Tadpole Cancellation in the Topological String

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Strings '08, CERN

based on: **arXiv:0712.2775**
arXiv:0705.409, arXiv:0709.2390 (with Andrew Neitzke)

Introduction and Motivation

The Topological String is valuable as

(a) a toy model for string dynamics: D-branes, non-perturbative effects, Open/Closed duality, S-duality, M-theory, . . .

(b) a tool for studying supersymmetric observables in (ordinary) string theory: (higher-derivative) $\mathcal{N} = 1, 2$ F-terms, string dualities, counting BPS states, . . .

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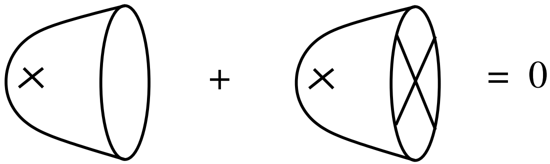
This talk is concerned with topological string on compact Calabi-Yau threefolds with D-branes and orientifolds.

Main Line of Investigation

The most celebrated consistency condition of string theory is anomaly cancellation in 10-d type I (and heterotic) string, discovered by Green and Schwarz in 1984. Upon compactification, this is more usefully phrased as tadpole cancellation, the vanishing of one-point functions of unphysical Ramond-Ramond (topform) potentials:



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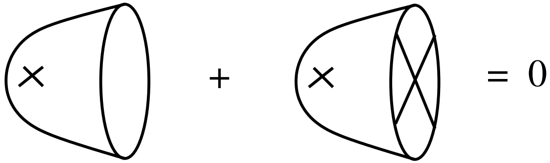

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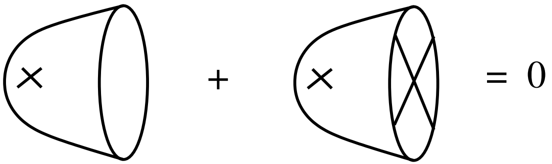

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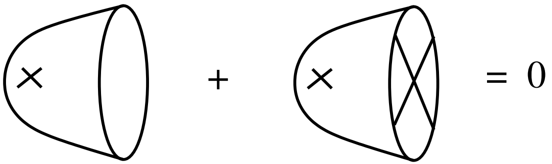

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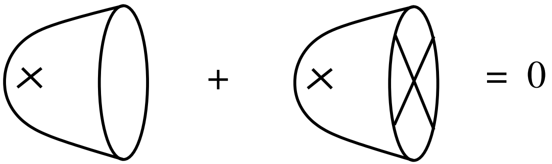

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2. Tadpoles created by background D-branes can be cancelled using anti-branes or orientifolds. In the superstring, supersymmetry requires the use of orientifolds. Somewhat surprisingly, it is also **best to cancel tadpoles using orientifolds** in the topological string, even without supersymmetry.

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Spacetime interpretation: Topological amplitudes admit BPS interpretation only in orientifold case. Explanation from say supergravity is so far missing.

Original Motivation

In this millenium, the open-closed topological string has been solved by Vafa and collaborators in several cases of non-compact Calabi-Yau manifolds.
Two classes:

- Toric Calabi-Yau solved by topological vertex (Aganagic, Klemm, Mariño, Vafa)
- Certain "conifold-like" Calabi-Yau manifolds related to matrix models according to Dijkgraaf-Vafa conjecture (See M. Mariño's talk).

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Problem: Compute loop amplitudes in topological string on genuine compact Calabi-Yau manifolds. Understand role of open-closed duality. Extract general lessons for string theory.

I. Tadpole Cancellation in the Topological String

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Recall definition of topological string (Witten 1988).

• Start from unitary $\mathcal{N}=(2,2)$ superconformal field theory of central charge $\hat{c}=3$, for example obtained from sigma-model on Calabi-Yau threefold.

• Identify generators of (topologically twisted) superconformal algebra with BRST operator and anti-ghost of "bosonic string" in which *ghost and matter do not decouple*. For example, in "B-model"

$$\begin{array}{ccc} (Q,\bar{Q}) & \leftrightarrow & (G^+,\bar{G}^+) \\ (b_0,\bar{b}_0) & \leftrightarrow & (G^-,\bar{G}^-) \\ (bc,\bar{b}\bar{c}) & \leftrightarrow & (J,\bar{J}) \end{array}$$

• Define topological string amplitudes by integrating over moduli space of Riemann surfaces

$$\mathcal{F}^{(g)} = \int_{\mathcal{M}^{(g)}} \langle | \prod_{a=1}^{3g-3} (G^-,\mu_a)|^2 \rangle$$

Four Different Topological Models

	Q	b_0	moduli
A-model	$G^+ + \bar{G}^-$	$G^- + \bar{G}^+$	Kähler t
anti A-model	$G^- + \bar{G}^+$	$G^+ + \bar{G}^-$	$\bar{t}$
B-model	$G^+ + \bar{G}^+$	$G^- + \bar{G}^-$	Complex structure z
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Mirror Symmetry relates A-model with B-model (and anti A-model with anti B-model), in general changing the target space.

In a unitary $\mathcal{N} = 2$ CFT, worldsheet CPT relates A-model with anti A-model, and B-model with anti B-model. In particular, from the point of view of (say) B-model, the anti-ghost cohomology (cohomology of BRST operator of anti B-model) is non-empty.

Anomalies

The B-model-BRST trivial states from anti B-model fail to decouple in general.

$\rightsquigarrow$ Topological amplitudes of the B-model depend on the complex structure moduli in a non-holomorphic way (BCOV 1993). This is an anomaly and arises from the boundary of the moduli space of Riemann surfaces.

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Again from the point of view of B-model, the *mixed BRST-anti-ghost* cohomology (cohomology of BRST operator of A-model) is also non-empty. The marginal operators are precisely the Kähler moduli. BCOV showed in 1993 that closed string amplitudes do not depend on those "wrong" moduli from the "other" topological model.

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This statement has to be revisited in the presence of background D-branes . . .

D-branes in Topological String (Witten 1993)

For sigma-model on (three-dimensional, simply-connected) Calabi-Yau:

A-branes: Lagrangian submanifolds with flat bundle

B-branes: Complex submanifolds with holomorphic bundle

Basic Fact

Topological charges of topological branes are naturally carried by the “other” model. (Ooguri-Oz-Yin, 1996)

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Example: (For simply connected CY) An A-brane is Lagrangian submanifold L, representing 3-cycle Γ. These naturally couple to three-forms, among which the complex structure deformations. Topological D-brane charge is measured by:

$$\mathrm{ch}(L) = \int_{\Gamma} (\text{3-form})$$

This definition supports the index theorem (cmp, Polchinski, 1995)

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Analogy: Mixed BRST-anti-ghost cohomology of topological string $\leftrightarrow$ Compact RR-potentials of superstring compactification

Do we have to cancel the tadpoles?

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Tadpole cancellation in topological string: (J.W. 2007, Cook-Ooguri-Yang 2008, one-loop: Klemm-Vafa (unpub.)) *A- and B-model decouple only for amplitudes with vanishing total D-brane charge.*

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Two ways to cancel tadpoles:

- Study dependence on open string moduli
  - Continuous moduli: Operator insertion on boundary
  - Discrete moduli: brane-anti-brane configuration
- Include orientifolds (preferred)

II. Extended Holomorphic Anomaly

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Strategy:

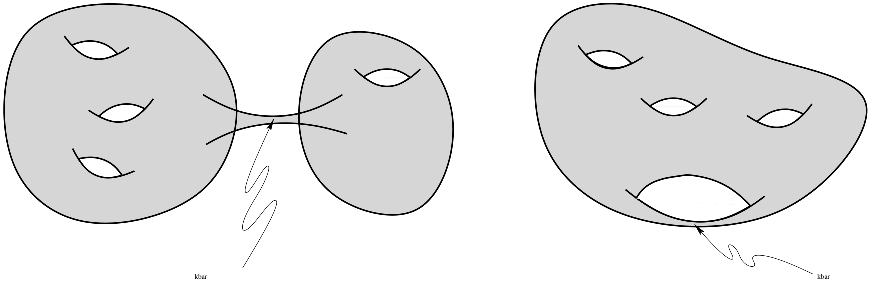
• Use holomorphic anomaly equation (Bershadsky-Cecotti-Ooguri-Vafa, 1993) and modular invariance to reduce to a finite-dimensional problem.

• Determine integration constants from physical requirements at singularities in moduli space ($e.g.$, large volume, conifold, orbifold), or some other duality.

Example: $\mathcal{F}^{(g)}$ on the quintic can be computed in this way up to $g = 51$ loops (Huang-Klemm-Quackenbush, 2006), to all orders for certain local models (Eynard-Orantin, Mariño, 2007)

BCOV: Anomalous contributions from boundary of moduli space, $\partial\mathcal{M}^{(g)}$.

$$\bar{\partial}_{\bar{i}}\mathcal{F}^{(g)} = \frac{1}{2}\sum_{g_1+g_2=g} C_{\bar{i}}^{jk}\mathcal{F}_j^{(g_1)}\mathcal{F}_k^{(g_2)} + \frac{1}{2}C_{\bar{i}}^{jk}\mathcal{F}_{jk}^{(g-1)},$$



Recursive in perturbative expansion $\chi = 2g-2$
Determines $\mathcal{F}^{(g)}$ up to finite number of constants
Origin: Unitarity of underlying $\mathcal{N}=(2,2)$ worldsheet theory; non-empty anti-ghost cohomology

Extension to open/unoriented strings

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Genus g, number of boundary components h, some background D-brane(s)

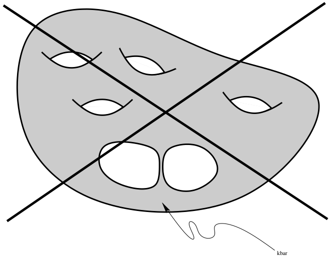
$$\mathcal{F}^{(g,h)} = \int_{\mathcal{M}^{(g,h)}} [dm][dl] \, \Big\langle \prod_{a=1}^{3g+h-3} \Big(\int \mu_a G^- \Big) \Big(\int \bar{\mu}_{\bar{a}} \bar{G}^- \Big) \prod_{b=1}^{h} \lambda_b (G^- + \bar{G}^-) \Big\rangle_{\Sigma_{g,h}}$$

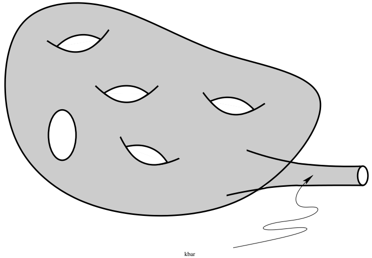
Problem: Moduli space $\mathcal{M}^{(g,h)}$ is real, has codimension-one boundaries.

Conditions: 1. Tadpole Cancellation

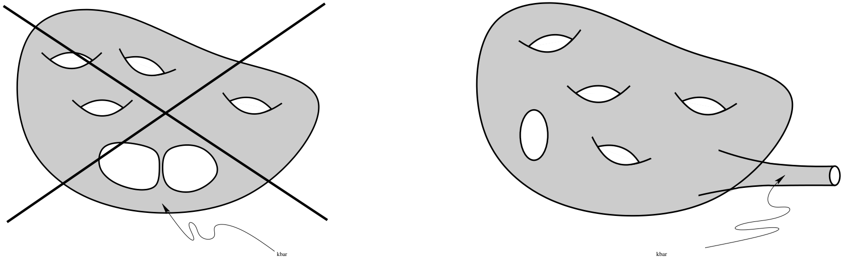
2. $\mathcal{F}^{(g,h)}$ do not depend on continuous open string moduli

Then: $\bar{\partial} \mathcal{F}^{(g,h)}$ receives additional contributions only from degeneration in which length of boundary component shrinks to zero.





$$\bar{\partial}_{\bar{i}}\mathcal{F}^{(g,h)} = (\mathrm{BCOV}) - \Delta_{\bar{i}}^{j}\mathcal{F}_{j}^{(g,h-1)}$$



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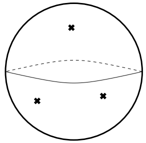
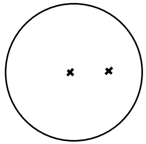
Tree-level data:

Closed string: Three-point function on the sphere $\quad C_{ijk} \sim \partial^3\mathcal{F}^{(0)} \sim$ 

Open string: Two-point function on the disk $\quad \Delta_{ij} \sim \partial^2\mathcal{F}^{(0,1)} \sim$ 

III. Tadpole Cancellation in Topological Orientifolds

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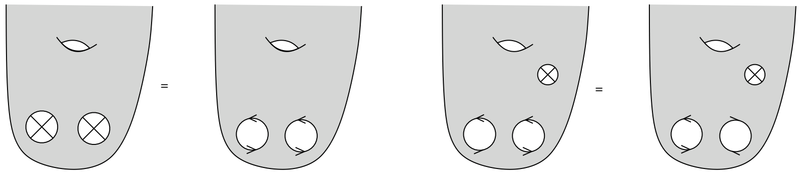
Extended holomorphic anomaly equation reaches full potential only under *inclusion of unoriented strings*.

Tadpole cancellation necessary for satisfactory *BPS interpretation* of topological string (on compact CY). At present, only (compelling) numerical evidence in examples.

Digression: Klein surfaces

Open + unoriented Riemann surfaces are classified by genus g, number of boundary components h and number of crosscaps c. Order of perturbation theory $\chi = 2g + h + c - 2$. Equivalence $2c \sim g$.



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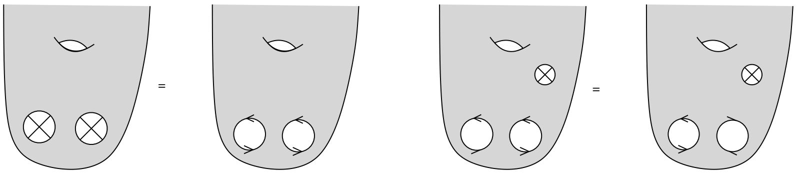

Equivalently, can think of doubled surface with involution $\rightsquigarrow$ Klein surfaces. Topological string as before.

Conventions: Orientable surface: $\mathcal{F}^{(g,h)}$
Even number of crosscaps: $\mathcal{K}^{(g,h)}$
Odd number of crosscaps: $\mathcal{R}^{(g,h)}$

How the various Klein surfaces can degenerate?

$$\partial_{\bar{i}}\mathcal{R}^{(g,h)} \underset{\text{closed}}{\supset} \sum_{\substack{g_1+g_2=g\\ h_1+h_2=h}} C_{\bar{i}}^{jk}\mathcal{K}_j^{(g_1,h_1)}\mathcal{R}_k^{(g_2,h_2)} + \sum_{\substack{g_1+g_2=g\\ h_1+h_2=h}} C_{\bar{i}}^{jk}\mathcal{F}_j^{(g_1,h_1)}\mathcal{R}_k^{(g_2,h_2)} + \tfrac{1}{2}C_{\bar{i}}^{jk}\mathcal{R}_{jk}^{(g-1,h)} + \tfrac{1}{2}B_{\bar{i}}^{jk}\mathcal{R}_{jk}^{(g-1,h)}$$

Non-orientable Riemann surfaces with an even number of crosscaps, $\Sigma^{(g,h)_k}$, have several more possible types of closed string degenerations:

$$\partial_{\bar{i}}\mathcal{K}^{(g,h)} \underset{\text{closed}}{\supset} \sum_{\substack{g_1+g_2=g\\ h_1+h_2=h}} C_{\bar{i}}^{jk}\mathcal{K}_j^{(g_1,h_1)}\mathcal{F}_k^{(g_2,h_2)} + \frac{1}{2}\sum_{\substack{g_1+g_2=g-1\\ h_1+h_2=h}} C_{\bar{i}}^{jk}\mathcal{R}_j^{(g_1,h_1)}\mathcal{R}_k^{(g_2,h_2)} + \frac{1}{2}\sum_{\substack{g_1+g_2=g\\ h_1+h_2=h}} C_{\bar{i}}^{jk}\mathcal{K}_j^{(g_1,h_1)}\mathcal{K}_k^{(g_2,h_2)} + \frac{1}{2}C_{\bar{i}}^{jk}\mathcal{K}_{jk}^{(g-1,h)} + \frac{1}{2}B_{\bar{i}}^{jk}\mathcal{K}_{jk}^{(g-1,h)} + \frac{1}{2}B_{\bar{i}}^{jk}\mathcal{F}_{jk}^{(g-1,h)}$$

Finally, tadpole contribution:

$$
\begin{aligned}
\partial_{\bar{i}}\big(\mathcal{F}^{(g,h)} + \mathcal{R}^{(g,h-1)}\big) &\underset{\text{tadpole}}{\supset} -\sqrt{2}\Delta_{\bar{i}}^{j}\mathcal{F}_j^{(g,h-1)} \\
\partial_{\bar{i}}\big(\mathcal{K}^{(g,h)} + \mathcal{R}^{(g,h-1)}\big) &\underset{\text{tadpole}}{\supset} -\sqrt{2}\Delta_{\bar{i}}^{j}\mathcal{K}_j^{(g,h-1)} \\
\partial_{\bar{i}}\big(\mathcal{K}^{(g,h)} + \mathcal{R}^{(g-1,h+1)}\big) &\underset{\text{tadpole}}{\supset} -\sqrt{2}\Delta_{\bar{i}}^{j}\mathcal{R}_j^{(g-1,h)}
\end{aligned}
$$

Result: Define total amplitude at order χ in perturbation theory

$$\mathcal{G}^{(\chi)} = \frac{1}{2^{\frac{\chi}{2}+1}}\Big[\mathcal{F}^{(g_\chi)} + \sum_{2g+h-2=\chi}\mathcal{F}^{(g,h)} + \sum_{2g+h-1=\chi}\mathcal{R}^{(g,h)} + \sum_{2g+h-2=\chi}\mathcal{K}^{(g,h)}\Big]$$

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$$\mathcal{G}^{(\chi)} = \frac{1}{2^{\frac{\chi}{2}+1}}\Big[\mathcal{F}^{(g_\chi)} + \sum_{2g+h-2=\chi} \mathcal{F}^{(g,h)} + \sum_{2g+h-1=\chi} \mathcal{R}^{(g,h)} + \sum_{2g+h-2=\chi} \mathcal{K}^{(g,h)}\Big]$$

This satisfies extended holomorphic anomaly from before ($\chi > 0$, ^{P}: orientifold projection, Δ now disk+crosscap.)

$$\partial_{\bar{i}}\mathcal{G}^{(\chi)} = \frac{1}{2}\sum_{\chi_1+\chi_2=\chi-2} C^{P}{}_{\bar{i}}^{jk}\mathcal{G}_j^{(\chi_1)}\mathcal{G}_k^{(\chi_2)} + \frac{1}{2}C^{P}{}_{\bar{i}}^{jk}\mathcal{G}_{jk}^{(\chi-2)} - \Delta^{P}{}_{\bar{i}}^{j}\mathcal{G}_j^{(\chi-1)}$$

BPS interpretation

Topological string amplitudes are related to BPS state counting (Gopakumar-Vafa 1998)

$$\sum_{g} \lambda^{2g-2} \lim_{\bar{t}\to\infty} \mathcal{F}^{(g)}(t,\bar{t}) = \sum_{g,d,k} n_d^{(g)} \frac{1}{k}\left(2\sinh\frac{\lambda k}{2}\right)^{2g-2} q^{dk}$$

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- holomorphic limit in A-model. t: Kähler modulus, $q \sim e^t$
- $d \in H_2(X,\mathbb{Z})$: charge under $\mathcal{N}=2$ vectormultiplet
- λ: topological string coupling
- $g :\sim SU(2)_L \subset SO(4)$ 5d spin
- $n_d^{(g)}$: Integers counting "net" number of M2/D2 BPS states with quantum numbers d, g.

Open/unoriented amplitudes (Ooguri-Vafa 2000, J.W. 2007)

In example (Real Quintic)

$$\sum_{\chi} \lambda^{\chi}\Big(\mathcal{G}^{(\chi)}(t,\epsilon) - \frac{1}{2}\mathcal{F}^{(g_\chi)}(t)\Big) = \sum_{\chi,d,k} n_g^{(\hat{g},\mathrm{real})}\frac{1}{k}\Big(2\sinh\frac{\lambda k}{2}\Big)^{\chi} q^{kd/2}\epsilon^{kd}$$

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- ϵ : discrete open string modulus (discrete Wilson line on Lagrangian L)
- $d \in H_2(X, L; \mathbb{Z})$
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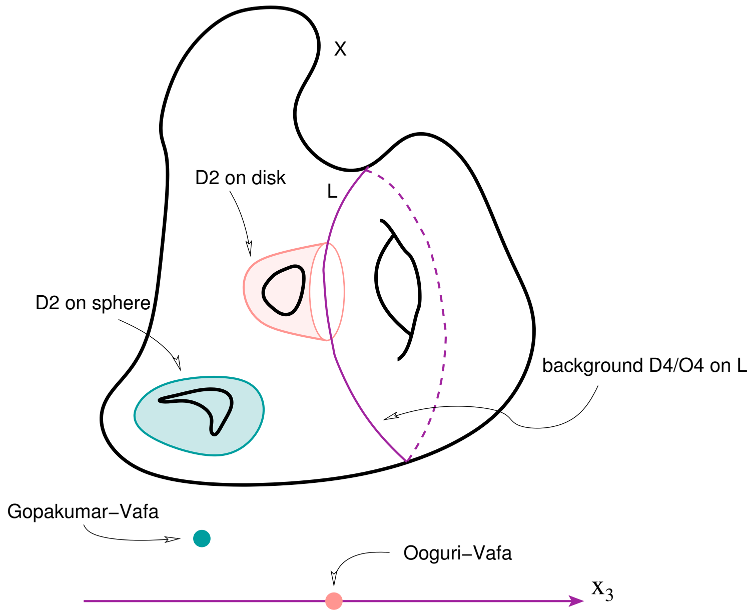
This statement arises from detailed computations in A- and B-model and is essentially mathematically rigorous.

What Real Topological String is Counting?

→ BPS states (solitons) in $1+1$-dimensional theory on D4-brane wrapped on L. Carry vectormultiplet charge, as well as topological charge asscoiated with (discrete) open string moduli.

→ Mathematics: Real enumerative invariants (Welschinger, Solomon,. . .)



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1. D6-branes+O6-planes wrapped on 3-cycles and filling 4d spacetime. Tadpole cancellation of type IIA requires vanishing total D6-brane charge, where

$$Q_6(\text{O6-plane}) = 4$$

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2. D4-branes+O4-planes wrapped on 3-cycle and extended along $1+1$-dimensional subspace of spacetime (Ooguri-Vafa setup). Since RR-flux can escape to infinity, there is naively no tadpole cancellation condition. But for the record, note that

$$Q_4(\text{O4-plane}) = 1$$

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To compute this, we note that from the topological string point of view, the O-planes of interest are half-dimensional, and the parity twisted Witten index for D-brane wrapped on L can be computed by geometric intersection with O-plane cycle $\Gamma_{\rm O}$.

$$\mathrm{Tr}_{L,P(L)}(-1)^F = \Gamma \cap \Gamma_{\rm O}$$

$$\Rightarrow Q_{\rm top}(\text{top. O-plane}) = 1$$

(agrees with result from CS/top. string on conifold duality Sinha-Vafa)

Conclusions

Tadpoles of topological string are cancelled in Ooguri-Vafa setup precisely when O4/D4 charge cancels locally.

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Tadpoles of topological string are cancelled in Ooguri-Vafa setup precisely when O4/D4 charge cancels locally.

Tadpoles of topological string are not cancelled when tadpoles of superstring are cancelled in O6/D6 "braneworld" setup.

Note that "Ooguri-Vafa string" supporting the relevant BPS states is charged under axions in $\mathcal{N} = 2$ hypermultiplets. It appears that BPS state counting is only well defined when that axionic charge vanishes.

Can one justify this from supergravity?

or else

What is BPS state counting when backreaction by Ooguri-Vafa string is taken into account?

V. Speculations

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Background Independence

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Witten (1993) has interpreted the holomorphic anomaly equation as an embodiment of background independence of the topological string. Naively, $\mathcal{F}^{(g)}$ should be holomorphic functions on moduli space. Consider worldsheet deformations

$$\delta S \sim (X^I + y^I) \int d^2x d^2\theta \phi_I + \bar{X}^I \int d^2x d^2\bar{\theta} \bar{\phi}_I$$

Topological theory should depend only on X^I, not on the $\bar{X}^I$. We need to adjust $\bar{X}^I$ to keep unitarity of $\mathcal{N} = 2$ worldsheet theory. This specifies the "background" around which one expands the topological string.

$\rightsquigarrow$ Holomorphic anomaly controls dependence of $\mathcal{F}^{(g)}$ on $\bar{X}^I$.

Consider total topological string amplitude

$$Z_{\rm top}(X^I, \bar{X}^I; y_I) \sim \exp\left[\sum_{g,n} \frac{\lambda^{2g-2}}{n!} \mathcal{F}^{(g)}_{i_1,\ldots,i_n}(X^I, \bar{X}^I) y^{i_1} \cdots y^{i_n}\right]$$

In appropriate variables (including $Z_{\rm top} \to \Psi_{\rm closed}$), holomorphic anomaly equation takes the "holomorphic" form, similar to "heat equation" (BCOV, E. Verlinde, Günaydin-Neitzke-Pioline)

$$\left[\frac{\partial}{\partial X^I} - \frac{1}{2} C_{IJK} \frac{\partial^2}{\partial y_J \partial y_K}\right] \Psi_{\rm closed} = 0\,,$$
$$\frac{\partial}{\partial \bar{X}^I} \Psi_{\rm closed} = 0\,,$$

As before, $C_{IJK} \sim \partial_I \partial_J \partial_K \mathcal{F}^{(0)}$ is three-point function on the sphere (Yukawa coupling).

This heat equation is equivalent to implementation of infinitesimal Bogliubov transformation when changing the holomorphic polarization in geometric quantization of symplectic vector space $H^3(Y)$ (B-model).

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Upshot: Topological string amplitudes $\mathcal{F}^{(g)}$ depend on the background complex structure order by order in perturbation theory.

But the total topological partition function admits an interpretation as a *background independent quantum state* in the auxiliary Hilbert space $\mathcal{H}_W$ from quantization of $H^3(Y)$.

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But the total topological partition function admits an interpretation as a *background independent quantum state* in the auxiliary Hilbert space $\mathcal{H}_W$ from quantization of $H^3(Y)$.

Puzzles:

- What is significance of $\mathcal{H}_W$?
- What selects $\Psi_{\mathrm{closed}} \in \mathcal{H}_W$?
- What are the other states?
- Relation to background independence in physical string?

Extended holomorphic anomaly sheds new light on these issues....

As it turns out (and this is not speculation), the extension of holomorphic anomaly equation to open/unoriented strings is equivalent to extending the heat equation by a "convection term," (Cook-Ooguri-Yang, 2007, Neitzke-J.W. 2007),

$$\left[\frac{\partial}{\partial X^I} - \frac{1}{2}C_{IJK}\ \frac{\partial^2}{\partial y_J \partial y_K} - i\Delta_{IJ}\frac{\partial}{\partial y_J}\right]\Psi_{\rm open} = 0\,,$$

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The vector field Δ_{IJ} is integrable: $\Delta_{IJ} = \partial_I \partial_J \mathcal{T}$, $\mathcal{T} \sim$ disk $(+$ crosscap$)$. $\rightsquigarrow$ The convection term can be absorbed by a *shift of variables*

$$\Psi^{\Delta}(X^I, y_I) = \Psi_{\rm open}(X^I, y_I - i\Delta_I)$$

Note that this is not a shift of background (as the y_I correspond to the fluctuations), but is in accord with general lines of research related to open/closed string correspondence ($e.g.$, geometric transitions). (Perhaps closest in AdS/CFT context is recent work by Kruczenski.)

Speculation 1. Significance of $\mathcal{H}_W$

After the shift, the open topological string partition function can be interpreted as a state $\Psi^\Delta \in \mathcal{H}_W$ in the same Hilbert space as the closed topological string.

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Semi-classically,

$$\Psi_{\text{open}} \sim \exp \lambda^{-1} \int^{C} \Omega, \qquad C : \text{holomorphic curve representing topological D-brane}$$

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Known facts about holomorphic curves in Calabi-Yau lead to the Conjecture: *The collection of all D-branes furnishes a basis of the entire Witten-Hilbert space $\mathcal{H}_W$.*

Speculation 2. Finite number of states

There is a fairly well-understood mathematical sense that the number of holomorphic curves in X of fixed topology is finite. As a result, there is only a finite number of possible D-brane configurations that cancel the tadpoles of any given orientifold plane.

This selects a finite number of quantum states Ψ^{Δ} in background independent Hilbert space $\mathcal{H}_W$.

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(Speculative) Conclusion: A "new" condition on the topological string (tadpole cancellation) reduces the number of physically relevant states to a finite number.

This would be a pretty realization of a basic idea about the ordinary physical string.