Harmony of Scattering Amplitudes: From Gauge Theory to $N = 8$ Supergravity

Strings, June 26, 2009
Zvi Bern, UCLA

Overview + results from papers with:
J.J. Carrasco, L. Dixon, D. Forde, H. Ita, H. Johansson,
D. Kosower, V. Smirnov, M. Spradlin, R. Roiban and A. Volovich.
Outline

This talk will present some recent developments in understanding scattering amplitudes in gauge and gravity theories.

- **Harmony:** Examples of remarkable relations in gauge and gravity theories.
- **QCD:** Brief look at applications of new ideas to LHC physics.
- **Supersymmetric gauge theory:** Resummation of certain planar $N = 4$ super-Yang-Mills scattering amplitudes to all loop orders. A new symmetry: dual conformal invariance.
- **Quantum gravity:** Simplicity and reexamination of standard wisdom on ultraviolet properties of quantum gravity. Four-loop confirmation of very strong UV cancellations.
Why are Feynman diagrams clumsy for high-loop or high-multiplicity processes?

• Vertices and propagators involve gauge-dependent off-shell states. An important origin of the complexity.

\[ \int \frac{d^4p}{(2\pi)^4} \]

• To get at root cause of the trouble we must rewrite perturbative quantum field theory.

• All steps should be in terms of gauge invariant on-shell states. On-shell formalism. \[ p^2 \neq m^2 \]

ZB, Dixon, Dunbar, Kosower
On-Shell Recursion for Tree Amplitudes

Britto, Cachazo, Feng and Witten

Consider amplitude under complex shifts of the momenta

\[ p_1^{\mu}(z) = p_1^{\mu} - zq^{\mu} \quad p_n^{\mu}(z) = p_n^{\mu} + zq^{\mu} \quad q^2 = 0, \quad p \cdot q = 0 \]

\[(p_i^{\mu}(z))^2 = 0 \quad \text{complex momenta} \]

If \( A(z) \to 0, \quad z \to \infty \)

\[ \oint_{C_\infty} \frac{A(z)}{z} \, dz = 0 \quad \Rightarrow \quad A(z = 0) = -\sum_{\alpha} \text{Res}_\alpha \frac{A(z)}{z} \]

\[ A(z) = \sum_{\alpha} \frac{c_\alpha}{z - z_\alpha} \]

on-shell amplitude

Sum over residues gives the on-shell recursion relation

Poles in \( z \) come from kinematic poles in amplitude.

Same construction works in gravity

Brandhuber, Travaglini, Spence; Cachazo, Svrcek;
Benincasa, Boucher-Veronneau, Cachazo; Arkani-Hamed and Kaplan, Hall
Modern Unitarity Method

Two-particle cut:

Generalized unitarity as a practical tool:
Bern, Dixon and Kosower

Three-particle cut:

Systematic assembly of complete amplitudes from cuts for any number of particles or loops.

Different cuts merged to give an expression with correct cuts in all channels.

Generalized cut interpreted as cut propagators not canceling.
Method of Maximal Cuts

A refinement of unitarity method for constructing complete higher-loop amplitudes is “Method of Maximal Cuts”.
Systematic construction in any massless theory.

To construct the amplitude we use cuts with maximum number of on-shell propagators:

Then systematically release cut conditions to obtain contact terms:

Related to subsequent leading singularity method which uses hidden singularities.

[Cachazo and Skinner; Cachazo; Cachazo, Spradlin, Volovich; Spradlin, Volovich, Wen]
Examples of Harmony
Gravity vs Gauge Theory

Consider the gravity Lagrangian

$$L_{\text{gravity}} = \frac{2}{\kappa^2} \sqrt{-g} \, R$$

$$\kappa^2 = 32\pi G_{\text{Newton}}$$

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

Infinite number of complicated interactions

flat metric

Comparing to Yang-Mills Lagrangian on which QCD is based

$$L_{\text{YM}} = \frac{1}{g^2} F^2$$

Only three and four point interactions

bloody mess

Gravity seems so much more complicated than gauge theory.

Does not look harmonious!
Three Vertices

Three gluon vertex:

$$V_{3_{\mu\nu\sigma}}^{abc} = -g f^{abc}(\eta_{\mu\nu}(k_1 - k_2)_\rho + \eta_{\nu\rho}(k_1 - k_2)_\mu + \eta_{\rho\mu}(k_1 - k_2)_\nu)$$

Three graviton vertex:

$$G_{3_{\mu\alpha,\nu\beta,\sigma\gamma}}^{\alpha,\beta,\gamma}(k_1, k_2, k_3) =$$

\[
\text{sym}\left[ -\frac{1}{2} P_3 (k_1 \cdot k_2 \eta_{\mu\alpha} \eta_{\nu\beta} \eta_{\sigma\gamma}) - \frac{1}{2} P_6 (k_{1\nu} k_{1\beta} \eta_{\mu\alpha} \eta_{\sigma\gamma}) + \frac{1}{2} P_3 (k_1 \cdot k_2 \eta_{\mu\nu} \eta_{\alpha\beta} \eta_{\sigma\gamma}) \\
+ P_6 (k_1 \cdot k_2 \eta_{\mu\alpha} \eta_{\nu\sigma} \eta_{\beta\gamma}) + 2 P_3 (k_{1\nu} k_{1\gamma} \eta_{\mu\alpha} \eta_{\beta\sigma}) - P_3 (k_{1\beta} k_{2\mu} \eta_{\alpha\nu} \eta_{\sigma\gamma}) \\
+ P_3 (k_{1\sigma} k_{2\gamma} \eta_{\mu\nu} \eta_{\alpha\beta}) + P_6 (k_{1\sigma} k_{1\gamma} \eta_{\mu\nu} \eta_{\alpha\beta}) + 2 P_6 (k_{1\nu} k_{2\gamma} \eta_{\beta\mu} \eta_{\alpha\sigma}) \\
+ 2 P_3 (k_{1\nu} k_{2\mu} \eta_{\beta\sigma} \eta_{\gamma\alpha}) - 2 P_3 (k_1 \cdot k_2 \eta_{\alpha\nu} \eta_{\beta\sigma} \eta_{\gamma\mu}) \right]
\]

\[k_i^2 = E_i^2 - \vec{k}_i^2 \neq 0\]

About 100 terms in three vertex
Naïve conclusion: Gravity is a nasty mess.
Not very harmonious!
Simplicity of Gravity Amplitudes

On-shell three vertices contain all information:

\[ k_i^2 = 0 \]

\[ -g f^{abc} (\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic}) \]

\[ i\kappa (\eta_{\mu\nu}(k_1 - k_1)_\rho + \text{cyclic}) \]
\[ \times (\eta_{\alpha\beta}(k_1 - k_2)_\gamma + \text{cyclic}) \]

“square” of Yang-Mills vertex.

Any gravity scattering amplitude constructible solely from on-shell 3 vertex.

• BCFW on-shell recursion for tree amplitudes.
  Britto, Cachazo, Feng and Witten; Brandhuber, Travaglini, Spence; Cachazo, Svrcek; Benincasa, Boucher-Veronneau, Cachazo; Arkani-Hamed and Kaplan, Hall

• Unitarity method for loops.
  ZB, Dixon, Dunbar and Kosower; ZB, Dixon, Kosower; Britto, Cachazo, Feng; ZB, Morgan; Buchbinder and Cachazo; ZB, Carrasco, Johansson, Kosower; Cachzo and Skinner.
Consider the gravity Lagrangian

\[ L_{\text{gravity}} = \frac{2}{\kappa^2} \sqrt{-g} \, R \]

\[ \kappa^2 = 32\pi G_{\text{Newton}} \]

\[ g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \]

Compare to Yang-Mills Lagrangian

\[ L_{\text{YM}} = \frac{1}{g^2} F^2 \]

Gravity seems so much more complicated than gauge theory.

Does not look harmonious!
Harmony of Color and Kinematics

ZB, Carrasco, Johansson

Color factors based on a Lie algebra: 

$[T^a, T^b] = i f^{abc} T^c$

Color factors satisfy Jacobi identity:

$[[T^a, T^b], T^c] + [[T^b, T^c], T^a] + [[T^c, T^a], T^b] = 0$

Use $1 = s/s = t/t = u/u$ to assign 4-point diagram to others.

$A_4^{\text{tree}} = g^2 \left( \frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u} \right)$

$s = (k_1 + k_2)^2$

$t = (k_1 + k_4)^2$

$u = (k_1 + k_3)^2$

Color factors satisfy Jacobi identity:

Numerator factors satisfy similar identity:

$C_u = c_s - c_t$

$n_u = n_s - n_t$

Color and kinematics are singing same tune!
Harmony of Color and Kinematics

At higher points similar structure:

\[ A_{5}^{\text{tree}} = \sum_{i=1}^{15} \frac{c_i \, n_i}{D_i} \]

\[ c_3 \equiv f_{a_3a_4b} f_{ba_5c} f_{ca_1a_2}, \quad c_5 \equiv f_{a_3a_4b} f_{ba_2c} f_{ca_1a_5}, \quad c_8 \equiv f_{a_3a_4b} f_{ba_1c} f_{ca_2a_5} \]

\[ c_3 - c_5 + c_8 = 0 \iff n_3 - n_5 + n_8 = 0 \]

Claim: We can always find a rearrangement so color and kinematics satisfy the same Jacobi constraint equations.

- Color and kinematics sing same tune!
- Nontrivial constraints on amplitudes.
Higher-Point Gravity and Gauge Theory

Gauge theory: \( A_n^{\text{tree}} = i g^{n-2} \sum_i \frac{c_i n_i}{D_i} \)

Einstein Gravity: \( M_n^{\text{tree}} = i \kappa^{n-2} \sum_i \frac{n_i^2}{D_i} \)

Same relations between \( N=4 \) sYM and \( N = 8 \) sugra

Claim: This is unproven but it is correct! Related to KLT relations

Another recent relation:

Gauge: \( A^{\text{tree}}(1, \ldots, n) = A^{\text{MHV}}(1, \ldots, n) \sum_{\{\alpha\}} R_\alpha(\lambda_i, \tilde{\lambda}_i, \eta_i) \)

Gravity: \( M^{\text{tree}}(1, \ldots, n) = \sum_{\text{perms}} [A^{\text{MHV}}(1, \ldots, n)]^2 \sum_{\{\alpha\}} [R_\alpha(\lambda_i, \tilde{\lambda}_i, \eta_i)]^2 G_\alpha(\lambda_i, \tilde{\lambda}_i) \)

Gravity and gauge theory kinematic numerators sing same tune!

Cries out for a unified description of the sort given by string theory. Symmetries of YM have echo in gravity!
We can use Jacobi-like identities to obtain non-planar from planar contributions at higher loops!

For example for three loop (super) Yang-Mills

Symmetries of planar obviously restrict the structure of non-planar as well (e.g. dual conformal symmetry)
In 2004 Ed Witten demonstrated that twistor space reveals a hidden structure in scattering amplitudes.

Penrose twistor transform:

\[ \tilde{A}(\lambda_i, \mu_i) = \int \prod_i \frac{d^2 \tilde{\lambda}_i}{(2\pi)^2} \exp \left( \sum_j \mu_j \tilde{\lambda}_{j\dot{a}} \right) A(\lambda_i, \tilde{\lambda}_i) \]

Witten’s twistor-space link:

\( N = 4 \) scattering amplitudes \( \leftrightarrow \) Topological String Theory
Witten conjectured that in twistor–space gauge theory amplitudes have delta-function support on curves of degree:

\[ d = q - 1 + L, \quad q = \# \text{ negative helicities}, \quad L = \# \text{ loops}, \]

Structures imply an amazing simplicity in the scattering amplitudes.

Gravity and gauge theory share same structure! Derivative of delta-function support instead of delta-function support

See Nima’s talk for a modified version: “ambi-twistor space” Remarkable simplicity exposed in twistor space via BCFW
Applications to LHC Physics
In 1948 Schwinger computed anomalous magnetic moment of the electron.

60 years later typical examples:

$$pp \rightarrow W, Z + 2 \text{ jets} \quad pp \rightarrow VVV \quad V = Z, W$$

Amusing numbers of diagrams:

- $6g: 10,860$
- $7g: 168,925$
- $8g: 3,017,490$

Much worse difficulty: integral reduction generates nasty dets.

$$\frac{1}{\det(k_i \cdot k_j)^n}$$

Gram determinant singularities
Application: State of the Art QCD for the LHC

Berger, ZB, Dixon, Febres Cordero, Forde, Gleisberg, Ita, Kosower, Maitre (BlackHat collaboration)

Excellent agreement between NLO theory and experiment. Triumph of on-shell methods!

Have similar predictions for LHC.

Apply on-shell methods

Complexity has prevented calculations via Feynman diagrams.
Amplitudes and AdS/CFT
Since ‘t Hooft’s paper thirty years ago on the planar limit of QCD we have dreamed of solving QCD in this limit. This is too hard. $N = 4$ sYM is much more promising.

- Special theory because of AdS/CFT correspondence.
- Maximally supersymmetric.

Remarkable relation

To make this link need to evaluate $N = 4$ super-Yang-Mills amplitudes to all loop orders. Seems impossible even with modern methods.
The planar four-point two-loop amplitude undergoes fantastic simplification.

\[-st \, A_4^{\text{tree}} \left\{ \begin{array}{c} \begin{array}{c} 4 \\ 3 \\ 2 \\ 1 \\ 4 \\ 3 \\ 2 \\ 1 \\ s \\ t \end{array} \end{array} \right\} \]

\[M_4^{2-\text{loop}}(s, t) = \frac{1}{2} \left( M_4^{1-\text{loop}}(s, t) \right)^2 + f(\epsilon) M_4^{1-\text{loop}}(s, t) \bigg|_{\epsilon \rightarrow 2\epsilon} - \frac{1}{2} \zeta_2\]

\[M_4^{\text{loop}} = A_4^{\text{loop}} / A_4^{\text{tree}} \quad f(\epsilon) = -\zeta_2 - \zeta_3\epsilon - \zeta_4\epsilon^2 \]

This gives two-loop four-point planar amplitude as iteration of one-loop amplitude.

Three loop satisfies similar iteration relation. Rather nontrivial.

\[D = 4 - 2\epsilon \]

ZB, Dixon, Smirnov
Why not be bold and guess scattering amplitudes for all loop and all legs, at least for simple helicity configurations?

\[ \mathcal{A}_n = A_{n,\text{tree}} A_{n,\text{divergent}} \exp \left[ \frac{1}{4} \gamma_K F_n^{1-\text{loop}} + C \right] \]

- Limit of collinear momenta gives us key analytic information, at least for MHV amplitudes, restricting form of anzatz
- IR singularities agree with Magnea and Sterman formula.

Gives a definite prediction for all values of coupling given BES integral equation for the cusp anomalous dimension.

Anastasiou, ZB, Dixon, Kosower
ZB, Dixon and Smirnov

F4 = \frac{1}{4} \gamma_K \ln^2 \left( \frac{s_{12}}{s_{23}} \right) + \text{const}

Beisert, Eden, Staudacher
In a beautiful paper Alday and Maldacena confirmed the conjecture for 4 gluons at strong coupling from an AdS string theory computation. Minimal surface calculation.

Very suggestive link to Wilson loops even at weak coupling.

For MHV amplitudes: \[ F_{4}^{1\text{-loop}} = \frac{1}{2} \ln^2(s/t) + \frac{2\pi^2}{3} \]

\[ A_4 = A_4^{\text{tree}} A_4^{\text{divergent}} \exp \left[ \frac{1}{4} \gamma_K F_{4}^{1\text{-loop}} + C \right] \]

- All-loop resummed amplitude
- IR divergences
- Cusp anomalous dimension
- Finite part of one-loop amplitude

Identification of new symmetry: “dual conformal symmetry”

- ZB, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich;
- Anastasiou, Brandhuber, Heslop, Khoze, Spence, Travagli,
- Berkovits and Maldacena

See Maldacena’s talk
For various technical reasons it is hard to solve for minimal surface for large numbers of gluons.

Alday and Maldacena realized certain terms can be calculated at strong coupling for an infinite number of gluons.

\[ T/L \to \infty \]

Trouble also in the Regge limit.

\[ s/t \to \infty \]

Bartels, Lipatov, Sabio Vera

Explicit computation at 2-loop six points.

Need to modify conjecture!

Disagrees with BDS conjecture

Dual conformal invariance and equivalence to Wilson loops persists

Can the BDS conjecture be repaired for six and higher points?
Important new information from regular polygons should serve as a guide.

Explicit solution at eight points

\[ A_{BDS} = -\frac{1}{4} \sum_{i=1}^{n} \sum_{j=1, j \neq i, i-1}^{n} \log \frac{x_j^+ - x_i^+}{x_{j+1}^+ - x_i^+} \log \frac{x_j^- - x_{i-1}^-}{x_j^- - x_i^-} \]

\[ A = A_{div} + A_{BDS} + R \]

\[ R = -\frac{1}{2} \log(1 + \chi^-) \log(1 + \frac{1}{\chi^+}) + \frac{7\pi}{6} + \int_{-\infty}^{\infty} dt \frac{|m| \sinh t}{\tanh(2t + 2i\phi)} \log \left(1 + e^{-2\pi|m| \cosh t}\right) \]

Solution valid only for strong coupling and special kinematics, but it’s explicit!

Alday and Maldacena (2009)

Can we figure out the discrepancy?

See Maldacena’s talk
A key to understanding the structure of amplitudes is symmetry. Planar $N = 4$ YM has three distinct interlocked symmetries:

- Supersymmetry
- Conformal symmetry
- Dual conformal symmetry

See Henn’s talk

Yangian structure

Drummond, Henn Plefka; Bargheer, Beisert, Galleas, Loebbert, McLoughlin

Relatively simple at tree level for generic kinematics

**Conformal:**

$$k_{a\dot{a}} = \sum_i \frac{\partial^2}{\partial \lambda^a_i \partial \bar{\lambda}^{\dot{a}}_i}, \quad k_{a\dot{a}} A_n^{\text{tree}} = 0$$

**Dual Conformal:**

$$K^{a\dot{a}} = \sum \left[ x_i^{ab} \bar{x}_i^{\dot{a}\dot{b}} \frac{\partial}{\partial x_i^{\dot{b}}} + \ldots \right], \quad K^{a\dot{a}} A_n^{\text{tree}} = 0$$

**Holomorphic Anomaly:**

$$\frac{\partial}{\partial \bar{\lambda}^{\dot{a}}} \frac{1}{\langle \lambda, \mu \rangle} = \pi \delta^2(\langle \lambda, \mu \rangle) \epsilon_{ab} \bar{\mu}^b$$

Complex variables $z, \bar{z}$ not independent at $z = 0$. Collinear momenta means extra contribution, but simple at tree level.

What about loop level?
If we assume that IR singularities and dim reg breaks dual conformal symmetry at multi-loops identically as at one loop, get a remarkably nice anomalous dual conformal Ward Identity

\[ K^{\alpha \dot{\alpha}} F_n = \sum_{i=1}^{n} [2x^{\alpha \dot{\alpha}}(x \cdot \partial x_i) - x_i^2 \partial^\alpha \partial_{\dot{\alpha}}] F_n = \frac{1}{2} \gamma_K \sum_{i=1}^{n} x_i^{\alpha \dot{\alpha}} \ln\left( \frac{x_{i,i+2}^2}{x_{i-1,i+1}^2} \right) \]

Solution gives exactly BDS ansatz

\[ F_4 = \frac{1}{4} \gamma_K \ln^2\left( \frac{x_{13}^2}{x_{24}^2} \right) + \text{const} \]
\[ F_5 = -\frac{1}{8} \gamma_K \sum_{i=1}^{5} \ln\left( \frac{x_{i,i+2}^2}{x_{i,i+3}^2} \right) \ln\left( \frac{x_{i+1,i+3}^2}{x_{i+2,i+4}^2} \right) + \text{const} \]
\[ k_i = x_{i+1} - x_i \]

Starting at six points any function of conformal cross ratios annihilated by \( K \)

\[ u_1 = \frac{x_{13}^2 x_{26}^2}{x_{14}^2 x_{36}^2}, \quad u_2 = \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2}, \quad u_3 = \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2} \]

No unique solution!

Can we use the ordinary conformal symmetry to help?

\[ k^{\alpha \dot{\alpha}} F_n = ??? \]

How does conformal symmetry act on finite part of high-loop amplitudes?

IR singularities and holomorphic anomaly complicates this.

Surprise is not complexity of multiloop conformal Ward identity.

Surprise is simplicity of the dual conformal Ward identity.

Not even obvious for general amplitudes at one loop.
Nair's on-shell $N=4$ sYM superspace for MHV amplitudes:

Combine MHV amplitudes in single superamplitude

$$\mathcal{A}_{n}^{\text{MHV}}(p_1, \eta_1; \ldots; p_n, \eta_n) = i(2\pi)^4 \delta^{(4)}(\sum_{i=1}^{n} p_i) \delta^{(8)}(\sum_{i=1}^{n} \lambda_i^\alpha \eta_i^A)$$

Two basic approaches for trees: Use either a BCFW or MHV vertex approach build a generating function for all $N=4$ sYM amplitudes.

- Exposes dual superconformal invariance.
- Used in proof of “no-triangle property” of $N=8$ sugra.
- Used in four loop calculation of $N=4$ sYM and $N=8$ supergravity
- Theories with fewer susys.

Major annoyance: Only half the susy manifest, chiral superspace.
We can expose all susy cancellations, but it’s clumsier than we would like.
Some $N=4$ YM contributions:

- **IR singularities in $D = 4$ QCD:** Once we are able to integrate non-trivial 3 and 4 loop non-planar integrals, we will find out which recent conjectures for soft anomalous dimension matrix are correct. 
  
  Dixon, Magnea, Sterman, Gardi and Magnea, Becher and Neubert; Dixon

- **UV divergences in $D > 4$:** Used to investigate UV divergences higher dimensions. Certain additional subleading color vanishings of UV divergences. Compare to string theory. See Michael Green’s talk

- **Key input to four loop $N=8$ supergravity:** Use the squaring relation in the cuts.

Applications:

50 distinct planar and non-planar diagrammatic topologies.

Susy cancellations manifest!

UV finite for $D < \frac{6}{L} + 4$, $L = 4$
UV Properties of $N = 8$ Supergravity
Is a UV finite theory of gravity possible?

\[ \kappa = \sqrt{32\pi G_N} \rightarrow \text{Dimensionful coupling} \]

**Gravity:**

\[ \int \prod_{i=1}^{L} \frac{d^D p_i}{(2\pi)^D} \frac{(\kappa p^\mu_i p^\nu_j)}{\text{propagators}} \]

**Gauge theory:**

\[ \int \prod_{i=1}^{L} \frac{d^D p_i}{(2\pi)^D} \frac{(g p^\nu_j)}{\text{propagators}} \]

Extra powers of loop momenta in numerator means integrals are badly behaved in the UV.

Much more sophisticated power counting in supersymmetric theories but this is the basic idea.

Reasons to focus on \( N = 8 \) maximal supergravity: Cremmer and Julia

- With more susy suspect better UV properties.
- High symmetry implies simplicity. Much simpler than expected. May be “simplest theory”. See Nima’s talk
We are interested in UV finiteness of $N = 8$ supergravity because it would imply a new symmetry or non-trivial dynamical mechanism.

The discovery of either would have a fundamental impact on our understanding of gravity.

- Here we only focus on order-by-order UV finiteness.
- Non-perturbative issues and viable models of Nature are not the goal for now.
Unfortunately, in the absence of further mechanisms for cancellation, the analogous $N = 8 \ D = 4$ supergravity theory would seem set to diverge at the three-loop order.

Howe, Stelle (1984)

There are no miracles… It is therefore very likely that all supergravity theories will diverge at three loops in four dimensions. … The final word on these issues may have to await further explicit calculations.

Marcus, Sagnotti (1985)

The idea that all supergravity theories diverge has been widely accepted for over 25 years
Divergences in Gravity

One loop:

Vanish on shell

\[ R^2, \quad R_{\mu\nu}^2, \quad R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho} \]

vanishes by Gauss-Bonnet theorem

Pure gravity 1-loop finite (but not with matter)

‘t Hooft, Veltman (1974)

Two loop: Pure gravity counterterm has non-zero coefficient:

\[ R^3 \equiv R^\lambda_{\mu\nu} R_{\mu\nu}^\rho R^{\rho\sigma\tau} \]

Any supergravity:

Goroff, Sagnotti (1986); van de Ven (1992)

\[ R^3 \] is not a valid supersymmetric counterterm.

Produces a helicity amplitude (−,+,+,+) forbidden by susy.

Grisaru (1977); Tomboulis (1977)

The first divergence in any supergravity theory can be no earlier than three loops.

\[ R^4 \] squared Bel-Robinson tensor expected counterterm

1) Discovery of remarkable cancellations at 1 loop –
   the “no-triangle property”. **Nontrivial cancellations!**
   ZB, Dixon, Perelstein, Rozowsky; ZB, Bjerrum-Bohr, Dunbar; Bjerrum-Bohr, Dunbar, Ita, Perkins, Risager; Bjerrum-Bohr, Vanhove; Arkani-Hamed Cachazo, Kaplan

2) *Every* explicit loop calculation to date finds \( N = 8 \) supergravity
   has identical power counting as \( N = 4 \) super-Yang-Mills theory,
   which is UV finite. Green, Schwarz and Brink; ZB, Dixon, Dunbar, Perelstein, Rozowsky; Bjerrum-Bohr, Dunbar, Ita, Perkins; Risager; ZB, Carrasco, Dixon, Johanson, Kosower, Roiban.

3) Interesting hint from string dualities. Chalmers; Green, Vanhove, Russo
   – Dualities restrict form of effective action. May prevent
divergences from appearing in \( D = 4 \) supergravity, although
issues with decoupling of towers of massive states and indirect.

4) Interesting string non-renormalization theorem from Berkovits.
   Suggests divergence delayed to nine loops, but needs to be
redone directly in field theory not string theory. Green, Vanhove, Russo
### Where is First $D=4$ UV Divergence in $N=8$ Sugra?

**Various opinions:**

<table>
<thead>
<tr>
<th>Loops</th>
<th>Statement</th>
<th>References</th>
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| 3 loops | Conventional superspace power counting | Green, Schwarz, Brink (1982)  
Howe and Stelle (1989)  
Marcus and Sagnotti (1985) |
| 5 loops | Partial analysis of unitarity cuts; *If* $N=6$ harmonic superspace exists; algebraic renormalisation argument | Bern, Dixon, Dunbar,  
Perelstein, Rozowsky (1998)  
Howe and Stelle (2003, 2009) |
| 6 loops | *If* $N=7$ harmonic superspace exists | Howe and Stelle (2003) |
| 7 loops | *If* $N=8$ harmonic superspace exists; lightcone gauge locality arguments | Grisaru and Siegel (1982);  
Kallosh (2009) |
| 8 loops | Explicit identification of potential susy invariant counterterm with full non-linear susy | Kallosh; Howe and Lindström (1981) |
| 9 loops | Assume Berkovits’ superstring non-renormalization theorems can be carried over to $D=4$ $\mathcal{N}=8$ supergravity and extrapolate to 9 loops. | Green, Russo, Vanhove (2006)  
Berkovits, Green, Vanhove, to appear |

**Note many of the statements are contradictory.**

Note: no divergence demonstrated above. Arguments based on lack of susy protection!

**To end debate, we need solid results!**
Need following cuts:

For cut (g) have:

\[ \sum_{N=8 \text{ states}} M_4^{\text{tree}}(1,2,l_3,l_1) \times M_5^{\text{tree}}(-l_1,-l_3,q_3,q_2,q_1) \times M_5^{\text{tree}}(3,4,-q_1,-q_2,-q_3) \]

Use Kawai-Lewellen-Tye tree relations

\[ M_4^{\text{tree}}(1,2,l_3,l_1) = -i s_{12} A_4^{\text{tree}}(1,2,l_3,l_1) A_4^{\text{tree}}(2,1,l_3,l_1) \]

\[ M_5^{\text{tree}}(-l_1,-l_3,q_3,q_2,q_1) = i s_{l_1 q_1} s_{l_3 q_3} A_5^{\text{tree}}(-l_1,-l_3,q_3,q_2,q_1) A_5^{\text{tree}}(-l_1,q_1,q_3,-l_3,q_2) + \{l_1 \leftrightarrow l_3\} \]

\[ N = 8 \text{ supergravity cuts are sums of products of } \]
\[ N = 4 \text{ super-Yang-Mills cuts} \]
Three loops is not only UV finite it is “superfinite”—cancellations beyond those needed for finiteness in $D = 4$. Finite for $D < 6$

No term more divergent than the total amplitude. *All* cancellation exposed!

Identical manifest power count as $N = 4$ super-Yang-Mills
In particular, they [non-renormalization theorems and algebraic formalism] suggest that maximal supergravity is likely to diverge at four loops in $D = 5$ and at five loops in $D = 4$, unless other infinity suppression mechanisms not involving supersymmetry or gauge invariance are at work.

$D^6R^4$ is expected counterterm in $D = 5$.

We have the tools to decisively decide this. Bottles of wine at stake!

Widespread agreement ultraviolet finiteness of maximal supergravity requires a mechanism beyond known one of supersymmetry – little else is agreed upon by the experts.

Bossard, Howe, Stelle (2009)
Get 50 distinct diagrams or integrals (ones with two- or three-point subdiagrams not needed).

\[ M_{4\text{-loop}} = \left( \frac{\kappa}{2} \right)^{10} stu M_{4\text{tree}}^\text{tree} \sum_{S_4} \sum_{i=1}^{50} c_i I_i \]

Integral

leg perms

symmetry factor

arXiv submission has mathematica files with all 50 diagrams
Four-Loop Construction

$$I_i = \int d^D l_1 d^D l_2 d^D l_3 d^D l_4 \frac{N_i(l_j, k_j)}{l_1^2 l_2^2 l_3^2 l_4^2 l_5^2 l_6^2 l_7^2 l_8^2 l_9^2 l_{10}^2 l_{11}^2 l_{12}^2 l_{13}^2}$$

Determine numerators from 2906 maximal and near maximal cuts

Completeness of expression confirmed using 26 generalized cuts

11 most complicated cuts shown
UV Finiteness at Four Loops

\[ I_i = \int d^D l_1 d^D l_2 d^D l_3 d^D l_4 \frac{N(l_j, k_j)}{\prod_{m=1}^{13} l_m^2} \]

\[ N_i \sim O(k^4 l^8) \]

\[ k_i: \text{external momenta} \]
\[ l_i: \text{loop momenta} \]

For technical reasons, unlike 3 loops, easier to find a solution where individual terms have worse \( O(k^4 l^8) \) behavior than complete amplitude which behaves as \( O(k^8 l^4) \).

UV finite for \( D = 4 \) and 5
actually finite for \( D < 5.5 \)

1. Shows potential supersymmetry explanation of Bossard, Howe, Stelle does not work.
2. The cancellations are stronger at 4 loops than at 3 loops, which is in turn stronger than at 2 loops.
   Rather surprising from traditional susy viewpoint.
**L-Loops \( N = 4 \) Super-Yang-Mills Warmup**

From 2 particle cut:

\[
[(k_1 + k_2)^2]^{(L-2)}
\]

numerator factor

\[
[(l + k_4)^2]^{(L-2)}
\]

numerator factor

**Power counting this gives**

UV finiteness for:

\[ D < \frac{6}{L} + 4 \]

**Power count of UV behavior follows from supersymmetry alone**

A bit better than more conventional superspace power counts of \( N = 4 \) sYM

- Confirmed by explicit calculation through \( L = 5 \).
- Confirmed by Howe and Stelle using \( N = 4 \) harmonic superspace.
- Through \( L = 6 \) agrees with Berkovits, Green and Vanhove, who use low-energy limit of open string in Berkovits’ pure spinor formalism.

See Michael Green’s talk
**N = 8 Supergravity No-Triangle Property**

ZB, Dixon, Perelstein, Rozowsky; ZB, Bjerrum-Bohr and Dunbar; Bjerrum-Bohr, Dunbar, Ita, Perkins, Risager; Proofs by Bjerrum-Bohr and Vanhove; Arkani-Hamed, Cachazo and Kaplan.

**One-loop D = 4 theorem:** Any one loop amplitude is a linear combination of scalar box, triangle and bubble integrals with rational coefficients:

\[ A_{n}^{1\text{-loop}} = \sum_{i} d_i I_4^{(i)} + \sum_{i} c_i I_3^{(i)} + \sum_{i} b_i I_2^{(i)} \]

- In \( N = 4 \) Yang-Mills *only box* integrals appear. No triangle integrals and no bubble integrals.
- The “no-triangle property” is the statement that same holds in \( N = 8 \) supergravity. Non-trivial constraint on analytic form of amplitudes.
- Unordered nature of gravity is important for this property.
$N = 8$ $L$-Loop UV Cancellations

From 2 particle cut:

$$[(k_1 + k_2)^2]^{2(L-2)}$$

numerator factor

$L$-particle cut

$$[(l + k_4)^2]^{2(L-2)}$$

numerator factor

- Numerator violates one-loop “no-triangle” property.
- Too many powers of loop momentum in one-loop subamplitude.
- After cancellations behavior is same as in $N = 4$ Yang-Mills!

- UV cancellation exist to all loop orders! (not a proof of finiteness)
- These all-loop cancellations not explained by supersymmetry alone or by Berkovits’ string theory non-renormalization theorem discussed in Green’s talk.
Adding legs generically does not worsen power count.

Add an extra leg:
1. extra $\kappa p^\mu p^\nu$ in vertex
2. extra $1/p^2$ from propagator

Cutting propagators exposes lower-loop higher-point amplitudes.

• Higher-point divergences should be visible in high-loop four-point amplitudes.
• Very recent paper from Kallosh argues against $D=4$, $L=4$ counterterm ($R^5$) from susy alone. Consistent with this.
• A proof of UV finiteness would need to systematically rule out higher-point divergences.
First consider tree level

\[ k^{\mu}_1 \rightarrow k^{\mu}_1 + \frac{\bar{z}}{2} \langle k_1^- | \gamma^{\mu} | k_2^- \rangle, \quad k^{\mu}_2 \rightarrow k^{\mu}_2 - \frac{\bar{z}}{2} \langle k_1^- | \gamma^{\mu} | k_2^- \rangle \]

\( m \) propagators and \( m+1 \) vertices between legs 1 and 2

\[ \text{Yang-Mills scaling: } \bar{z}^{m+1} \times \frac{1}{\bar{z}^m} \times \frac{1}{\bar{z}^2} \sim \frac{1}{\bar{z}} \quad \text{well behaved} \]

\[ \text{gravity scaling: } \bar{z}^{2(m+1)} \times \frac{1}{\bar{z}^m} \times \frac{1}{\bar{z}^4} \sim \bar{z}^{m-2} \quad \text{poorly behaved} \]

Summing over all Feynman diagrams, correct gravity scaling is:

\[ M_n^{\text{tree}}(\bar{z}) \sim \frac{1}{\bar{z}^2} \]

Remarkable tree-level cancellations. Better than gauge theory!

\[ \bar{z}^{n-5} \text{ cancels to } \frac{1}{\bar{z}^2} \]

Bedford, Brandhuber, Spence, Travaglini; Cachazo and Svrcek; Benincasa, Boucher-Veronneau, Cachazo; Arkani-Hamed, Kaplan; Hall
Proposal: This continues to higher loops, so that most of the observed \( N = 8 \) multi-loop cancellations are \textit{not} due to susy but in fact are generic to gravity theories!

All loop finiteness of \( N = 8 \) supergravity would follow from a combination of susy cancellations on top of novel but generic cancellations present even in pure Einstein gravity.
All-loop UV finiteness. No susy or string non-renormalization explanation.

Berkovits string theory non-renormalization theorem points to good $L = 5, 6$ behavior. Needs to be redone in field theory!

from feeding 2, 3 and 4 loop calculations into iterated cuts.

finiteness unproven
Scattering amplitudes have a surprising simplicity and rich structure. Remarkable progress in a broad range of topics: AdS/CFT, quantum gravity and LHC physics.

- **N=4 supersymmetric gauge theory:**
  - Scattering amplitudes open an exciting new venue for studying Maldacena’s AdS/CFT conjecture.
  - Example valid to all loop orders, matching strong coupling!
  - Can we repair BDS conjecture at 6 points and beyond?
  - New symmetries. Dual conformal invariance and Yangians. [See Henn’s talk](#)

- **Quantum gravity:** Surprisingly simple structures emerge.
  - Gravity as the “square” of gauge theory.
  - Is a point-like perturbatively UV finite quantum gravity theory possible? Explicit four-loop evidence!
  - Better descriptions? [See talks from Arkani-Hamed and Green](#)

Expect many more exciting developments in scattering amplitudes in the coming years.