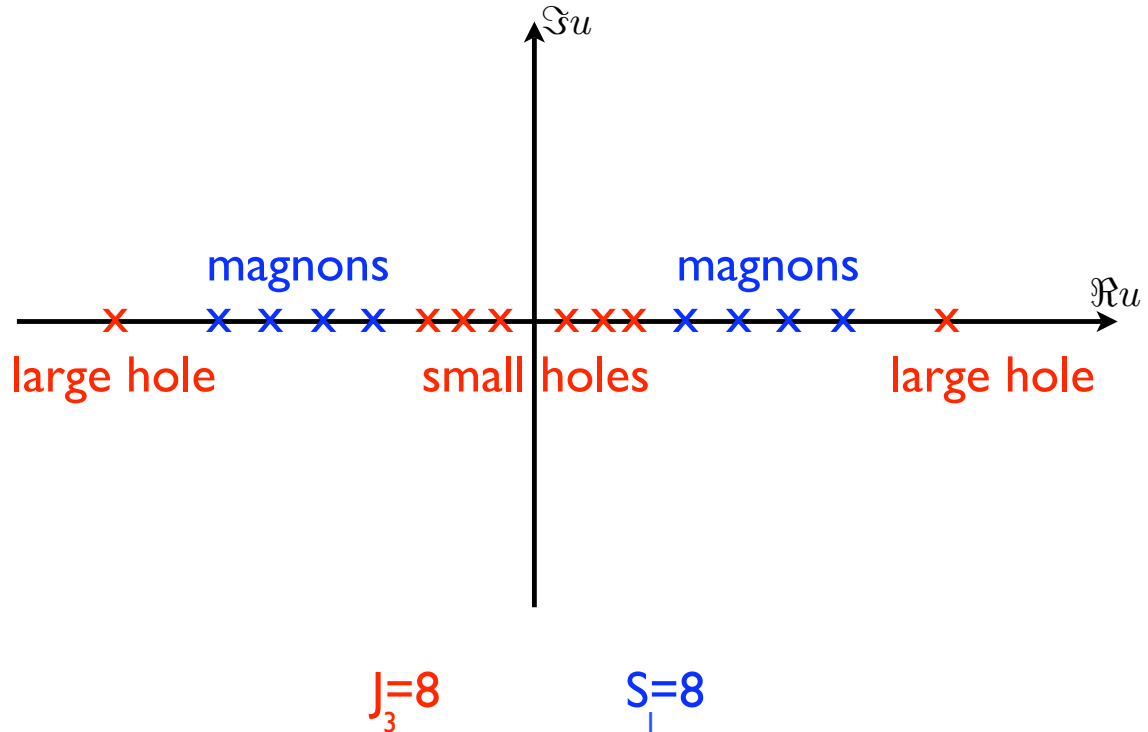
$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} \right)^{J_3} = \prod_{\substack{j=1 \\ j \neq k}}^{S_1} \frac{u_k - u_j - i}{u_k - u_j + i} \implies \left(\frac{\tilde{u} + \frac{i}{2}}{\tilde{u} - \frac{i}{2}} \right)^{J_3} = \prod_{\substack{j=1 \\ j \neq k}}^{S_1} \frac{\tilde{u} - u_j - i}{\tilde{u} - u_j + i}$$

Then there are J_3 further hidden solutions \tilde{u} of the Bethe equations. These are the rapidities of the \mathcal{Z} -particles in the background of the \mathcal{D} 's:

$$\mathrm{Tr} \left((\mathcal{D}^{s_1} \mathcal{Z})(\mathcal{D}^{s_2} \mathcal{Z}) \dots (\mathcal{D}^{s_{J-1}} \mathcal{Z})(\mathcal{D}^{s_J} \mathcal{Z}) \right).$$

Magnon and Hole Root Distributions

For the lowest state in the $\mathfrak{sl}(2)$ twist operator sector we have



One-Loop Bethe Ansätze for Magnons and Holes

The S_1 one-loop Bethe equations for the S_1 magnon rapidities u_k (or Bethe roots) read

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} \right)^{J_3} = \prod_{\substack{j=1 \\ j \neq k}}^{S_1} \frac{u_k - u_j - i}{u_k - u_j + i}.$$

For the J_3 hole roots \tilde{u}_k one needs to write a non-linear integral equation (NLIE, or Destri-DeVega equation). However, for large S_1 we have

$$2^{2i\tilde{u}_k} \left(\frac{\Gamma(\frac{1}{2} - i\tilde{u}_k)}{\Gamma(\frac{1}{2} + i\tilde{u}_k)} \right)^{J_3} \simeq \prod_{\substack{j=1 \\ j \neq k}}^{J_3} \frac{\Gamma(+i(\tilde{u}_k - \tilde{u}_j))}{\Gamma(-i(\tilde{u}_k - \tilde{u}_j))}.$$

One-Loop Dispersion Laws for Magnons and Holes

The additive dispersion law for magnons reads

$$\gamma_1 = \sum_{k=1}^{S_1} \frac{2}{u_k^2 + \frac{1}{4}}.$$

For the holes the dispersion law is not quite additive due to vacuum polarization effects. However, for large S_1 we have

$$\gamma_1 \simeq 2 \sum_{k=1}^{J_3} \left(\psi \left(\frac{1}{2} + i\tilde{u}_k \right) + \psi \left(\frac{1}{2} - i\tilde{u}_k \right) - 2\psi(1) \right) + 8 \log 2.$$

The two large holes scale as $\tilde{u} \simeq \pm S_1$, giving immediately $\gamma_1 \sim 8 \log S_1$. For exactly two or three holes the system is hyperintegrable. Can derive exact higher loop anomalous dimensions for arbitrary S_1 [Kotikov, Rej, Zieme, to appear].

Fine Structure of the One-Loop Anomalous Dimension

Consider the limit

$$S_1 \rightarrow \infty, J_3 \rightarrow \infty, \quad \text{with} \quad j := \frac{J_3}{\log S_1} = \text{fixed}.$$

The one-loop anomalous dimension becomes

$$\frac{\gamma_1(j)}{\log S_1} = 8 + 2 \int_{-1}^1 d\bar{u} \, \bar{\rho}_h(\bar{u}) \left(\psi\left(\frac{1}{2} + ia\bar{u}\right) + \psi\left(\frac{1}{2} - ia\bar{u}\right) - 2\psi(1) \right).$$

This leads to the expansion

$$\frac{\gamma_1(j)}{\log S_1} = 8 - 8j \log 2 + \frac{7}{12} j^3 \pi^2 \zeta(3) - \frac{7}{6} j^4 \pi^2 \log 2 \zeta(3) + \mathcal{O}(j^5)$$

Apparently a convergent series in j !

A Generalized Scaling Function for AdS/CFT

[Freyhult, Rej, MS, '07]

This suggests that the anomalous dimension γ of twist J_3 ops generates a generalized, two-parameter, bi-analytic scaling function $f(g, j)$

$$\Delta - S_1 - J_3 = \gamma = f(g, j) \log S_1 + O(S_1^0),$$

in the limit

$$S_1 \rightarrow \infty, J_3 \rightarrow \infty, \quad \text{with} \quad j := \frac{J_3}{\log S_1} = \text{fixed}.$$

Indeed true, as may be shown from the all-loop Bethe ansatz!

Interestingly, it does not appear to be possible to decouple magnons and holes beyond one-loop order.

Integral Equation for the Generalized Scaling Function

[Freyhult, Rej, MS, '07]

We find the following generalization of the linear integral equation for the universal scaling function

$$\hat{\sigma}(t) = \frac{t}{e^t - 1} \left[\hat{K}(t, 0) - \frac{j J_0(2 g t)}{8 t} - 4 \int_0^\infty dt' \hat{K}(t, t') \hat{\sigma}(t') \right]$$

The generalized universal scaling function $f(g, j)$ is then given by

$$f(g, j) = 16 \left(\hat{\sigma}(0) + \frac{j}{16} \right) .$$

The generalized kernel \hat{K} is even more involved as before.

Being bi-analytic, the equation should be exact in both g and j !

The $O(6)$ Sigma-Model from Planar $\mathcal{N} = 4$ SYM

The corresponding limit was also studied to one- and two-loop order on the string side.

[Frolov, Tirziu, Tseytlin '06; Roiban, Tseytlin '07]

It was then suggested that $f(g, j)$ may be **exactly** determined at strong coupling.

[Alday, Maldacena '06]

This was done by reducing the full sigma model to an integrable $O(6)$ sigma model. Its free energy is known from the thermodynamic Bethe ansatz (TBA), and was conjectured to be given, for $j \ll g$, by

$$O(6) \sigma\text{-model free energy} = f(g, j) - f(g, 0)$$

Very recently, [Basso, Korchemsky '08] this was proven by extracting the $O(6)$ TBA equations, including the exact expression for the **mass gap**, from the strong coupling limit of our above integral equation.

[Freyhult, Rej, MS, '07]

See also [Fioravanti, Grinza, Rossi, Buccheri '08].

AdS/CFT Interpolation Works

- This is the second example, after the cusp anomalous dimension, of a non-trivial quantity which smoothly **interpolates** between perturbative gauge theory and quantized string theory.
- It is fascinating to see how the “**knowledge**” of the $O(6)$ symmetry is restored when tracking the anomalous dimension of a twist operator $\mathcal{O} = \text{Tr}(\mathcal{D}^{S_1} \mathcal{Z}^{J_3})$ in a “**closed sector**” from weak to strong coupling!

Wrapping Restrictions

$\mathfrak{s}l(2)$
length

6	✓	✓	✓	✓	✓	✓	✓
5	✓	✓	✓	✓	✓	?	
4	✓	✓	✓	✓	✓	?	?
3	✓	✓	✓	✓	?	?	?
2	✓	✓	✓	See Janik's talk !	?	?	?
	1	2	3	4	5	6	7... loops

TBA

- So in the sense of the table just shown it appears that “exactly half” of the perturbative spectrum of $\mathcal{N} = 4$ gauge theory is now known.
- However, it has been known for some time that the asymptotic Bethe ansatz indeed does not properly include finite size effects. This was shown on the string side in [Schäfer-Nameki, Zamaklar, Zarembo '06] and in [Arutyunov, Frolov, Zamaklar, '06], [Astolfi, Forini, Grignani, Semenoff '07] and on the gauge side in [Kotikov, Lipatov, Rej, MS, Velizhanin, '07].
- For concrete ideas on the lower half of the table, see Janik's talk later today.

Crucial Open Problems

- Actually, what exactly is this system we are solving?
How can we define it, and prove its integrability?
- In other words, what is it we have been/currently are diagonalizing?
- How can we derive this system from the planar $\mathcal{N} = 4$ gauge theory?
- And the same question remains open for the σ -model on $AdS_5 \times S^5$.

Solvable Structures in the (Planar) AdS/CFT System

- Spectral Problem
- Gluon Amplitudes
- Wilson Loops
- High Energy Scattering (BFKL)

These are all related!

- Olive-Montonen Duality \rightarrow Solvability beyond the planar limit?

Integrability beyond the Spectral Problem

Integrability in planar gauge theories actually first appeared in the high-energy scattering context.

[Lipatov '93; Faddeev, Korchemsky '94]

Evidence is accumulating that it also rules the (planar) space-time scattering processes in $\mathcal{N} = 4$ gauge theory. Several talks on this at this conference. Is there a “spin chain” for gluon amplitudes?

Lipatov showed very recently that an integrable open spin chain appears in the Regge limit of gluon amplitudes.

[Lipatov, to appear]

An exciting new example for integrable AdS/CFT?

The planar $\text{AdS}_4/\text{CFT}_3$ system. See Juan Maldacena's talk.

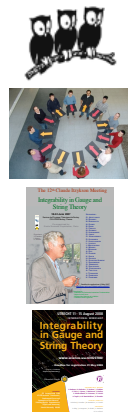
Has evolved very fast, **but** there is much less “data” than for $\text{AdS}_5/\text{CFT}_4$.

- Prove (or disprove?) full one-loop integrability, extending [Minahan, Zarembo '08]
- Is the CFT really integrable beyond one loop?
- The Lax-pair for the string σ -model was found [Arutyunov, Frolov '08]. But is the σ -model really quantum integrable?
- Are the Gromov-Vieira Bethe equations correct as is? [Gromov, Vieira '08]
- What is this $h(\lambda)$ function in the dispersion law?
- Finally, does this new model use the same “**trick**” to be integrable?

Conference Series: Integrability in Gauge and String Theory

This is becoming an exciting annual event:

- 2005: Paris (ENS Summer Institute)
- 2006: AEI Potsdam
- 2007: Paris (Itzykson Conference, Saclay and ENS)
- 2008: Utrecht



UTRECHT 11 - 15 August 2008

INTERNATIONAL WORKSHOP

Integrability in Gauge and String Theory

www.science.uu.nl/IGST08/

Deadline for registration 31 May 2008

[Faculty of Science
Physics and Astronomy]

Universiteit Utrecht



SPONSORS

ITP | Lorentz fonds | FOM
EC-RTN network "Constituents,
Fundamental Forces and
Symmetries of the Universe"
Stichting Physica | Department of
Physics and Astronomy,
Utrecht University | KNAW

SPEAKERS WILL INCLUDE

V. Bazhanov | N. Berkovits* | F. Cachazo* | L. Faddeev
S. Frolov | L. Lipatov | J. Maldacena* | R. Roiban
S. Schafer-Nameki | K. Schoutens | G. 't Hooft
A. Tseytlin | A.B. Zamolodchikov* | K. Zarembo

ADVISORY COMMITTEE

N. Beisert | V. Kazakov | M. Staudacher | G. 't Hooft

ORGANIZERS

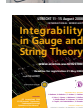
F. Alday | G. Arutyunov | B. de Wit | S. Vandoren

*To be confirmed

Conference Series: Integrability in Gauge and String Theory

This is becoming an exciting annual event:

- 2005: Paris (ENS Summer Institute)
- 2006: AEI Potsdam
- 2007: Paris (Itzykson Conference, Saclay and ENS)
- 2008: Utrecht
- 2009: AEI Potsdam (probably the week after Strings '09)



Outlook

Unique chance to participate in the first exact solution of a
four-dimensional Yang-Mills theory!