Harmony of Scattering Amplitudes: From Gauge Theory to $N = 8$ Supergravity

Strings, June 26, 2009

Zvi Bern, UCLA

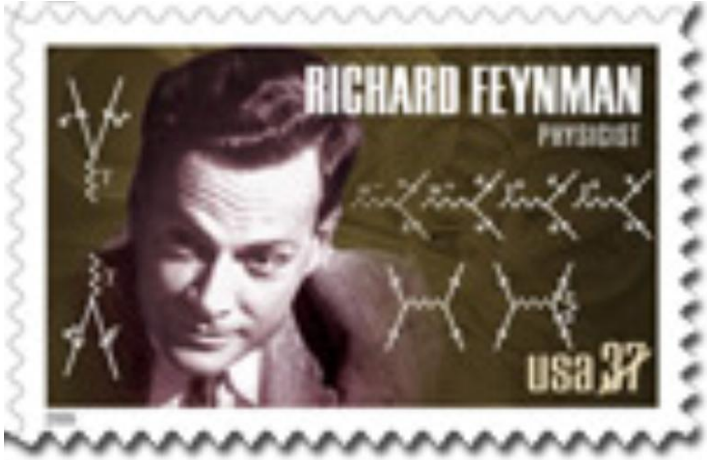
Overview + results from papers with:
J.J. Carrasco, L. Dixon, D. Forde, H. Ita, H. Johansson,
D. Kosower, V. Smirnov, M. Spradlin, R. Roiban and A. Volovich.

Outline

This talk will present some recent developments in understanding scattering amplitudes in gauge and gravity theories.

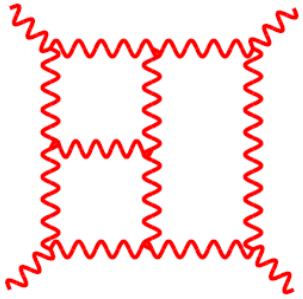
- **Harmony:** Examples of remarkable relations in gauge and gravity theories.
- **QCD:** Brief look at applications of new ideas to LHC physics
- **Supersymmetric gauge theory:** resummation of certain planar $N = 4$ super-Yang-Mills scattering amplitudes to *all* loop orders. A new symmetry: dual conformal invariance
- **Quantum gravity:** Simplicity and reexamination of standard wisdom on ultraviolet properties of quantum gravity. Four-loop confirmation of very strong UV cancellations.

Why are Feynman diagrams clumsy for high-loop or high-multiplicity processes?



- Vertices and propagators involve gauge-dependent off-shell states.
  An important origin of the complexity.

$$\int \frac{d^4p}{(2\pi)^4}$$



$p^2 \neq m^2$

- To get at root cause of the trouble we must rewrite perturbative quantum field theory.

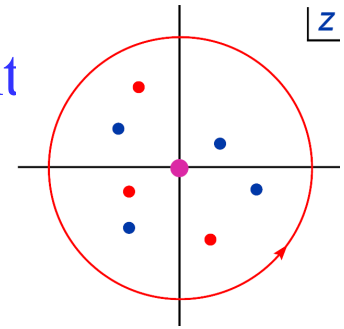
- **All steps should be in terms of gauge invariant on-shell states. On-shell formalism.** $p^2 = m^2$

ZB, Dixon, Dunbar, Kosower

On-Shell Recursion for Tree Amplitudes

Britto, Cachazo, Feng and Witten

Consider amplitude under complex shifts of the moment

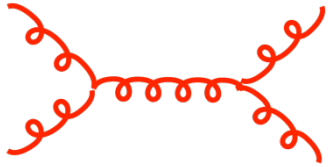


$$p_1^\mu(z) = p_1^\mu - zq^\mu \quad p_n^\mu(z) = p_n^\mu + zq^\mu \qquad q^2 = 0,\ p\cdot q = 0$$

$$(p_i^\mu(z))^2 = 0 \qquad \text{complex momenta}$$

If $A(z) \to 0, \quad z \to \infty$

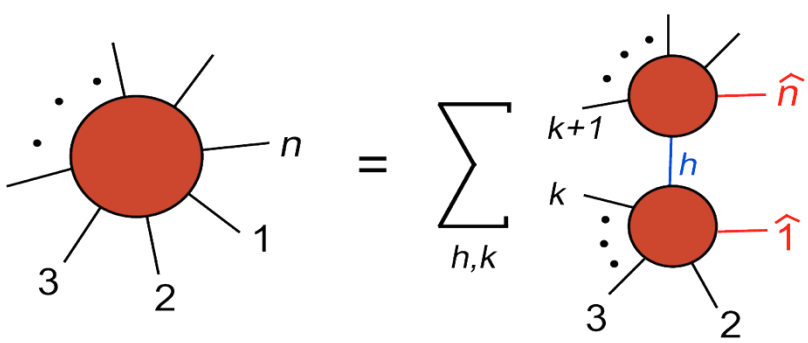
A(z) is amplitude with shifted momenta



$$\oint_{C_\infty} \frac{A(z)}{z} dz = 0 \quad \Rightarrow A(z=0) = -\sum_\alpha \mathrm{Res}_\alpha \frac{A(z)}{z}$$

$$A(z) = \sum_\alpha \frac{c_\alpha}{z - z_\alpha}$$

on-shell amplitude



Sum over residues gives the on-shell recursion relation

Poles in *z* come from kinematic poles in amplitude.

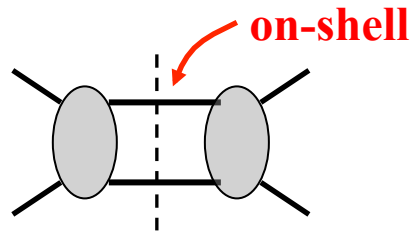
Same construction works in gravity

Brandhuber, Travaglini, Spence; Cachazo, Svrcek;
Benincasa, Boucher-Veronneau, Cachazo; Arkani-Hamed and Kaplan, Hall

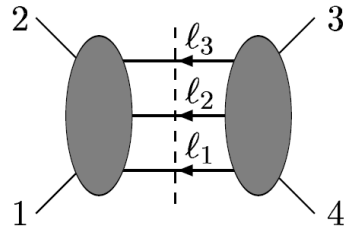
Bern, Dixon, Dunbar and Kosower

Modern Unitarity Method

Two-particle cut:



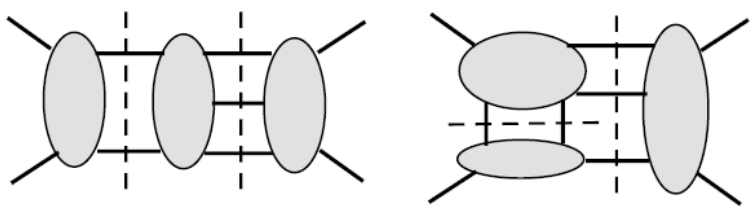
Three-particle cut:



Systematic assembly of complete amplitudes from cuts for any number of particles or loops.

Generalized unitarity as a practical tool:

Bern, Dixon and Kosower
Britto, Cachazo and Feng



Different cuts merged to give an expression with correct cuts in all channels.

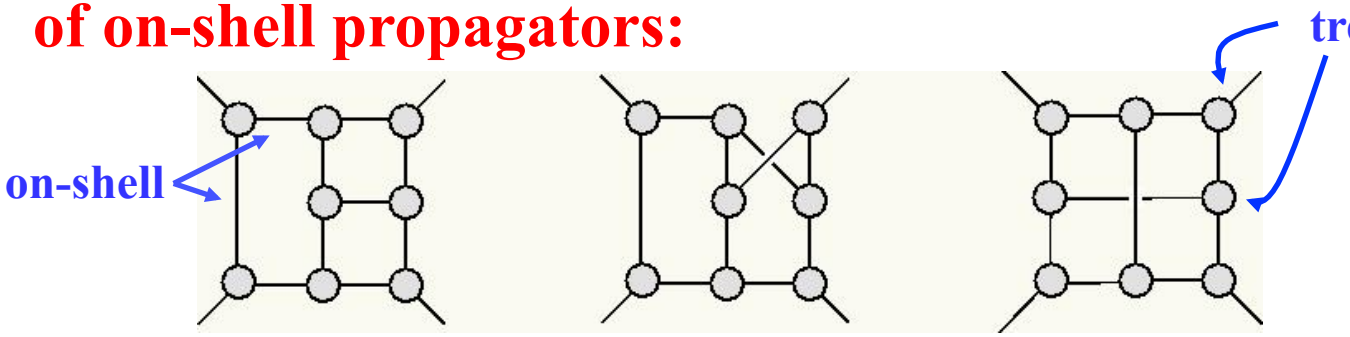
Generalized cut interpreted as cut propagators not canceling.

Method of Maximal Cuts

ZB, Carrasco, Johansson, Kosower

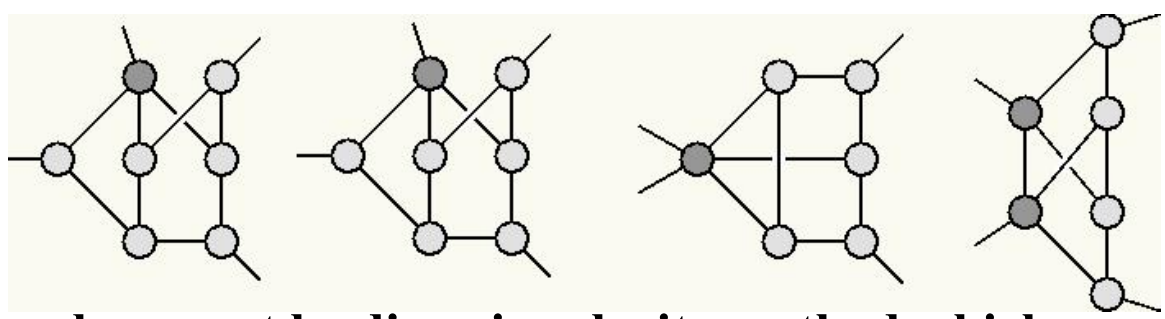
A refinement of unitarity method for constructing complete higher-loop amplitudes is "Method of Maximal Cuts". Systematic construction in any massless theory.

To construct the amplitude we use cuts with maximum number of on-shell propagators:



Maximum number of propagator placed on-shell.

Then systematically release cut conditions to obtain contact terms:

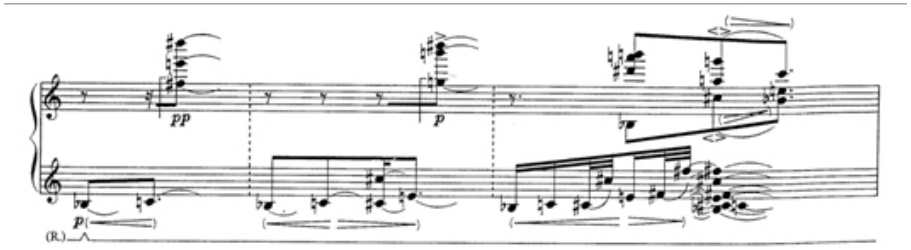


Fewer propagators placed on-shell.

Related to subsequent leading singularity method which uses hidden singularities.

Cachazo and Skinner; Cachazo; Cachazo, Spradlin, Volovich; Spradlin, Volovich, Wen

Examples of Harmony



Gravity vs Gauge Theory

Consider the gravity Lagrangian

$$L_{\text{gravity}} = \frac{2}{\kappa^2}\sqrt{-g}\,R$$

$$\kappa^2 = 32\pi\, G_{\text{Newton}}$$

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

flat metric

graviton field

metric

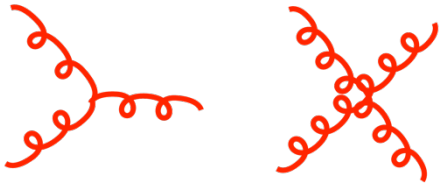
Infinite number of complicated interactions

$+ \cdots$

bloody mess

Compare to Yang-Mills Lagrangian on which QCD is based

$$L_{\text{YM}} = \frac{1}{g^2}F^2$$

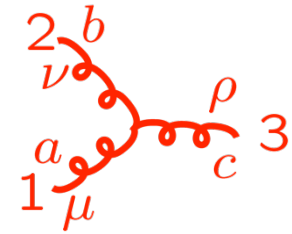


Only three and four point interactions

Gravity seems so much more complicated than gauge theory.

Does not look harmonious!

Three Vertices



Three gluon vertex:

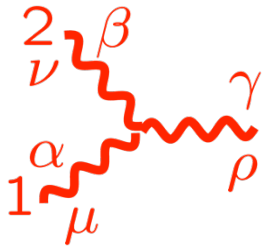
$$V_{3\,\mu\nu\sigma}^{abc} = -gf^{abc}(\eta_{\mu\nu}(k_1-k_2)_\rho+\eta_{\nu\rho}(k_1-k_2)_\mu+\eta_{\rho\mu}(k_1-k_2)_\nu)$$

$$k_i^2 = E_i^2 - \vec{k}_i^2 \neq 0$$

Three graviton vertex:

$$G_{3\mu\alpha,\nu\beta,\sigma\gamma}(k_1,k_2,k_3) =$$

$$\begin{aligned}
\mathrm{sym}[& -\tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \tfrac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) + \tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}) \\
& + P_6(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma}) \\
& + P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma}) \\
& + 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})]
\end{aligned}$$



About 100 terms in three vertex

Naïve conclusion: Gravity is a nasty mess.

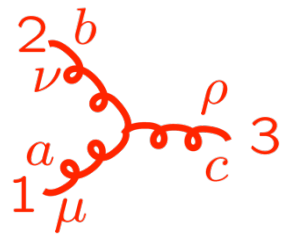
Not very harmonious!

Simplicity of Gravity Amplitudes

***On-shell* three vertices contain all information:**

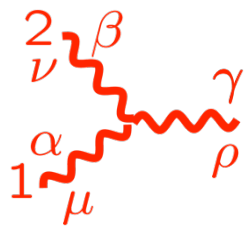
$$k_i^2 = 0$$

gauge theory:



$$-gf^{abc}(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$$

gravity:



$$i\kappa(\eta_{\mu\nu}(k_1 - k_1)_\rho + \text{cyclic})$$
$$\times\, (\eta_{\alpha\beta}(k_1 - k_2)_\gamma + \text{cyclic})$$

"square" of Yang-Mills vertex.

Any gravity scattering amplitude constructible solely from *on-shell* 3 vertex.

- **BCFW on-shell recursion for tree amplitudes.**

  Britto, Cachazo, Feng and Witten; Brandhuber, Travaglini, Spence; Cachazo, Svrcek; Benincasa, Boucher-Veronneau, Cachazo; Arkani-Hamed and Kaplan, Hall

- **Unitarity method for loops.**

  ZB, Dixon, Dunbar and Kosower; ZB, Dixon, Kosower; Britto, Cachazo, Feng; ZB, Morgan; Buchbinder and Cachazo; ZB, Carrasco, Johansson, Kosower; Cachzo and Skinner.

Gravity vs Gauge Theory

Consider the gravity Lagrangian

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$$\kappa^2 = 32\pi\, G_{\text{Newton}}$$

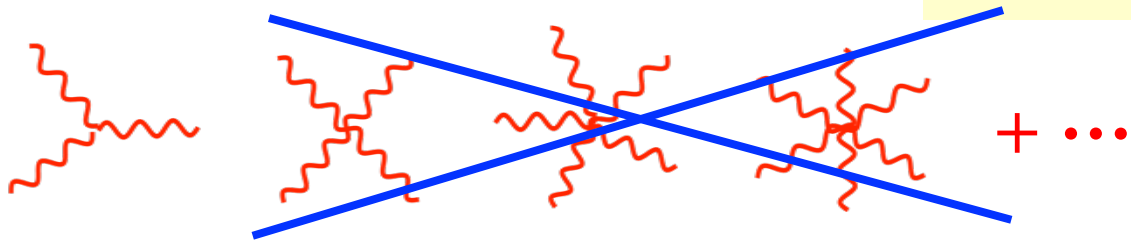
$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

flat metric

graviton field

metric

Infinite number of irrelevant interactions!

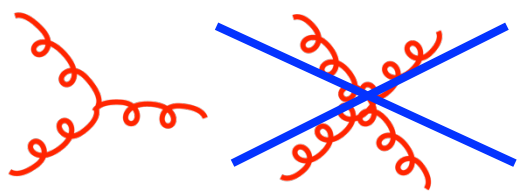


$+ \cdots$

Simple relation to gauge theory

Compare to Yang-Mills Lagrangian

$$L_{\text{YM}} = \frac{1}{g^2}F^2$$



Only three-point interactions

Gravity seems ~~so much~~ no more complicated than gauge theory.

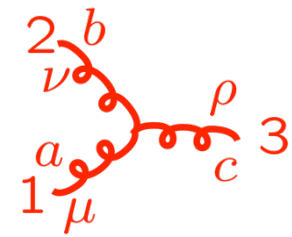
Does ~~not~~ look harmonious!

Harmony of Color and Kinematics

ZB, Carrasco, Johansson

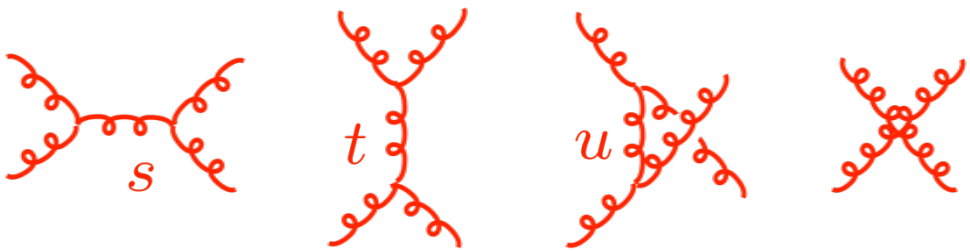
coupling constant — color factor — momentum dependent kinematic factor

$$-g f^{abc}(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$$



Color factors based on a Lie algebra: $[T^a, T^b] = i f^{abc} T^c$

Jacobi identity $[[T^a, T^b], T^c] + [[T^b, T^c], T^a] + [[T^c, T^a], T^b] = 0$



Use $1 = s/s = t/t = u/u$ to assign 4-point diagram to others.

$$\mathcal{A}_4^{\text{tree}} = g^2\left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}\right)$$

$s = (k_1 + k_2)^2$ $\quad u = (k_1 + k_3)^2$

$t = (k_1 + k_4)^2$

Color factors satisfy Jacobi identity: $c_u = c_s - c_t$

Numerator factors satisfy similar identity: $n_u = n_s - n_t$

Color and kinematics are singing same tune!

Harmony of Color and Kinematics

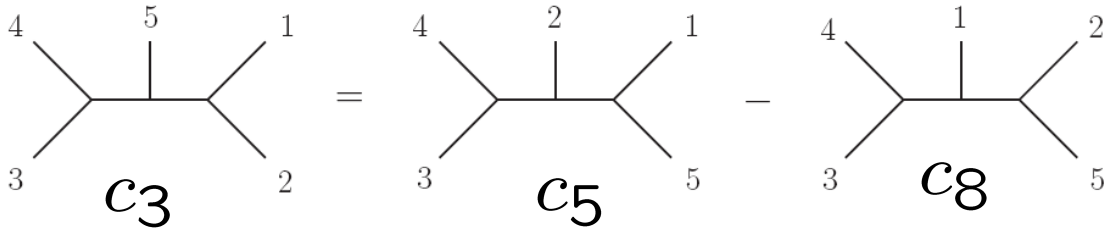
At higher points similar structure:

$$\mathcal{A}_5^{\text{tree}} = \sum_{i=1}^{15} \frac{c_i\, n_i}{D_i}$$

color factor

kinematic numerator factor

Feynman propagators



$$c_3 \equiv f^{a_3 a_4 b} f^{b a_5 c} f^{c a_1 a_2}\,, \qquad c_5 \equiv f^{a_3 a_4 b} f^{b a_2 c} f^{c a_1 a_5}\,, \qquad c_8 \equiv f^{a_3 a_4 b} f^{b a_1 c} f^{c a_2 a_5}$$

$$\boxed{c_3 - c_5 + c_8 = 0 \quad \Leftrightarrow \quad n_3 - n_5 + n_8 = 0}$$

Claim: We can always find a rearrangement so color and kinematics satisfy the *same* Jacobi constraint equations.

- **Color and kinematics sing same tune!**
- **Nontrivial constraints on amplitudes.**

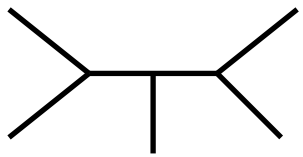
Higher-Point Gravity and Gauge Theory

Gauge theory: $\mathcal{A}_n^{\text{tree}} = i g^{n-2} \sum_i \frac{c_i\, n_i}{D_i}$

ZB, Carrasco, Johansson

sum over diagrams with only 3 vertices

Einstein Gravity: $\mathcal{M}_n^{\text{tree}} = i \kappa^{n-2} \sum_i \frac{n_i^2}{D_i}$



Same relations between N=4 sYM and N = 8 sugra

Claim: This is unproven but it is correct! Related to KLT relations

Another recent relation:

Elvang, Freedman
Drummond, Spradlin, Volovich, Wen

Gauge: $\mathcal{A}^{\text{tree}}(1,\ldots,n) = A^{\text{MHV}}(1,\ldots,n) \sum_{\{\alpha\}} R_\alpha(\lambda_i, \tilde{\lambda}_i, \eta_i)$

Dual conformal invariants

Gravity: $\mathcal{M}^{\text{tree}}(1,\ldots,n) = \sum_{\text{perms}} [A^{\text{MHV}}(1,\ldots,n)]^2 \sum_{\{\alpha\}} [R_\alpha(\lambda_i, \tilde{\lambda}_i, \eta_i)]^2 G_\alpha(\lambda_i, \tilde{\lambda}_i)$

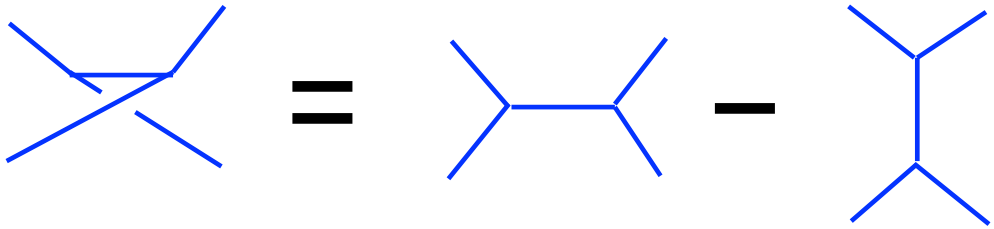
Gravity and gauge theory kinematic numerators sing same tune!

Cries out for a unified description of the sort given by string theory. Symmetries of YM have echo in gravity!

Non-Planar from Planar

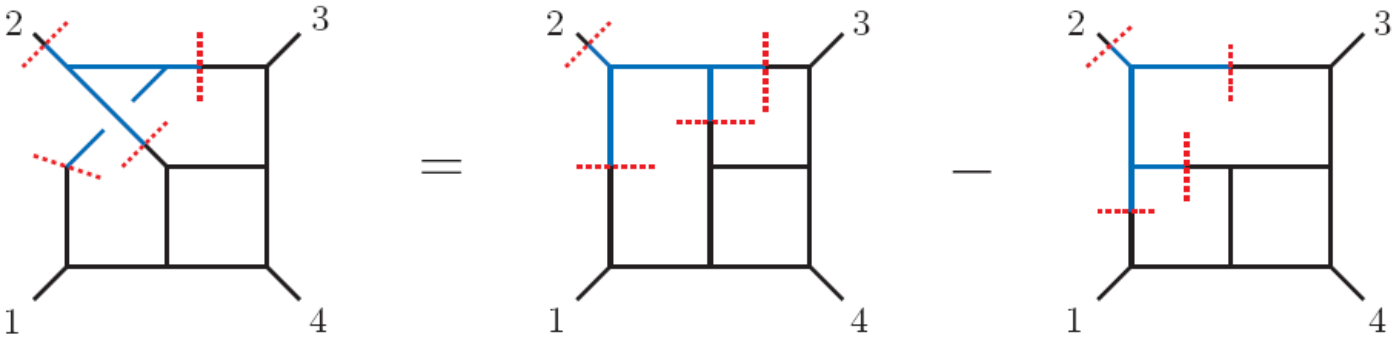
ZB, Carrasco, Johansson

We can use Jacobi-like identities to obtain non-planar from planar contributions at higher loops!



Satisfied by Yang-Mills numerator factors

For example for three loop (super) Yang-Mills



key tool for high loop calculations

non-planar **planar**

Symmetries of planar obviously restrict the structure of non-planar as well (*e.g.* dual conformal symmetry)

Twistors

In 2004 Ed Witten demonstrated that twistor space reveals a hidden structure in scattering amplitudes.

Penrose twistor transform:

Early work from Nair

$$\widetilde{A}(\lambda_i, \mu_i) = \int \prod_i \frac{d^2\widetilde{\lambda}_i}{(2\pi)^2} \exp\left(\sum_j \mu_j^{\dot{a}} \widetilde{\lambda}_{j\dot{a}}\right) A(\lambda_i, \widetilde{\lambda}_i)$$

Witten's twistor-space link:

Witten; Roiban, Spradlin and Volovich

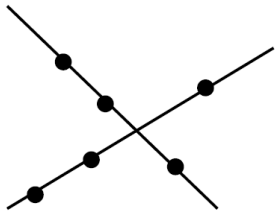
***N* = 4 scattering amplitudes** $\longleftrightarrow$ **Topological String Theory**

Amazing Simplicity

Witten conjectured that in twistor–space gauge theory amplitudes have delta-function support on curves of degree:

$$d = q - 1 + L, \quad q = \#\ \text{negative helicities}, \quad L = \#\ \text{loops},$$

Connected picture



Disconnected picture

CSW diagrams
MHV vertices

Structures imply an amazing simplicity in the scattering amplitudes.

Witten
Roiban, Spradlin and Volovich
Cachazo, Svrcek and Witten
Gukov, Motl and Neitzke
Bena Bern and Kosower

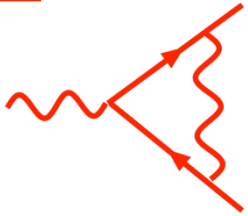
Gravity and gauge theory share same structure! Derivative of delta-function support instead of delta-function support

See Nima's talk for a modified version: "ambi-twistor space"
Remarkable simplicity exposed in twistor space via BCFW

Applications to LHC Physics

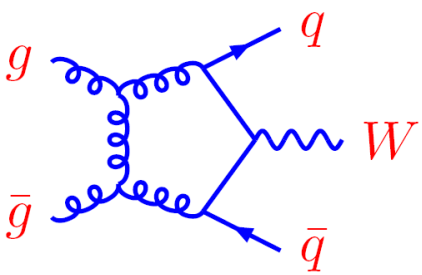
State-of-the-Art Loop Calculations

In 1948 Schwinger computed anomalous magnetic moment of the electron.

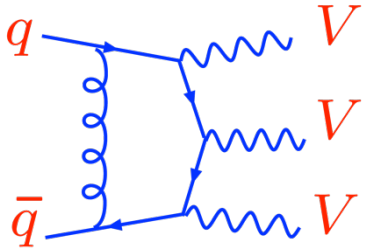


60 years later typical examples:

$pp \to W, Z + 2$ jets



$pp \to VVV \qquad V = Z, W$



Amusing numbers of diagrams:

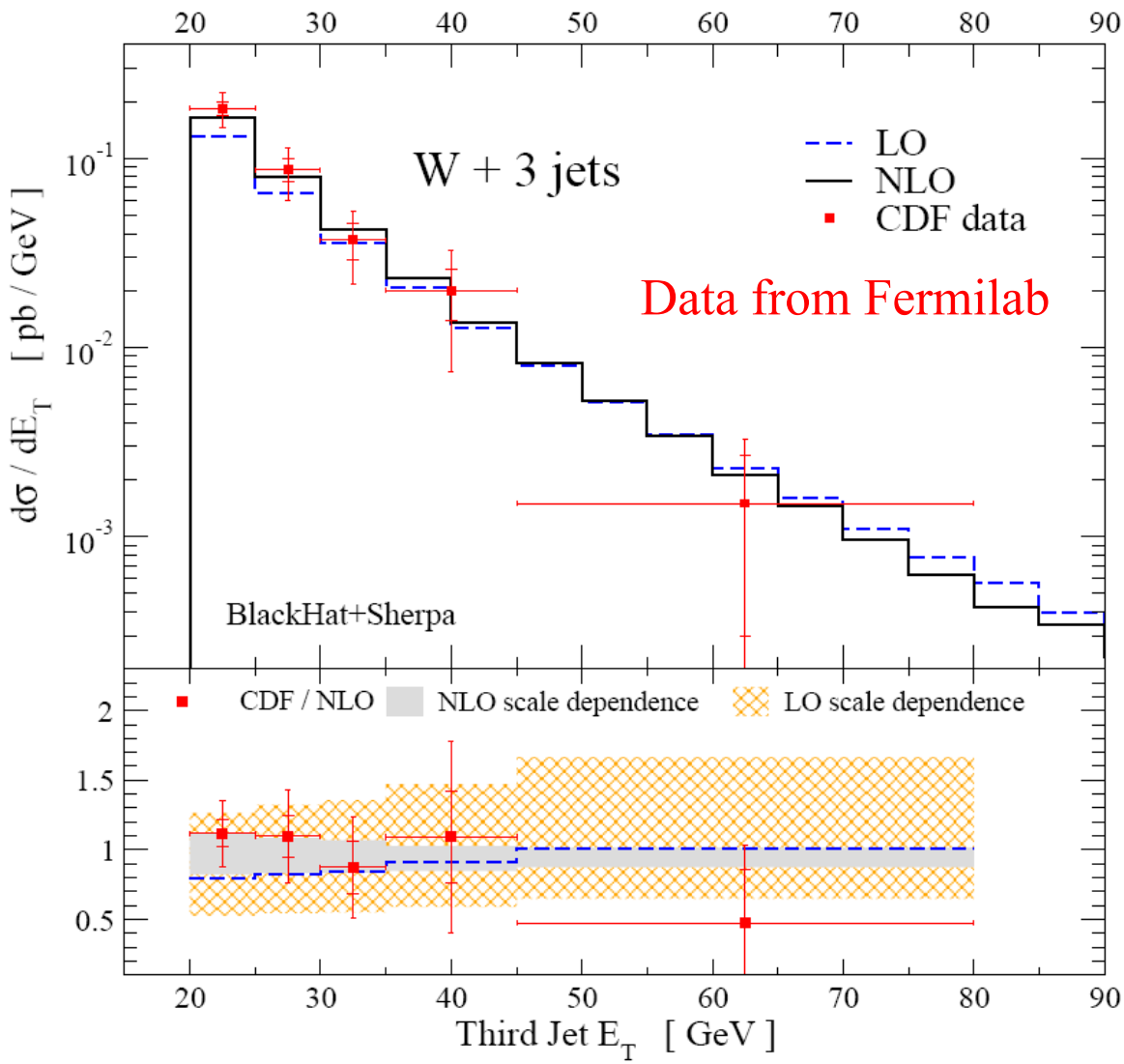
6g: 10,860 7g: 168,925 8g: 3,017,490

Much worse difficulty: integral reduction generates nasty dets.

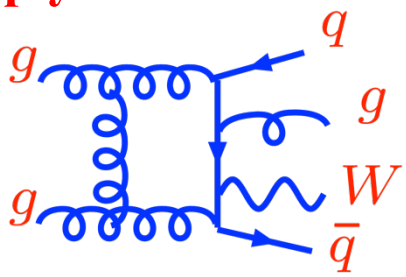
$$\frac{1}{\det(k_i \cdot k_j)^n}$$ Gram determinant singularities

Application: State of the Art QCD for the LHC

Berger, ZB, Dixon, Febres Cordero, Forde, Gleisberg, Ita, Kosower, Maitre **(BlackHat collaboration)**



Apply on-shell methods



Complexity has prevented calculations via Feynman diagrams.

Excellent agreement between NLO theory and experiment. Triumph of on-shell methods!

Have similar predictions for LHC.

Amplitudes and AdS/CFT

$N = 4$ Super-Yang-Mills to All Loops

Since 't Hooft's paper thirty years ago on the planar limit of QCD we have dreamed of solving QCD in this limit. This is too hard. $N = 4$ sYM is much more promising.

- **Special theory because of AdS/CFT correspondence.**
- **Maximally supersymmetric.**

Remarkable relation

Alday and Maldacena

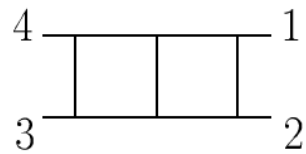
scattering at strong coupling in $N = 4$ sYM $\longleftrightarrow$ classical string theory in AdS space

To make this link need to evaluate $N = 4$ super-Yang-Mills amplitudes to *all* loop orders. Seems impossible even with modern methods.

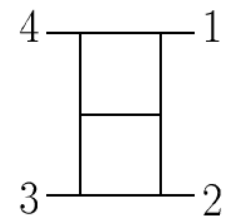
Loop Iteration of the $N = 4$ Amplitude

The planar four-point two-loop amplitude undergoes fantastic simplification.

$$-\,st\,A_4^{\text{tree}}\left\{ s \right.$$



$$+\;t$$



$$\left. \right\}$$

ZB, Rozowsky, Yan

$$M_4^{2\text{-loop}}(s,t) = \frac{1}{2}\left(M_4^{1\text{-loop}}(s,t)\right)^2 + f(\epsilon)\,M_4^{1\text{-loop}}(s,t)\big|_{\epsilon\to 2\epsilon} - \frac{1}{2}\zeta_2^2$$

$$M_4^{\text{loop}} = A_4^{\text{loop}}/A_4^{\text{tree}} \qquad f(\epsilon) = -\zeta_2 - \zeta_3\epsilon - \zeta_4\epsilon^2$$

Anastasiou, ZB, Dixon, Kosower

$f(\epsilon)$ is universal function related to IR singularities $\qquad D = 4 - 2\epsilon$

This gives two-loop four-point planar amplitude as iteration of one-loop amplitude.

Three loop satisfies similar iteration relation. Rather nontrivial.

ZB, Dixon, Smirnov

All-Loop Generalization

Why not be bold and guess scattering amplitudes for all loop and all legs, at least for simple helicity configurations?

$$\mathcal{A}_n = A_n^{\text{tree}} A_n^{\text{divergent}} \exp\left[\frac{1}{4}\gamma_K F_n^{\text{1-loop}} + C\right]$$

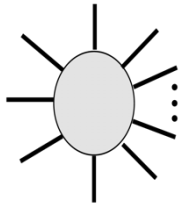
constant independent of kinematics.

all-loop resummed amplitude

IR divergences

cusp anomalous dimension

finite part of one-loop amplitude



$$F_4 = \frac{1}{4}\gamma_K \ln^2\left(\frac{s_{12}^2}{s_{23}^2}\right) + \text{const}$$

"BDS conjecture"

Anastasiou, ZB, Dixon, Kosower
ZB, Dixon and Smirnov

- **Limit of collinear momenta gives us key analytic information, at least for MHV amplitudes, restricting form of anzatz**
- **IR singularities agree with Magnea and Sterman formula.**

Gives a definite prediction for *all* values of coupling given BES integral equation for the cusp anomalous dimension.

Beisert, Eden, Staudacher

Alday and Maldacena Strong Coupling

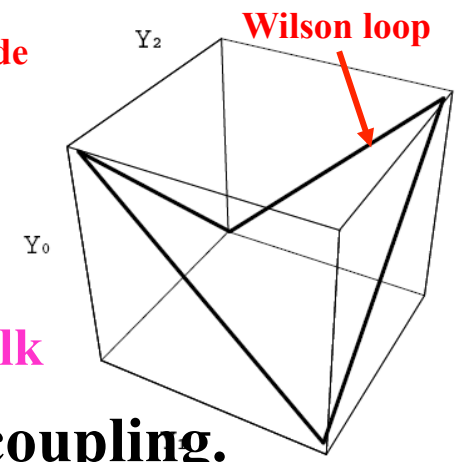
For MHV amplitudes: $F_4^{1\text{-loop}} = \frac{1}{2}\ln^2(s/t) + \frac{2\pi^2}{3}$ ZB, Dixon, Smirnov

constant independent of kinematics.

$$\mathcal{A}_4 = A_4^{\text{tree}} A_4^{\text{divergent}} \exp\left[\frac{1}{4}\gamma_K F_4^{1\text{-loop}} + C\right]$$

all-loop resummed amplitude **IR divergences** **cusp anomalous dimension** **finite part of one-loop amplitude**

In a beautiful paper Alday and Maldacena confirmed the conjecture for 4 gluons at *strong coupling* from an AdS string theory computation. Minimal surface calculation. **See Maldcena's talk**



Very suggestive link to Wilson loops even at weak coupling.

Drummond, Korchemsky, Henn, Sokatchev ; Brandhuber, Heslop and Travaglini
ZB, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich;
Anastasiou, Branhuber, Heslop, Khoze, Spence, Travagli,

- **Identication of new symmetry: "dual conformal symmetry"**

Drummond, Henn, Korchemsky, Sokatchev;
Beisert, Ricci, Tseytlin, Wolf; Brandhuber, Heslop, Travaglini;
Berkovits and Maldacena

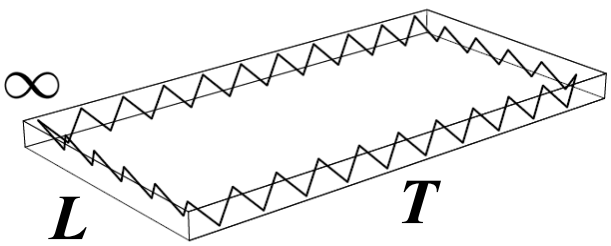
All-loop Trouble at Higher Points

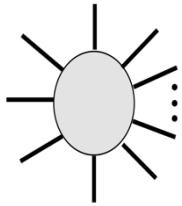
For various technical reasons it is hard to solve for minimal surface for large numbers of gluons.

Alday and Maldacena realized certain terms can be calculated at strong coupling for an infinite number of gluons.

Disagrees with BDS conjecture

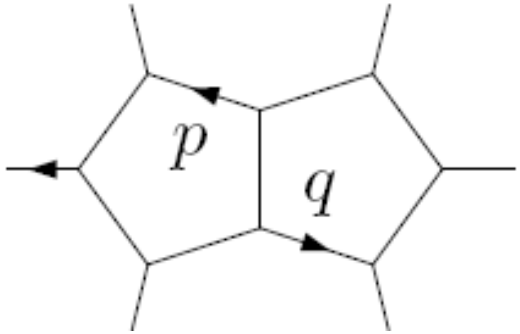
$T/L \to \infty$





Trouble also in the Regge limit. $s/t \to \infty$

Bartels, Lipatov, Sabio Vera



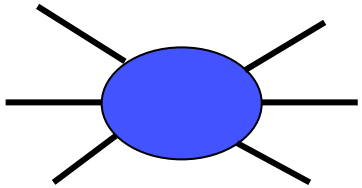
Explicit computation at 2-loop six points.
Need to modify conjecture!

ZB, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich
Drummond, Henn, Korchemsky, Sokatchev

Dual conformal invariance and equivalence to Wilson loops persists

Can the BDS conjecture be repaired for six and higher points?

In Search of the Holy Grail



log of the amplitude

discrepancy

$$A^{\text{truth}} = A^{\text{div}} + A^{\text{BDS}} + R$$

Can we figure out the discrepancy?

Important new information from regular polygons should serve as a guide.

Explicit solution at eight points

$$k_i = x_{i+1} - x_i$$

$$A_{BDS} = -\frac{1}{4}\sum_{i=1}^{n}\sum_{j=1, j\neq i, i-1}^{n} \log\frac{x_j^+ - x_i^+}{x_{j+1}^+ - x_i^+}\log\frac{x_j^- - x_{i-1}^-}{x_j^- - x_i^-}$$

Alday and Maldacena (2009)

$$A = A_{div} + A_{BDS} + R$$

$$R = -\frac{1}{2}\log(1+\chi^-)\log(1+\frac{1}{\chi^+}) + \frac{7\pi}{6} + \int_{-\infty}^{\infty} dt \frac{|m|\sinh t}{\tanh(2t+2i\phi)}\log\left(1+e^{-2\pi|m|\cosh t}\right)$$

Solution valid only for strong coupling and special kinematics, but it's explicit!

See Maldacena's talk

Symmetry and Ward Identities

A key to understanding the structure of amplitudes is symmetry.

Planar $N = 4$ YM has three distinct interlocked symmetries:

- **Supersymmetry**
- **Conformal symmetry**
- **Dual conformal symmetry**

} **Yangian structure**

See Henn's talk

Drummond, Henn Plefka; Bargheer, Beisert, Galleas, Loebbert, McLoughlin

Relatively simple at tree level for generic kinematics

dual variables

$$k_i = x_{i+1} - x_i$$

conformal: $$k_{a\dot{a}} = \sum_i \frac{\partial^2}{\partial \lambda_i^a \partial \tilde{\lambda}_i^{\dot{a}}}, \qquad k_{a\dot{a}} A_n^{\text{tree}} = 0$$

dual conformal: $$K^{a\dot{a}} = \sum \left[x_i^{a\dot{b}} x_i^{\dot{a}b} \frac{\partial}{\partial x_i^{b\dot{b}}} + \ldots \right], \qquad K_{a\dot{a}} A_n^{\text{tree}} = 0$$

$$\langle \lambda \mu \rangle \equiv \epsilon_{ab} \lambda^a \lambda^b$$

holomorphic anomaly: $$\frac{\partial}{\partial \tilde{\lambda}^{\dot{a}}} \frac{1}{\langle \lambda, \mu \rangle} = \pi \delta^2(\langle \lambda, \mu \rangle) \epsilon_{\dot{a}\dot{b}} \bar{\mu}^b$$

Cachazo, Svrcek and Witten
Beisert et al,
Korchemsky and Sokatchev

Complex variables $z, \bar{z}$ not independent at $z = 0$. Collinear momenta means extra contribution, but simple at tree level.

What about loop level?

Dual Conformal Ward Identity

Drummond, Henn, Korchemsky, Sokatchev

If we *assume* that IR singularities and dim reg breaks dual conformal symmetry at multi-loops identically as at one loop, get a remarkably nice anomalous dual conformal Ward Identity

$$K^{a\dot{a}} F_n = \sum_{i=1}^{n} [2x^{a\dot{a}}(x\cdot\partial_{x_i}) - x_i^2 \partial_{x_i}^{a\dot{a}}] F_n = \frac{1}{2}\gamma_K \sum_{i=1}^{n} x_{i,i+1}^{a\dot{a}} \ln\left(\frac{x_{i,i+2}^2}{x_{i-1,i+1}^2}\right)$$

Solution gives exactly BDS ansatz

See Henn's talk

$$F_4 = \frac{1}{4}\gamma_K \ln^2\left(\frac{x_{13}^2}{x_{24}^2}\right) + \text{const}$$

$$\mathcal{A}_n = A_n^{\text{tree}} A_n^{\text{divergent}} \exp\left[\frac{1}{4}\gamma_K F_n^{\text{1-loop}} + C\right]$$

$$F_5 = -\frac{1}{8}\gamma_K \sum_{i=1}^{5} \ln\left(\frac{x_{i,i+2}^2}{x_{i,i+3}^2}\right) \ln\left(\frac{x_{i+1,i+3}^2}{x_{i+2,i+4}^2}\right) + \text{const}$$

$$k_i = x_{i+1} - x_i$$

Starting at six points any function of conformal cross ratios annihilated by *K*

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}, \qquad u_2 = \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2}, \qquad u_3 = \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2}$$

No unique solution!

Can we use the ordinary conformal symmetry to help?

$$k^{a\dot{a}} F_n = ???$$

How does conformal symmetry act on finite part of high-loop amplitudes?

IR singularities and holomorphic anomaly complicates this.

Surprise is *not* complexity of multiloop conformal Ward identity.

Surprise is simplicity of the dual conformal Ward identity.

Not even obvious for general amplitudes at one loop.

Brandhuber, Heslop, Travaglini

Supersymmetry

Bianchi, Freedman, Elvang, Kiermaier; Arkanki-Hamed, Cachazo, Kaplan; Brandhuber, Spence, Travaglini; Drummond, Korchemsky, Henn, Sokatchev; ZB, Carrasco, Ita, Johansson, Roiban; Hall

Nair's on-shell *N*=4 sYM superspace for MHV amplitudes:

Combine MHV amplitudes in single superamplitude

spinor

$$\mathcal{A}_n^{\rm MHV}(p_1,\eta_1;\ldots;p_n,\eta_n) = i(2\pi)^4 \frac{\delta^{(4)}\left(\sum_{i=1}^n p_i\right)\,\delta^{(8)}\left(\sum_{i=1}^n \lambda_i^\alpha \eta_i^A\right)}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}$$

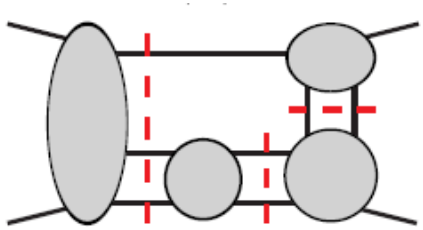
grassmann parameter

Two basic approaches for trees: Use either a BCFW or MHV vertex approach build a generating function for all *N*=4 sYM amplitudes.

Bianchi, Elvang, Freedman; Arkanki-Hamed, Cachazo, Kaplan; Brandhuber, Spence and Travaglini; Drummond Korchemsky, Henn, Sokatchev; Drummond and Henn; Kiermaier, Naculich

Powerful way to bookkeep sums over supermultiplet:

Supercut sewing for loops $\mathcal{C} = \int \left[\prod_{i=1}^k d^4\eta_i\right] \mathcal{A}^{\rm tree}_{(1)}\mathcal{A}^{\rm tree}_{(2)}\mathcal{A}^{\rm tree}_{(3)}\cdots\mathcal{A}^{\rm tree}_{(m)}$



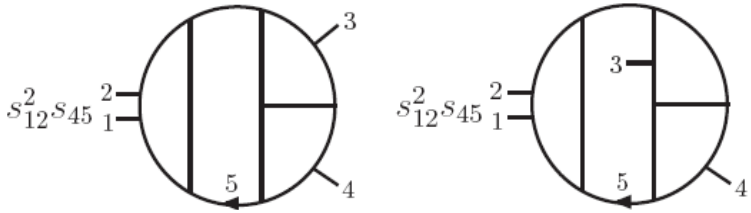
- **Exposes dual superconformal invariance.**
  Drummond, Korchemsky, Henn, Sokatchev; Brandhuber, Heslop, Travaglini; Hall
- **Used in proof of "no-triangle property" of *N* = 8 sugra.**
  Arkanki-Hamed, Cachazo, Kaplan
- **Used in four loop calculation of *N* = 4 sYM and *N* = 8 supergravity**
  ZB, Carrasco, Dixon, Johansson, Roiban; Elvang, Freedman, Kiermaier
- **Theories with fewer susys.** ZB, Carrasco, Ita, Johansson, Roiban

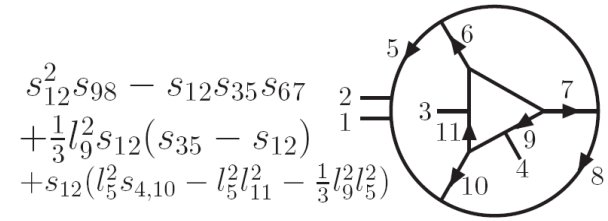
Major annoyance: Only half the susy manifest, chiral superspace.
We can expose *all* susy cancellations, but it's clumsier than we would like.

$N = 4$ sYM Four-Loop Non-Planar Amplitude

Some N =4 YM contributions:

ZB, Carrasco, Dixon, Johansson, Roiban





50 distinct planar and non-planar diagrammatic topologies.

$$I_i = \int \left[\prod_{p=1}^{4} \frac{d^D l_{n_p}}{(2\pi)^D} \right] \frac{N_i(l_j, k_j)}{\prod_{n=1}^{13} l_n^2}$$

Susy cancellations manifest!

UV finite for $D < \frac{6}{L} + 4, \;\; L = 4$

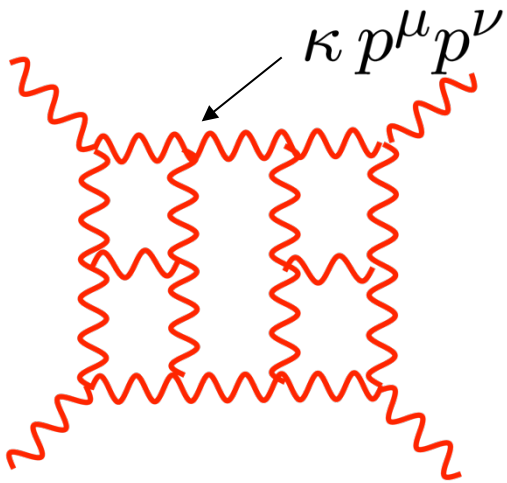
Applications:

- **IR singularities in $D = 4$ QCD:** **Once we are able to integrate non-trivial 3 and 4 loop non-planar integrals, we will find out which recent conjectures for soft anomalous dimension matrix are correct.**
  Dixon, Magnea, Sterman, Gardi and Magnea, Becher and Neubert; Dixon
- **UV divergences in $D > 4$:** **Used to investigate UV divergences higher dimensions. Certain additional subleading color vanishings of UV divergences. Compare to string theory.** See Michael Green's talk
- **Key input to four loop N=8 supergravity:** **Use the squaring relation in the cuts.**

UV Properties of $N = 8$ Supergravity

Is a UV finite theory of gravity possible?

$\kappa = \sqrt{32\pi G_N}$ ← Dimensionful coupling



Gravity: $$\int \prod_{i=1}^{L} \frac{dp_i^D}{(2\pi)^D} \frac{(\kappa\, p_j^\mu p_j^\nu)\cdots}{\text{propagators}}$$

Gauge theory: $$\int \prod_{i=1}^{L} \frac{d^D p_i}{(2\pi)^D} \frac{(g\, p_j^\nu)\cdots}{\text{propagators}}$$

Extra powers of loop momenta in numerator means integrals are badly behaved in the UV

Much more sophisticated power counting in supersymmetric theories but this is the basic idea.

Reasons to focus on $N = 8$ maximal supergravity: Cremmer and Julia

- **With more susy suspect better UV properties.**
- **High symmetry implies simplicity. Much simpler than expected. May be "simplest theory".** See Nima's talk

Finiteness of Point-Like Gravity Theory?

We are interested in UV finiteness of $N = 8$ supergravity because it would imply a new symmetry or non-trivial dynamical mechanism.

The discovery of either would have a fundamental impact on our understanding of gravity.

- **Here we only focus on order-by-order UV finiteness.**
- **Non-perturbative issues and viable models of Nature are *not* the goal for now.**

Opinions from the 80's

Unfortunately, in the absence of further mechanisms for cancellation, the analogous $N = 8$ $D = 4$ supergravity theory would seem set to diverge at the three-loop order.

Howe, Stelle (1984)

There are no miracles… It is therefore very likely that all supergravity theories will diverge at three loops in four dimensions. … The final word on these issues may have to await further explicit calculations. Marcus, Sagnotti (1985)

The idea that *all* supergravity theories diverge has been widely accepted for over 25 years

Divergences in Gravity

One loop: $R^2,\ R_{\mu\nu}^2,\ R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}$

Vanish on shell

vanishes by Gauss-Bonnet theorem

't Hooft, Veltman (1974)

Pure gravity 1-loop finite (but not with matter)

Two loop: Pure gravity counterterm has non-zero coefficient:

$$R^3 \equiv R^{\lambda\rho}{}_{\mu\nu}R^{\mu\nu}{}_{\sigma\tau}R^{\sigma\tau}{}_{\lambda\rho}$$

Any supergravity:

Goroff, Sagnotti (1986); van de Ven (1992)

R^3 is ***not*** a valid supersymmetric counterterm.

Produces a helicity amplitude $(-,+,+,+)$ forbidden by susy.

Grisaru (1977); Tomboulis (1977)

The first divergence in *any* supergravity theory can be no earlier than three loops.

R^4 **squared Bel-Robinson tensor expected counterterm**

Deser, Kay, Stelle (1977); Kaku, Townsend, van Nieuwenhuizen (1977), Ferrara, Zumino (1978)

Reasons to Reexamine This

1) **Discovery of remarkable cancellations at 1 loop – the "no-triangle property". Nontrivial cancellations!**
ZB, Dixon, Perelstein, Rozowsky; ZB, Bjerrum-Bohr, Dunbar; Bjerrum-Bohr, Dunbar, Ita, Perkins, Risager; Bjerrum-Bohr, Vanhove; Arkani-Hamed Cachazo, Kaplan

2) ***Every* explicit loop calculation to date finds $N = 8$ supergravity has identical power counting as $N = 4$ super-Yang-Mills theory, which is UV finite.** Green, Schwarz and Brink; ZB, Dixon, Dunbar, Perelstein, Rozowsky; Bjerrum-Bohr, Dunbar, Ita, PerkinsRisager; ZB, Carrasco, Dixon, Johanson, Kosower, Roiban.

3) **Interesting hint from string dualities.** Chalmers; Green, Vanhove, Russo
   – **Dualities restrict form of effective action. May prevent divergences from appearing in $D = 4$ supergravity, athough issues with decoupling of towers of massive states and indirect.**

4) **Interesting string non-renormalization theorem from Berkovits. Suggests divergence delayed to nine loops, but needs to be redone directly in field theory not string theory.** Green, Vanhove, Russo

Where is First D= 4 UV Divergence in N = 8 Sugra?

Various opinions:

3 loops	Conventional superspace power counting	**Green, Schwarz, Brink (1982)** **Howe and Stelle (1989)** **Marcus and Sagnotti (1985)**
5 loops	Partial analysis of unitarity cuts; *If* $\mathcal{N}$ = 6 harmonic superspace exists; algebraic renormalisation argument	**Bern, Dixon, Dunbar, Perelstein, Rozowsky (1998)** **Howe and Stelle (2003,2009)**
6 loops	*If* $\mathcal{N}$ = 7 harmonic superspace exists	**Howe and Stelle (2003)**
7 loops	*If* $\mathcal{N}$ = 8 harmonic superspace exists; lightcone gauge locality arguments	**Grisaru and Siegel (1982); Kallosh (2009)**
8 loops	Explicit identification of potential susy invariant counterterm with full non-linear susy	**Kallosh; Howe and Lindström (1981)**
9 loops	Assume Berkovits' superstring non-renormalization theorems can be carried over to D=4 $\mathcal{N}$ = 8 supergravity and extrapolate to 9 loops.	**Green, Russo, Vanhove (2006)** **Berkovits, Green, Vanhove, to appear**

Note many of the statements are contradictory.

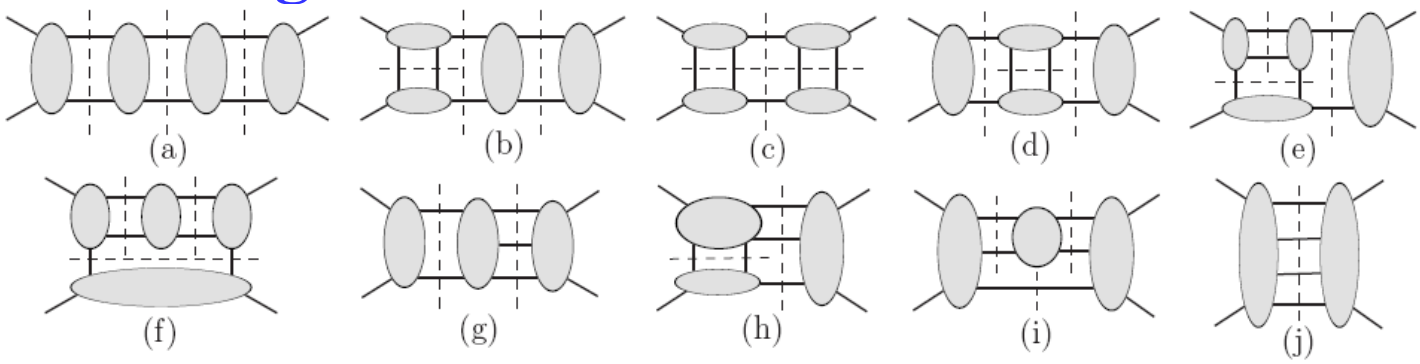
Note: no divergence demonstrated above. Arguments based on lack of susy protection!

To end debate, we need solid results!

Full Three-Loop Calculation

ZB, Carrasco, Dixon, Johansson, Kosower, Roiban

Need following cuts:



reduces everything to product of tree amplitudes

For cut (g) have:

$$\sum_{N=8 \text{ states}} M_4^{\text{tree}}(1,2,l_3,l_1) \times M_5^{\text{tree}}(-l_1,-l_3,q_3,q_2,q_1) \times M_5^{\text{tree}}(3,4,-q_1,-q_2,-q_3)$$

Use Kawai-Lewellen-Tye tree relations

$$M_4^{\text{tree}}(1,2,l_3,l_1) = -is_{12} A_4^{\text{tree}}(1,2,l_3,l_1) A_4^{\text{tree}}(2,1,l_3,l_1)$$

$$M_5^{\text{tree}}(-l_1,-l_3,q_3,q_2,q_1) = i\, s_{l_1 q_1} s_{l_3 q_3} A_5^{\text{tree}}(-l_1,-l_3,q_3,q_2,q_1)\, A_5^{\text{tree}}(-l_1,q_1,q_3,-l_3,q_2) + \{l_1 \leftrightarrow l_3\}\,,$$

supergravity

super-Yang-Mills

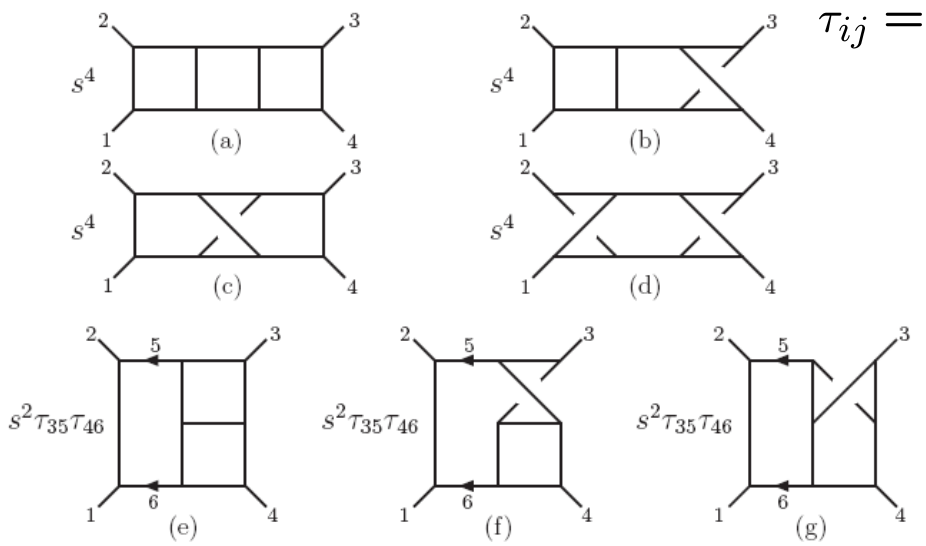
$N = 8$ supergravity cuts are sums of products of $N = 4$ super-Yang-Mills cuts

Complete Three-Loop Result

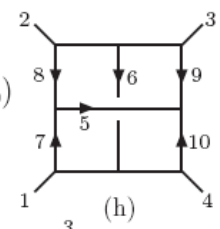
ZB, Carrasco, Dixon, Johansson, Kosower, Roiban; hep-th/0702112
ZB, Carrasco, Dixon, Johansson, Roiban arXiv:0808.4112 [hep-th]

$$M_4^{(3)} = \left(\frac{\kappa}{2}\right)^8 stu M_4^{\text{tree}} \sum_{S_3} \left[I^{(a)} + I^{(b)} + \tfrac{1}{2} I^{(c)} + \tfrac{1}{4} I^{(d)} + 2I^{(e)} + 2I^{(f)} + 4I^{(g)} + \tfrac{1}{2} I^{(h)} + 2I^{(i)} \right]$$

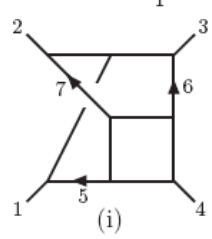
$$\tau_{ij} = 2k_i \cdot k_j$$



$$(s(\tau_{26}+\tau_{36}) + t(\tau_{15}+\tau_{25}) + st)^2$$
$$+(s^2(\tau_{26}+\tau_{36}) - t^2(\tau_{15}+\tau_{25}))(\tau_{17}+\tau_{28}+\tau_{39}+\tau_{4,10})$$
$$+s^2(\tau_{17}\tau_{28}+\tau_{39}\tau_{4,10}) + t^2(\tau_{28}\tau_{39}+\tau_{17}\tau_{4,10})$$
$$+u^2(\tau_{17}\tau_{39}+\tau_{28}\tau_{4,10})$$



$$(s\tau_{45} - t\tau_{46})^2 - \tau_{27}(s^2\tau_{45} + t^2\tau_{46})$$
$$-\tau_{15}(s^2\tau_{47} + u^2\tau_{46}) - \tau_{36}(t^2\tau_{47} + u^2\tau_{45})$$
$$+l_5^2 s^2 t + l_6^2 s t^2 - \tfrac{1}{3} l_7^2 stu$$



Three loops is not only UV finite it is "superfinite"—cancellations beyond those needed for finiteness in $D = 4$.

Finite for $D < 6$

No term more divergent than the total amplitude. *All* cancellation exposed!

$$\text{UVpole}_{D=6-2\epsilon} = \frac{1}{\epsilon}\frac{5\zeta_3}{(4\pi)^9}\left(\frac{\kappa}{2}\right)^8 (stu)^2 M_4^{\text{tree}}$$

Identical manifest power count as $N = 4$ super-Yang-Mills

More Recent Opinion

In particular, they [non-renormalization theorems and algebraic formalism] suggest that maximal supergravity is likely to diverge at four loops in $D = 5$ and at five loops in $D = 4$, unless other infinity suppression mechanisms not involving supersymmetry or gauge invariance are at work. Bossard, Howe, Stelle (2009)

D^6R^4 is expected counterterm in $D = 5$.

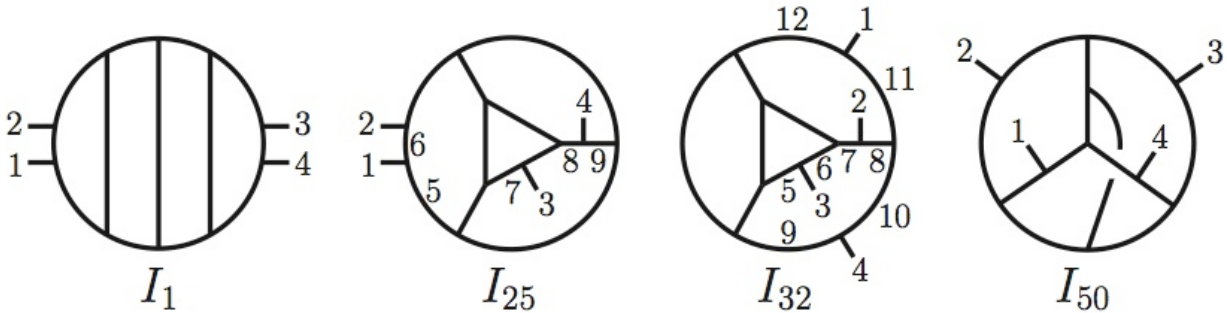
We have the tools to decisively decide this.
Bottles of wine at stake!

Widespread agreement ultraviolet finiteness of maximal supergravity requires a mechanism beyond known one of supersymmetry – little else is agreed upon by the experts.

Four-Loop Amplitude Construction

ZB, Carrasco, Dixon, Johansson, Roiban

Get 50 distinct diagrams or integrals (ones with two- or three-point subdiagrams not needed).



arXiv submission has mathematica files with all 50 diagrams

$$M_4^{\text{4-loop}} = \left(\frac{\kappa}{2}\right)^{10} stu M_4^{\text{tree}} \sum_{S_4} \sum_{i=1}^{50} c_i I_i$$

Integral

leg perms

symmetry factor

Four-Loop Construction

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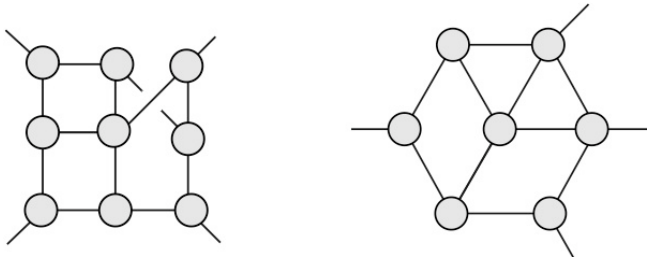
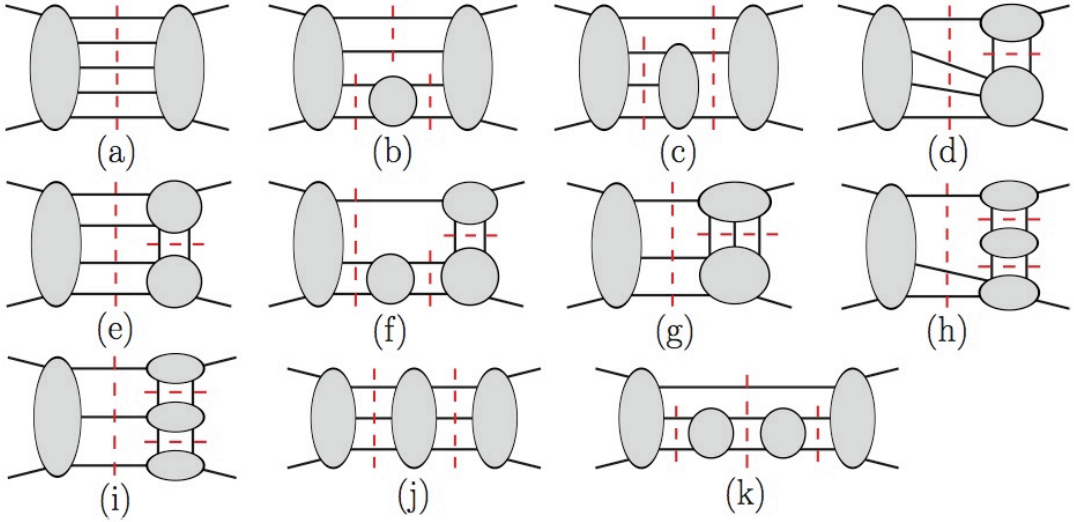
numerator

$$I_i = \int d^D l_1 d^D l_2 d^D l_3 d^D l_4 \frac{N_i(l_j, k_j)}{l_1^2 l_2^2 l_3^2 l_4^2 l_5^2 l_6^2 l_7^2 l_8^2 l_9^2 l_{10}^2 l_{11}^2 l_{12}^2 l_{13}^2}$$

Determine numerators from 2906 maximal and near maximal cuts



Completeness of expression confirmed using 26 generalized cuts



11 most complicated cuts shown

UV Finiteness at Four Loops

$$I_i = \int d^D l_1\, d^D l_2\, d^D l_3\, d^D l_4\, \frac{N(l_j, k_j)}{\prod_{m=1}^{13} l_m^2}$$

$$N_i \sim O(k^4 l^8)$$

k_i: external momenta

l_i : loop momenta

For technical reasons, unlike 3 loops, easier to find a solution where individual terms have worse $O(k^4 l^8)$ behavior than complete amplitude which behaves as $O(k^8 l^4)$.

UV finite for *D* = 4 and 5

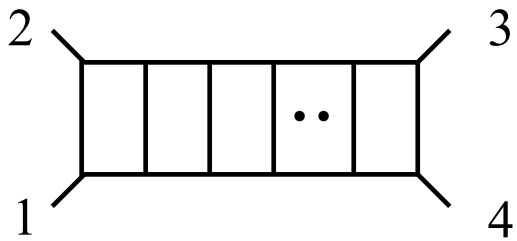
actually finite for *D* < 5.5

Integrals analyzed using a nice paper from Marcus and Sagnotti.

***D* = 4 finiteness a bit trivial following from susy alone**

1. **Shows potential supersymmetry explanation of Bossard, Howe, Stelle does *not* work.**
2. **The cancellations are stronger at 4 loops than at 3 loops, which is in turn stronger than at 2 loops. Rather surprising from traditional susy viewpoint.**

L-Loops $N = 4$ Super-Yang-Mills Warmup

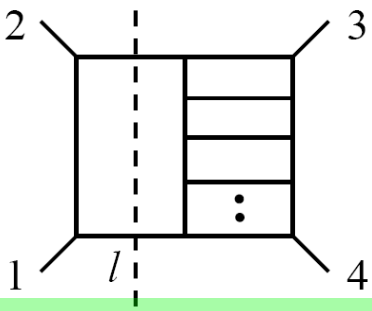


$$[(k_1 + k_2)^2]^{(L-2)}$$

numerator factor

ZB, Dixon, Dunbar, Perelstein, Rozowsky

From 2 particle cut:



$$[(l + k_4)^2]^{(L-2)}$$

numerator factor

Power counting this gives UV finiteness for :

$$D < \frac{6}{L} + 4$$

Power count of UV behavior follows from supersymmetry alone

A bit better than more conventional superspace power counts of $N = 4$ sYM

- **Confirmed by explicit calculation through $L = 5$.**
- **Confirmed by Howe and Stelle using $N = 4$ harmonic superspace.**
- **Through L = 6 agrees with Berkovits, Green and Vanhove, who use low-energy limit of open string in Berkovits' pure spinor formalism.**

See Michael Green's talk

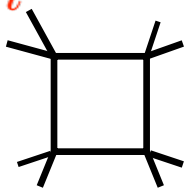
$N = 8$ Supergravity No-Triangle Property

ZB, Dixon, Perelstein, Rozowsky; ZB, Bjerrum-Bohr and Dunbar; Bjerrum-Bohr, Dunbar, Ita, Perkins, Risager; Proofs by Bjerrum-Bohr and Vanhove; Arkani-Hamed, Cachazo and Kaplan.

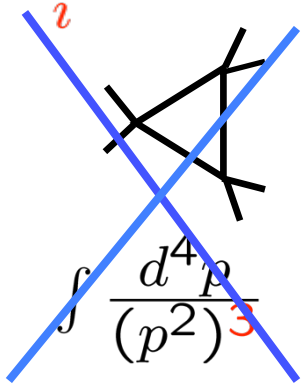
One-loop $D = 4$ theorem: Any one loop amplitude is a linear combination of scalar box, triangle and bubble integrals with rational coefficients:

Brown, Feynman; Passarino and Veltman, etc

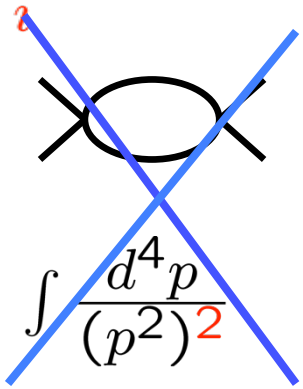
$$A_n^{\text{1-loop}} = \sum_i d_i I_4^{(i)} + \sum_i c_i I_3^{(i)} + \sum_i b_i I_2^{(i)}$$



$$\int \frac{d^4p}{(p^2)^4}$$



$$\int \frac{d^4p}{(p^2)^3}$$



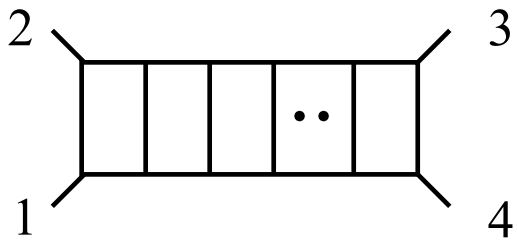
$$\int \frac{d^4p}{(p^2)^2}$$

- **In $N = 4$ Yang-Mills *only box* integrals appear. No triangle integrals and no bubble integrals.**
- **The "no-triangle property" is the statement that same holds in $N = 8$ supergravity. Non-trivial constraint on analytic form of amplitudes.**
- **Unordered nature of gravity is important for this property**

Bjerrum-Bohr and Vanhove

$N = 8$ L-Loop UV Cancellations

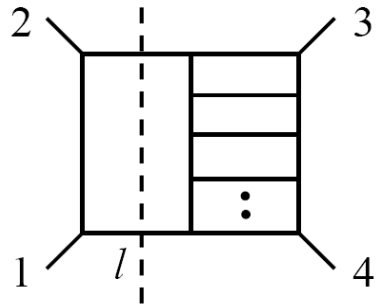
ZB, Dixon, Roiban



$$[(k_1 + k_2)^2]^{2(L-2)}$$

numerator factor

From 2 particle cut:

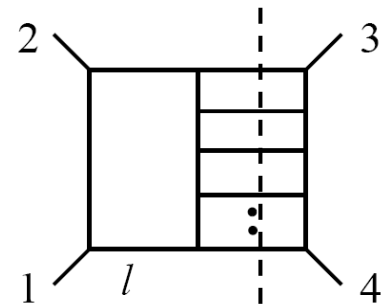


1 in $N = 4$ YM

$$[(l + k_4)^2]^{2(L-2)}$$

numerator factor

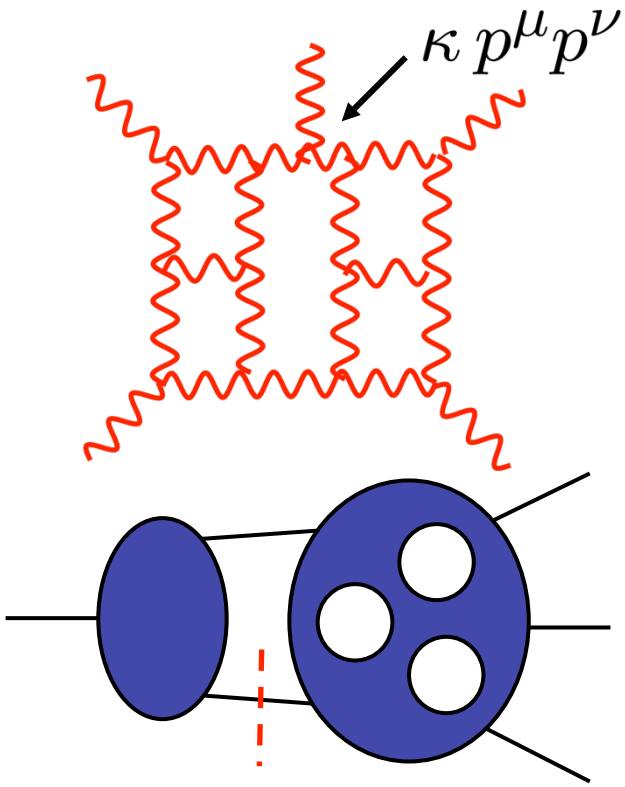
L-particle cut



- **Numerator violates one-loop "no-triangle" property.**
- **Too many powers of loop momentum in one-loop subamplitude.**
- **After cancellations behavior is same as in $N = 4$ Yang-Mills!**

- **UV cancellation exist to *all* loop orders! (not a proof of finiteness)**
- **These all-loop cancellations *not* explained by supersymmetry alone or by Berkovits' string theory non-renormalization theorem discussed in Green's talk.**

Higher-Point Divergences?



Add an extra leg:

1. **extra** $\kappa\, p^\mu p^\nu$ **in vertex**
2. **extra** $1/p^2$ **from propagator**

Adding legs generically does not worsen power count.

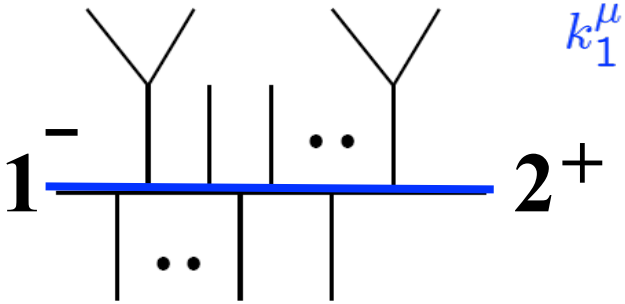
Cutting propagators exposes lower-loop higher-point amplitudes.

- **Higher-point divergences should be visible in high-loop four-point amplitudes.**
- **Very recent paper from Kallosh argues against D=4, L=4 counterterm (R^5) from susy alone. Consistent with this.**
- **A proof of UV finiteness would need to systematically rule out higher-point divergences.**

Origin of Cancellations

ZB, Carrasco, Forde, Ita, Johansson

First consider tree level



$$k_1^\mu \to k_1^\mu + \frac{z}{2}\langle k_1^- | \gamma^\mu | k_2^- \rangle, \qquad k_2^\mu \to k_2^\mu - \frac{z}{2}\langle k_1^- | \gamma^\mu | k_2^- \rangle$$

***m* propagators and *m*+1 vertices between legs 1 and 2**

Yang-Mills scaling: $z^{m+1} \times \frac{1}{z^m} \times \frac{1}{z^2} \sim \frac{1}{z}$ **well behaved**

vertices **propagators** **polarizations**

$z \longrightarrow \infty$

gravity scaling: $z^{2(m+1)} \times \frac{1}{z^m} \times \frac{1}{z^4} \sim z^{m-2}$ **poorly behaved**

Summing over all Feynman diagrams, correct gravity scaling is:

$$M_n^{\text{tree}}(z) \sim \frac{1}{z^2}$$

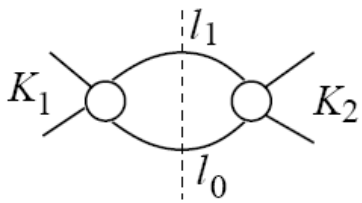
Remarkable tree-level cancellations. Better than gauge theory!

z^{n-5} cancels to $\frac{1}{z^2}$

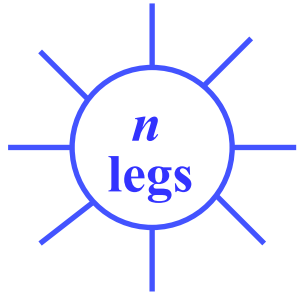
Bedford, Brandhuber, Spence, Travaglini; Cachazo and Svrcek; Benincasa, Boucher-Veronneau, Cachazo Arkani-Hamed, Kaplan; Hall

Loop Cancellations in Pure Gravity

Powerful new one-loop integration method due to Forde makes it much easier to track the cancellations. Allows us to link one-loop cancellations to tree-level cancellations.



Observation: Most of the one-loop cancellations observed in $N = 8$ supergravity leading to "no-triangle property" are already present in non-susy gravity.



$$(l^\mu)^{2n} \to (l^\mu)^{n+4} \times (l^\mu)^{-8}$$

maximum powers of loop momenta

Cancellation generic to Einstein gravity

Cancellation from $N = 8$ susy

ZB, Carrasco, Forde, Ita, Johansson

Proposal: This continues to higher loops, so that most of the observed $N = 8$ multi-loop cancellations are *not* due to susy but in fact are generic to gravity theories!

All loop finiteness of $N = 8$ supergravity would follow from a combination of susy cancellations on top of novel but generic cancellations present even in pure Einstein gravity.

Schematic Illustration of Status

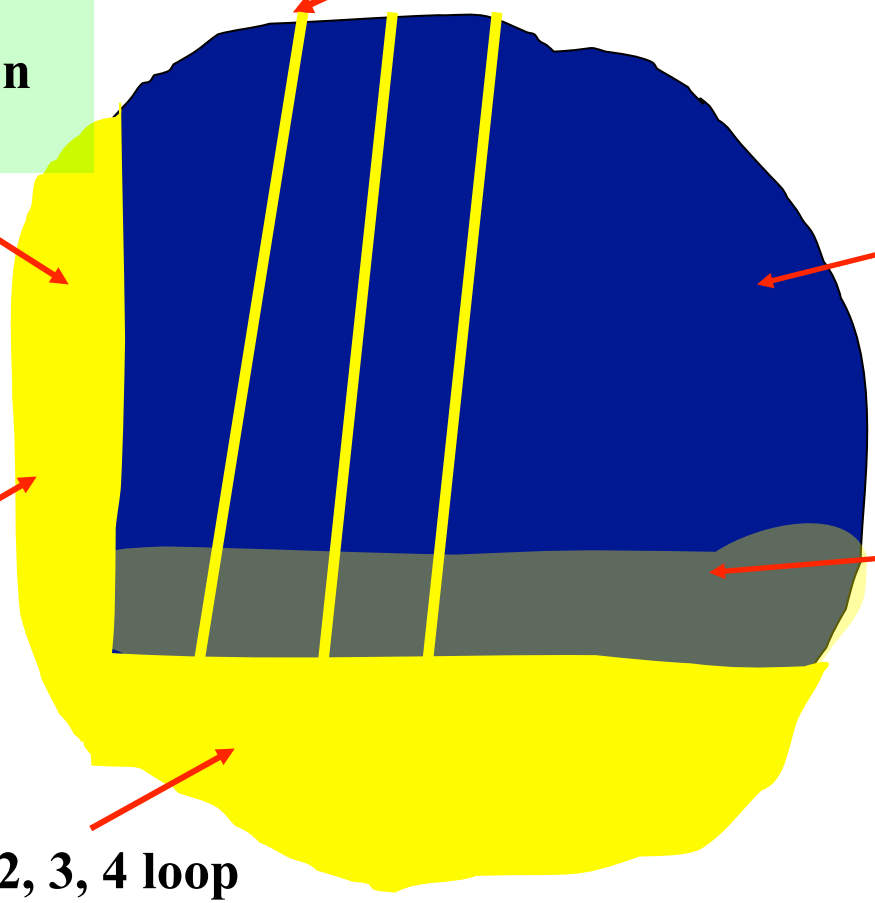
Same power count as N=4 super-Yang-Mills

UV behavior unknown

All-loop UV finiteness.
No susy or string non-renormalization explanation.

from feeding 2, 3 and 4 loop calculations into iterated cuts.

finiteness unproven



loops →

Berkovits string theory non-renormalization theorem points to good L = 5, 6 behavior. Needs to be redone in field theory!

No triangle property

explicit 2, 3, 4 loop computations

terms →

Summary

Scattering amplitudes have a surprising simplicity and rich structure. Remarkable progress in a broad range of topics: AdS/CFT, quantum gravity and LHC physics.

- ***N*=4 supersymmetric gauge theory:**
  - — **Scattering amplitudes open an exciting new venue for studying Maldacena's AdS/CFT conjecture.**
  - — **Example valid to all loop orders, matching strong coupling!**
  - — **Can we repair BDS conjecture at 6 points and beyond?**
  - — **New symmetries. Dual conformal invariance and Yangians.**

See Henn's talk

- **Quantum gravity: Surprisingly simple structures emerge.**
  - — **Gravity as the "square" of gauge theory.**
  - — **Is a point-like perturbatively UV finite quantum gravity theory possible? Explicit four-loop evidence!**
  - — **Better descriptions?**

See talks from Arkani-Hamed and Green

Expect many more exciting developments in scattering amplitudes in the coming years.