Holography in Flat Space:

Algebraic Geometry + the S-Matrix

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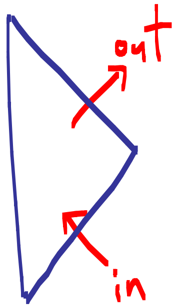
to appear soon

Goal: Discover Dual Theory for S-Matrix



Get S without evolution through spacetime

Evidence it exists: incredible properties of amplitudes, totally obscured by usual insistence on manifest locality.

"WEAK-WEAK" duality ⟹

Explicitly see the emergence of spacetime; "decode the hologram" perturbatively!

Kinematics

- $p^2=0 \Longrightarrow p_{\alpha\dot\alpha} = \lambda_\alpha \tilde\lambda_{\dot\alpha}$
- $M(t_i\lambda_i, t_i^{-1}\tilde\lambda_i, h_i) = t_i^{2h_i} M(\lambda_i,\tilde\lambda_i,h_i)$
- ex

$$M_{YM}^{++\cdots\overset{i}{-}\cdots\overset{j}{-}\cdots+} = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}$$

- Maximal SUSY

$$|\eta\rangle = e^{\bar{Q}\eta}|-\rangle, \quad |\tilde\eta\rangle = e^{Q\tilde\eta}|+\rangle$$

- $M\left(\lambda_i, \tilde\lambda_i, \begin{matrix}\eta_i \\ \text{or} \\ \tilde\eta_i\end{matrix}\right)$ no discrete labels!
- $M_n = \sum_{k=0}^{n} M_{n;k} \longleftarrow \tilde\eta$ charge

$M_{n;k=0,1,n-1,n} = 0$; $k=2$ "MHV"
$k=3$ "NMHV"
etc.

Twistor Space : Kinematics as simple as possible

$$M(..W..) = \int ..d^2\lambda .. e^{i\lambda\tilde{\mu}} M(..\lambda, \tilde{\lambda}, \tilde{\eta}, ..)$$

$$M(..Z..) = \int .. d^2\tilde{\lambda} .. e^{i\tilde{\lambda}\mu} M(..\lambda \tilde{\lambda}, \eta, ..)$$

$$W_A \equiv \left(\begin{array}{c} \tilde{\mu} \\ \tilde{\lambda} \\ \hline \tilde{\eta} \end{array}\right), \quad Z^A \equiv \left(\begin{array}{c} \lambda \\ \mu \\ \hline \eta \end{array}\right)$$

Conf. Grp : $SL(4,\mathbb{R})$

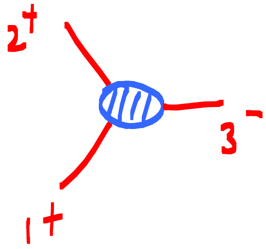
$$\mathcal{M}(tW) = t^{2(s-1)} \mathcal{M}(W)$$

$$\mathcal{M}(tZ) = t^{2(s-1)} \mathcal{M}(Z)$$

SYM: Functions of weight 0 on $\mathbb{RP}^{3|4}$

SUGRA: Functions of weight 2 on $\mathbb{RP}^{3|8}$

Twistor Space Amplitudes Amazingly Simple



YM: $\frac{[12]^3}{[13][23]}\,\delta^4(\Sigma\lambda\tilde{\lambda})$ $\quad$ $\text{sgn}\,W_1 Z_3\ \text{sgn}\,W_2 Z_3\ \text{sgn}\,[12]$

GR: $\left(\frac{[12]^3}{[13][23]}\right)^2 \delta^4(\Sigma\lambda\tilde{\lambda})$ $\quad$ $|W_1 Z_3|\,|W_2 Z_3|\ |\,[12]\,|$

Symmetries

- Cyclic/(perm. for grav)

$$\mathcal{M}[\lambda_i, \tilde{\lambda}_i, \tilde{\eta}_i]$$

$$\shortparallel$$

$$\mathcal{M}[\lambda_{i+1}, \tilde{\lambda}_{i+1}, \tilde{\eta}_{i+1}]$$

- Parity

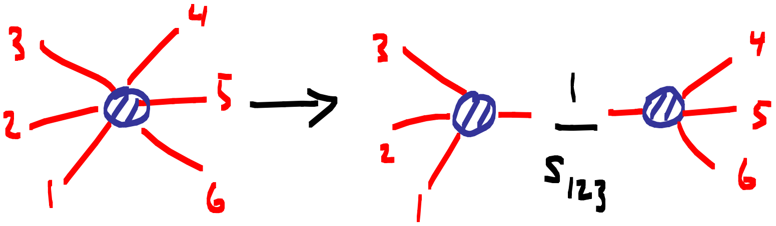
$$\int d^{\mathcal{N}}\tilde{\eta}_i \, e^{\eta_i \tilde{\eta}_i} \mathcal{M}(\lambda_i, \tilde{\lambda}_i, \tilde{\eta}_i)$$

$$\shortparallel$$

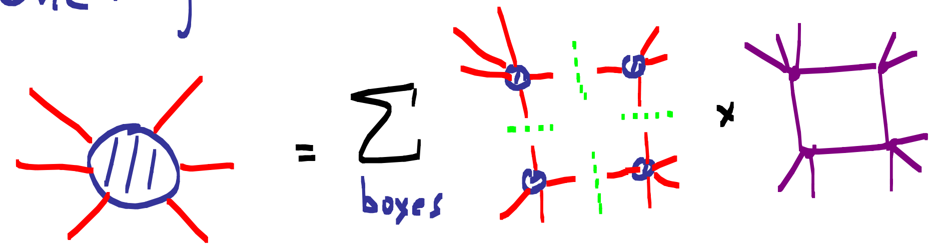
$$\mathcal{M}[\lambda \leftrightarrow \tilde{\lambda}, \tilde{\eta} \rightarrow \eta]$$

Singularity Structure: Imprint of Locality

- Trees

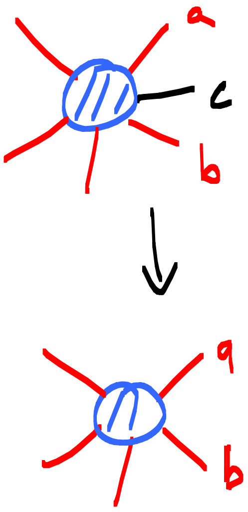


- One Loop



(+ subleading, vanish w/ maximal SUSY)

IR Limits



$$\times \frac{\langle ab\rangle}{\langle ac\rangle\langle bc\rangle}$$

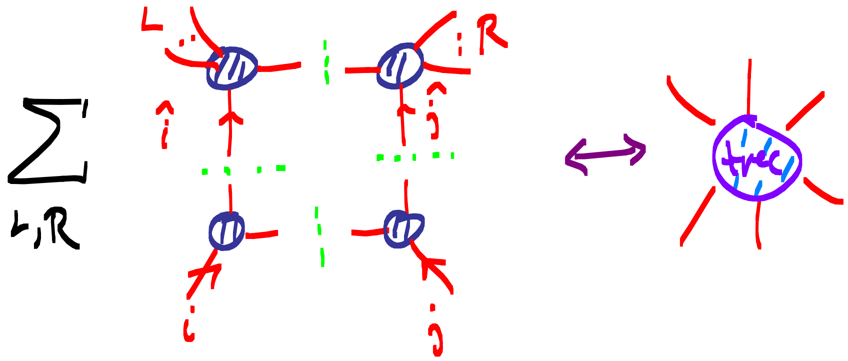
$$M_n^{\text{1-loop}}\Big|_{IR} =$$

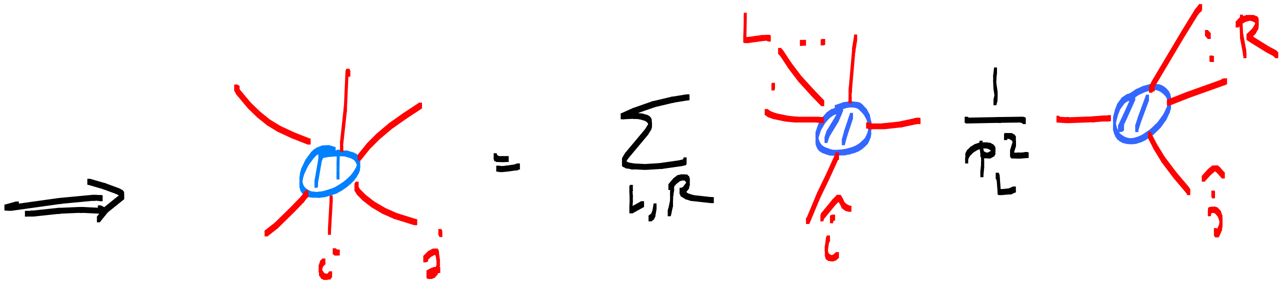
$$\left(\sum_i -\frac{1}{\epsilon^2}(-s_{i,i+1})^{\epsilon}\right) M_n^{\text{tree}}$$

[many eqns...]

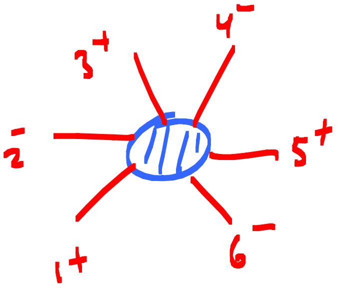
BCF

IR eqns @ 1-loop





BCFW 6pt NMHV



$$= \frac{\langle 46\rangle^4 [13]^4}{[12][23]\langle 45\rangle\langle 56\rangle} \frac{1}{(p_1+p_2+p_3)^2} \times \frac{1}{\langle 6|5+4|3]} \frac{1}{\langle 4|5+6|1]}$$

"Spurious" Poles: Don't occur in local theories!

$$+ \{i \to i+2\} + \{i \to i+4\}$$

Remarkable 6-term Id

$$\frac{\langle 46\rangle^4 [13]^4}{[12][23]\langle 45\rangle\langle 56\rangle}\frac{1}{(p_1+p_2+p_3)^2}\times\frac{1}{\langle 6|5+4|3]}\frac{1}{\langle 4|5+6|1]} + \{i\to i+2\} + \{i\to i+4\} = \frac{\langle 3|(2+4)|6]^4}{[23][34]\langle 56\rangle\langle 61\rangle}\frac{1}{(p_5+p_6+p_1)^2}\times\frac{1}{\langle 1|6+5|4]}\frac{1}{\langle 5|6+1|2]} + \{i\to i+2\} + \{i\to i+4\}$$

$$\text{Guarantees}\left\{\begin{array}{l}\text{Parity}\\ \text{Cyclicity}\\ \text{No Spurious Poles}\end{array}\right.$$

7-pt	12 terms
8-pt	20 terms
	40 terms
⋮	⋮

SOME POWERFUL MATHEMATICAL STRUCTURE IS AT WORK!

The Conjectured Duality

$$Q_{n,k} = \int \frac{d^{n\times k} C_{a\alpha}}{(12\cdots k)\cdots(n1\cdots k-1)} \prod_{\alpha=1}^{k} \delta^{4|4}[C_{a\alpha} W_a]$$

Integral over Grassmannian G(n,k)

$(m_1\cdots m_k) = \epsilon^{\alpha_1\cdots\alpha_k} C_{m_1\alpha_1}\cdots C_{m_k\alpha_k}$

Manifestly Cyclically Invariant

k linear relations on n twistors

- Claim: after we make this sharply defined – trivial to back to momentum space – multi-dimensional contour integral.
- Residues compute 1-loop leading singularities (+ hence all 1-loop amps) in $\mathcal{N}=4$ YM!
- (Includes all tree amps too by BCF logic)

- $$C = \overset{\longleftarrow\; m \;\longrightarrow}{\begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \\ c_{k1} & \cdots & c_{kn} \end{pmatrix}} \begin{matrix} \uparrow \\ k \\ \downarrow \end{matrix} \rightarrow \begin{matrix} k\text{-plane} \\ \text{in } n\text{-space} \\ \in G(k,n) \end{matrix}$$

- "Gauge Symmetry" $C_{a\alpha} \to C_{a\beta} L^{\beta}_{\alpha}$, any $k{\times}k$ L

- "Gauge Fix": columns I to some basis e.g.

$$C = \begin{pmatrix} 1 & 0 & 0 & c_{14} & \cdots & c_{17} \\ 0 & 1 & 0 & c_{24} & \cdots & c_{27} \\ 0 & 0 & 1 & c_{34} & \cdots & c_{37} \end{pmatrix} \text{ or } C = \begin{pmatrix} 1 & c_{21} & 0 & c_{41} & 0 & c_{61} & c_{71} \\ 0 & c_{23} & 1 & c_{43} & 0 & c_{63} & c_{73} \\ 0 & c_{25} & 0 & c_{45} & 1 & c_{65} & c_{75} \end{pmatrix}$$

Can Choose any GFing we like. [Different Charts on Grassmannian]

- Leaves us with $k(n-k)$ variables.

- Obvious mapping between k-plane and $\perp$ $(n-k)$ plane.

Given e.g.

$$C = \left(\mathbb{1}_{k\times k} \,\middle|\, c \right), \text{ define } *C = \left(\mathbb{1}_{n-k\times n-k} \,\middle|\, -c^T \right)$$

- Symmetry $k \leftrightarrow (n-k)$ is just parity

- After GFing, trivial to go back to momentum space:

$$Q_{n,k} = \int \frac{d^{(n-k)k} C}{(12..k) .. (n..k-1)} \delta^2[C\lambda]\,\delta^2[*C\tilde{\lambda}]\,\delta^4[*C\tilde{\eta}]$$

- Parity Manifest!

Note: for gluon amplitudes, convenient to gauge fix columns I corresponding to particles of negative helicity – then intgd is always the same, its just form of C that changes.

e.g. $1^- 2^- 3^+ 4^- 5^+ 6^+$

$$C = \begin{pmatrix} 1 & 0 & c_{31} & 0 & c_{51} & c_{61} \\ 0 & 1 & c_{32} & 0 & c_{52} & c_{62} \\ 0 & 0 & c_{34} & 1 & c_{54} & c_{64} \end{pmatrix}$$

$1^- 2^+ 3^- 4^+ 5^- 6^+$

$$C = \begin{pmatrix} 1 & c_{21} & 0 & c_{41} & 0 & c_{61} \\ 0 & c_{23} & 1 & c_{43} & 0 & c_{63} \\ 0 & c_{25} & 0 & c_{45} & 1 & c_{65} \end{pmatrix}$$

Diff. helicities: integrate same fn. on different charts of Grassmanian!

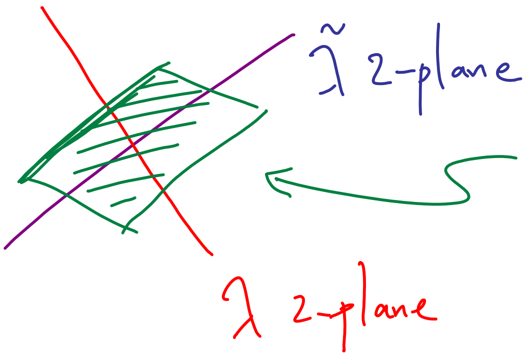
The condition $C\lambda = 0$, $\cancel{x} C\tilde{\lambda} = 0$

$$\iff \quad \lambda_i - c_{iI}\lambda_I = 0 \quad , \quad \tilde{\lambda}_I + c_{iI}\tilde{\lambda}_i = 0$$

Can only be satisfied if mom. conserved

$$\lambda_i\tilde{\lambda}_i + \lambda_I\tilde{\lambda}_I = 0 \quad !$$

Geometrically: $\vec{\lambda}_\alpha, \vec{\tilde{\lambda}}_{\dot\alpha}$ n vectors.



C k-plane, orthog. to λ, must contain $\tilde\lambda$

$\Longrightarrow$ $\lambda \cdot \tilde\lambda = 0$ = mom conservation!

So, we are left with $k(n-k)-(2n-4)$

$= (k-2)(n-k-2)$ parameters; solns of

$$\lambda_i - c_{iI}(\tau_A)\lambda_I = 0 \,, \quad \tilde{\lambda}_I + c_{iI}\tilde{\lambda}_i(\tau_A) = 0$$

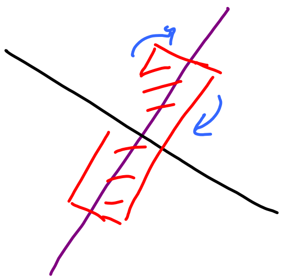
$$c_{iI}(\tau_A) = c^{*}_{iI} + d_{iIA}\tau^A \,, \quad A = 1, \ldots, (k-2)(n-k-2)$$

[one to one correspondence with $G(k-2, n-4)$]

Note: no such plane exists for $k = 0, 1, n-1, n$!
Thats why these amps vanish.

$k = 2,\ n-2$ [MHV + $\overline{\text{MHV}}$], plane fixed + we get right answer.

Otherwise



$(k-2)[n-k-2]$ directions k-plane can rotate in!

Factoring out $\delta^4(\sum_a p_a)$, we have

↑

$$Q_{n,k}(\lambda,\tilde{\lambda}) = J(\lambda,\tilde{\lambda}) \int \frac{d^{(k-2)(n-k-2)}\tau_A}{(12..k)(\tau)\,..\,(n1..k-1)(\tau)}$$

At this point everything can be fully complexified,

Easy to prove: each minor is of degree min$[k-2, n-k-2]$ in the τ_A's. $\left\{ k=3, \text{ all linear} \right\}$

First interesting case: 6 pt NMHV $(n=6, k=3)$.

Look @ $1^+ 2^- 3^+ 4^- 5^+ 6^-$

$$C = \begin{pmatrix} \times & 1 & \times & 0 & \times & 0 \\ \times & 0 & \times & 1 & \times & 0 \\ \times & 0 & \times & 0 & \times & 1 \end{pmatrix}$$

$$\lambda_i - c_{iI}\lambda_I = 0 \, , \quad \tilde{\lambda}_I + c_{iI}\tilde{\lambda}_i = 0$$

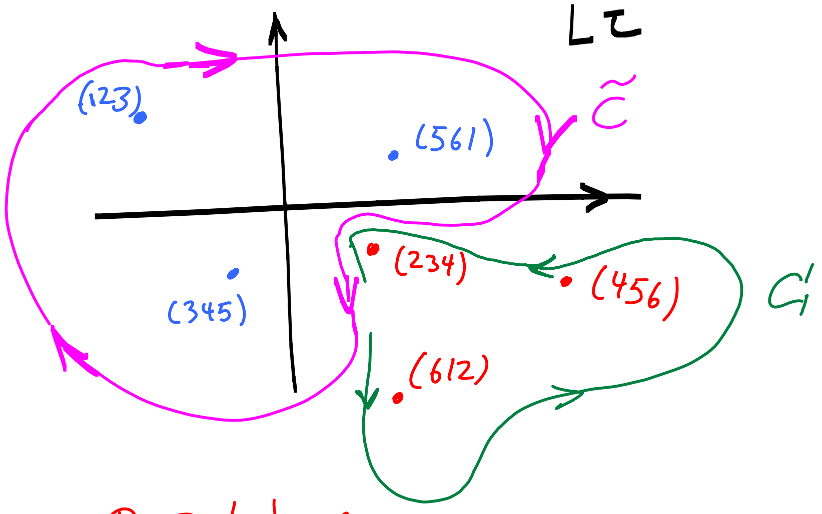
$$\Longrightarrow \quad c_{iI}(\tau) = c^*_{iI} + \epsilon_{ijk}\epsilon_{IJK}\langle JK\rangle[jk]\tau$$

Jacob. $\delta^4(\Sigma p_a) \to J(\lambda, \tilde{\lambda}) = 1.$

So

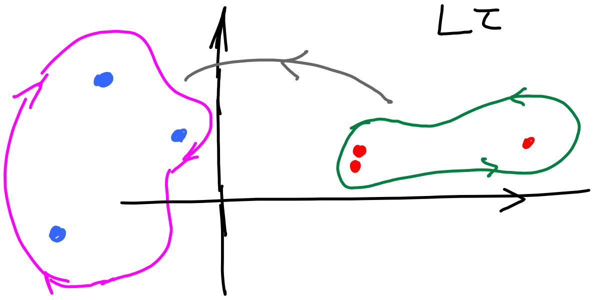
$$Q_{6,3} = \int d\tau \; \frac{1}{(123)\,(234)\cdots(612)\,(\tau)}$$

each factor linear in τ



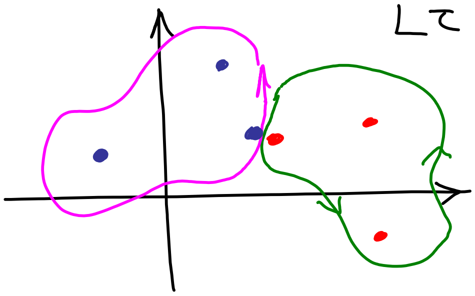
- ● residues : BCFW terms
- ● residues : P[BCFW] terms
- Cauchy : BCFW = P[BCFW] = Remarkable 6-term identity!

Spurious Poles



Contour can be deformed away from singularity

Physical Poles



Can't deform contour to avoid singularity

LOCALITY ↔ CONTOUR DEFORMATION

- For all other cases we have more than 1 complex variable. What is a residue?

$$f(z) = \frac{g(z)}{A_1(z) \cdots A_n(z)} \qquad z = (z_1, \ldots, z_n)$$

$$\underset{A_1(z_*) = \cdots = A_n(z_*) = 0}{\text{res}} f(z_*) = \frac{g(z_*)}{\det \left. \frac{\partial A_i}{\partial z_j} \right|_{z_*}}$$

- Also higher-dim gen of Cauchy's thm:

$$f(z) = \frac{g(z)}{A_1(z) \cdots A_n(z)}$$

[note a given $f(z)$ can be written in this form in many ways]

then

$$\sum_{\substack{z_* \\ A_1(z_*) = \cdots = A_n(z_*) = 0}}' \operatorname{res} f(z_*) = 0$$

{if deg g is small enough}

- First new case: $n=7$, $k=3$ (7 pt NMHV).

Look e.g. @ $1^+2^-3^+4^-5^+6^-7^-$.

$$Q_{7,3} = \int \frac{d\tau_1\, d\tau_2}{(123)\ \cdots\ (712)} \xleftarrow{} \text{each linear in } \tau_1, \tau_2$$

There are $\binom{7}{2} = 21$ residues.

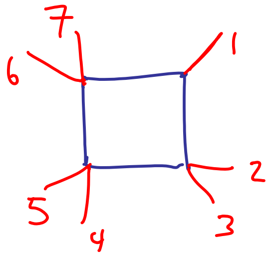
- They are computed just by solving linear equations, but can get interesting!

e.g. $(123)(456) = \dfrac{\left([7|(2+4)|3\rangle\langle 54\rangle + [76]\langle 65\rangle\langle 34\rangle\right)^4}{\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle[71]}$

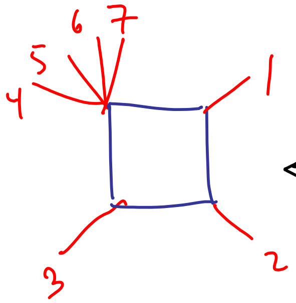
$$\times \frac{1}{[1|(2+3)|4\rangle[7|5+6|4\rangle\langle 4|(5+6)(7+1)|2\rangle\langle 4|(2+3)(7+1)|6\rangle}$$

+ 20 other horrible guys!

- Miraculously, these 21 residues exactly match the 21 objects first found in '04 by Zvi, Lance et. al. in the 1-loop 7 pt $\mathcal{N}=4$ amp!



$$\longleftrightarrow \quad (123)(671)$$



$$\longleftrightarrow \quad (123)\left[(345)+(567)+(712)\right] + (456)(567)$$

Now, the Global Residue Thm implies many identities. [Actually overkill in this case since all factors linear – just repeated use of Cauchy gives same thing].

$$(i) \longleftrightarrow [i\ i{+}1\ i{+}2]$$

$$(i)\cdot(j) \longleftrightarrow \mathrm{res}\Big|_{(i)=(j)=0}$$

$$f(\tau_1,\tau_2) = \frac{1}{\underbrace{(1)}_{A_1}\ \underbrace{[(2)(3)\cdots(7)]}_{A_2}} \Longrightarrow (1)\cdot(2) + \cdots + (1)\cdot(7) = 0$$

+ cyclic exhaust all residue id.

These "basic" identities have a direct physical interpetation: IR equations!

e.g. look at $\log((p_1+p_2+p_3)^2)$ term in 1-loop IR eqn $= (1)(2) + (1)(3) + \ldots + (1)(7) = 0$.

LOCALITY ↔ CONTOUR DEFORM.

Take

$$f(\tau_1,\tau_2) = \frac{1}{\underbrace{(1)(2)(3)(4)}_{A_1}\ \underbrace{(5)(6)(7)}_{A_2}}$$

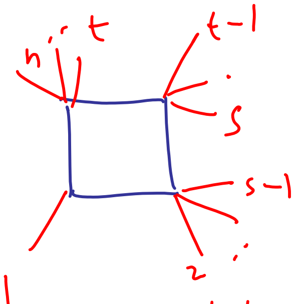
$4 \times 3 = 12$ term identity which guarantees cyclic.

+ absence of spurious poles for 7 pt tree amp!

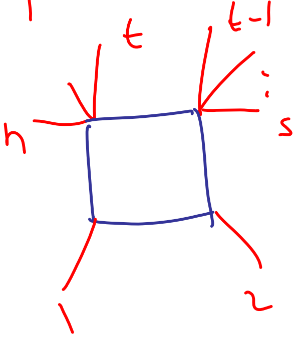
(Can actually show there is a unique such object → build local 7pt amp).

- We know the explicit map between residues + boxes for all NMHV amplitudes:

e.g.



$$(n-1\ n\ 1)\quad (1\,2\,3)\prod_{i=1}^{s-3}(i\ i{+}1\ i{+}2)\prod_{i=s}^{t-3}(i\ i{+}1\ i{+}2)\prod_{i=t}^{n-2}(i\ i{+}1\ i{+}2)$$



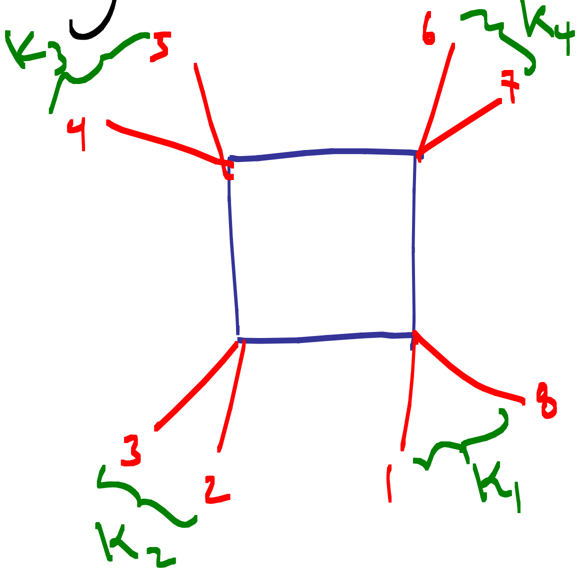
$$\Big[(n-1\ n\ 1)+(2\,3\,4)\Big]\prod_{i=3}^{t-3}(i\ i{+}1\ i{+}2)\prod_{i=1}^{n-2}(i\ i{+}1\ i{+}2)$$

All Gl. Res. Thm ident. linear comb. of

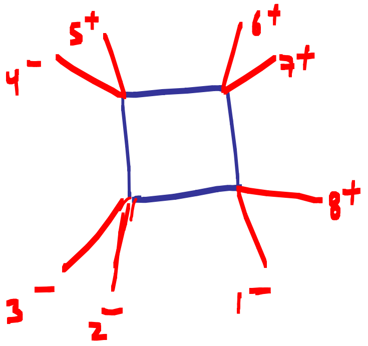
$$\sum_j (i_1)(i_2)\cdots(i_{n-6})(j) = 0$$

BUNCH OF 6 term id.

Simplest IR Finite 1-loop:



8 pt N^2MHV

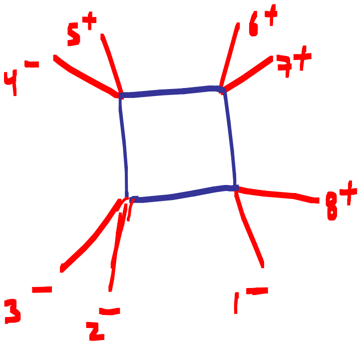


$$= \sum_{\ell^+,\ell^-} \frac{\langle 23\rangle^3 [67]^3 [4|\ell-P_{23}|4\rangle^3 \langle 1\ell\rangle^3}{\langle 45\rangle\langle 81\rangle\langle \ell 2\rangle\langle 3|\ell-P_{23}|4]}$$

$$\times \frac{1}{[4|P_{23}|\ell\rangle\langle \ell|P_{18}|7]\langle 5|\ell-P_{2345}|6] \quad \langle 8|(\ell-P_{18})(\ell-P_{2345})(\ell-P_{23})|4]}$$

where $\ell^2 = (\ell-K_1)^2 = (\ell-K_1-K_2)^2 = (\ell+K_4)^2$, quad. eqn with

$$\Delta = 1-2(\rho_1+\rho_2)+(\rho_1-\rho_2)^2 ; \quad \rho_1 = \frac{K_1^2 K_2^2}{K_{12}^2 K_{23}^2} , \quad \rho_2 = \frac{K_2^2 K_4^2}{K_{12}^2 K_{23}^2}$$



$$= \text{res}\left[\frac{1}{(1234)\cdot\cdot(8123)}\right]$$

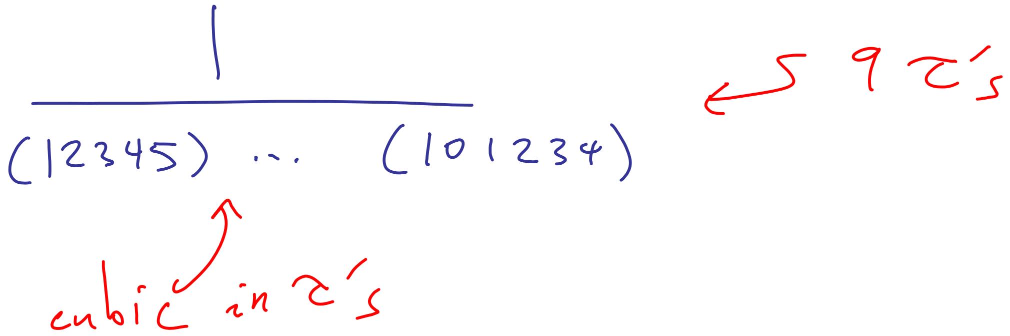
where $(1234)=(4567)=(6781)=(8123)=0$!

Very Non-trivial Check!

Actually these are quadratic in 4 τ's, get 2 solns, corresponding to $\underline{\ell_\pm}$!

- The object we have found seems to unify all 1-loop leading singularities. Very natural + beautiful mathematical structure — intersection theory + Schubert Calculus — seems to lie at the heart of tree + loop gluon scattering amps!

For ex: $n=10, k=5$



residue: simultaneous sol'n of 9 cubics in 9 variables…

help!

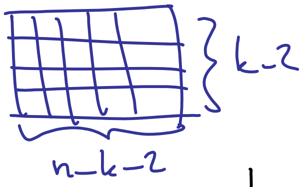
General answer: $D = (k-2)(n-k-2) = \dim G(k-3, n-4)$

\# self-int of σ_1^D

elem. Schubert cycle

$\parallel$

\# of appearances of



in $(\square)^D$

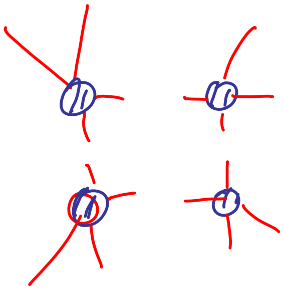
$$= \frac{1!\,2!\,\cdots(k-3)!\;D!}{(n-k-2)!\;\cdots\;(n-5)!} = 42 \text{ for } k=5,\ n=10$$

(WORKS!)

But perhaps our physical interpretation has something to offer mathematicians?

Naively: expect soln's of cubics to involve $\sqrt[3]{}$'s.

But in



only $\sqrt{}$'s appear!

[4D Locality again!]

So we predict that, no matter how high k, n get, that if the external momenta are in $\mathbb{Q}$, the solns are in $\mathbb{Q} + \sqrt{\mathbb{Q}}$, no higher roots!

We've checked that this is correct for $k = 5$, $n = 10$.

- All of these checks reveal the power of of a weak-weak duality! And we are beginning to see the sorts of structures that allow local spacetime physics to emerge holographically.

Outlook

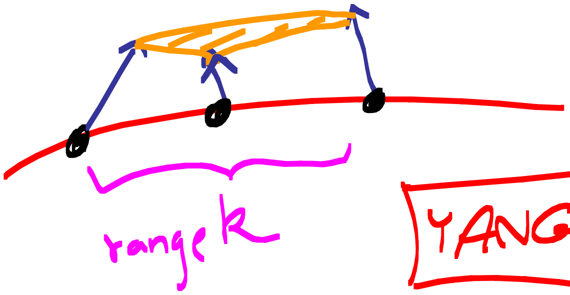
THE DUAL THRY
CLEARLY EXISTS

[Quantum!] S-Matrix

Twistors

Alg. Geometry

WHAT IS THE
PHYSICS?

- 

$\boxed{\text{YANGIAN?}}$

- Connection to Hodges recent work?

- $$Q_{n,k} = \int dX_\alpha^{(j)}\, dY_\beta^{(j)}\, dZ_\alpha\, dC_{a\alpha}\, e^{iS}$$

$$S = X_\alpha^{(j)} Y_\beta^{(j)} C_{j+\alpha-1,\beta} + W_a C_{a\alpha} Z_\alpha$$

- $N=8$ Will Be Much More Interesting!