Master Space and Hilbert Series

Davide Forcella

ENS Paris

STRINGS 2009

In collaboration with:

A. Zaffaroni, A. Hanany, A. Butti, Y.H. He, D. Vegh

Other important related works by:

S.T. Yau, D. Martelli, J. Sparks, S. Benvenuti, B. Feng,...

Introduction

I would like give you an overview of the use of the Master Space $\mathcal{F}^{\flat}$ and the Hilbert Series $H(q; ...)$ to analyse SUSY field theories. I will use the example of D3 branes at $\mathcal{N} = 1$ singularities χ.

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- Low energy dynamics: vacuum structure. In SUSY field theory a high degeneracy of vacua (moduli space $\mathcal{M}$) is a typical phenomenon. $\mathcal{M}$ is defined by the set of F-flat, D-flat conditions modulo gauge invariance.

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- There exist a nice interplay between $\mathcal{F}^\flat$ and $H(q;\mathcal{M})$.

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- $\lim_{t \to 0} H(q = e^{-bt}; \mathcal{M}) \sim P(q)/(1-q)^{\dim \mathcal{M}} + \ldots$
- The symmetries G of the variety $\mathcal{M}$ can be encoded into the Hilbert Series: $H(q; \mathcal{M}) = \sum_{\vec{l}} \chi^G_{\vec{l}}(q)$

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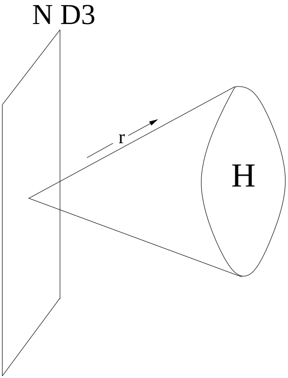
D3 branes at Singularities



Generic features: Quiver Gauge Theories

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- examples: $\mathbb{C}^3$, $\mathbb{C}^3/\Gamma$, $C(T^{1,1})$, $C(Y^{p,q})$, $C(L^{a,b,c})$,...

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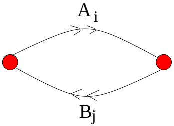
- Use the PE function: $PE_\nu[f(t)] := \exp\left(\sum_{k=1}^{\infty} \frac{\nu^k f(t^k)}{k}\right)$, to pass from $N = 1$ to $N > 1$

The conifold $\chi = C(T^{1,1})$



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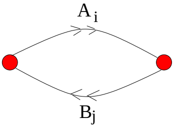

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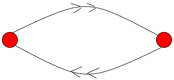

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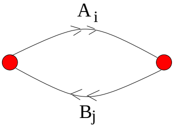

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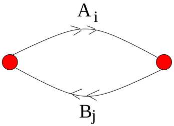

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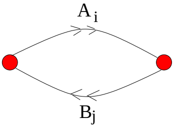

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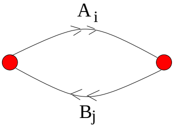

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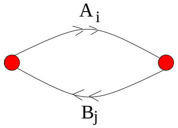

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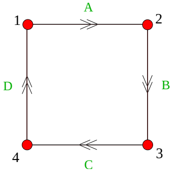

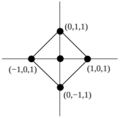

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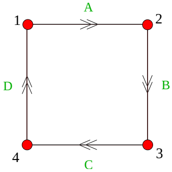

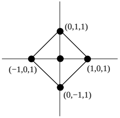

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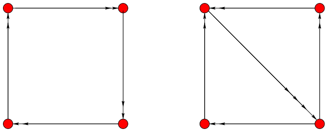

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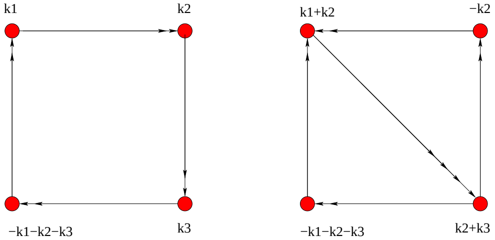
The two three dimensional quiver CS theories



(a) (b)

with superpotentials

$$W_{(a)} = \epsilon_{ij}\epsilon_{pq}\mathbf{A}_i\mathbf{B}_p\mathbf{C}_j\mathbf{D}_q$$
$$W_{(b)} = \varepsilon_{ij}\varepsilon_{kl}X_{13}^{ik}X_{32}^{l}X_{21}^{j} - \varepsilon_{ij}\varepsilon_{kl}X_{13}^{ik}X_{34}^{l}X_{41}^{j}$$

probe the same CY_4 singularities for specific values of k_i.

Conclusions and Outlooks

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Thanks for your Attention

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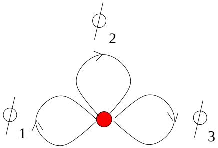

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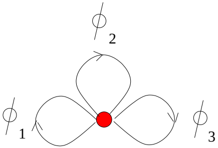

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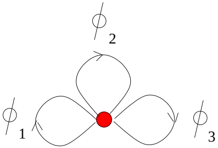

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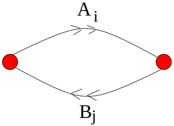

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