Scattering amplitudes in $\mathcal{N} = 4$ super Yang-Mills

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Based on work in collaboration with

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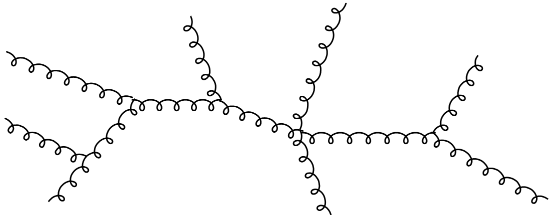
Motivation and outline

- ✔ Why $\mathcal{N}=4$ super Yang-Mills?
  - ✗ learn something about pure Yang-Mills

    e.g. tree-level gluon scattering amplitudes



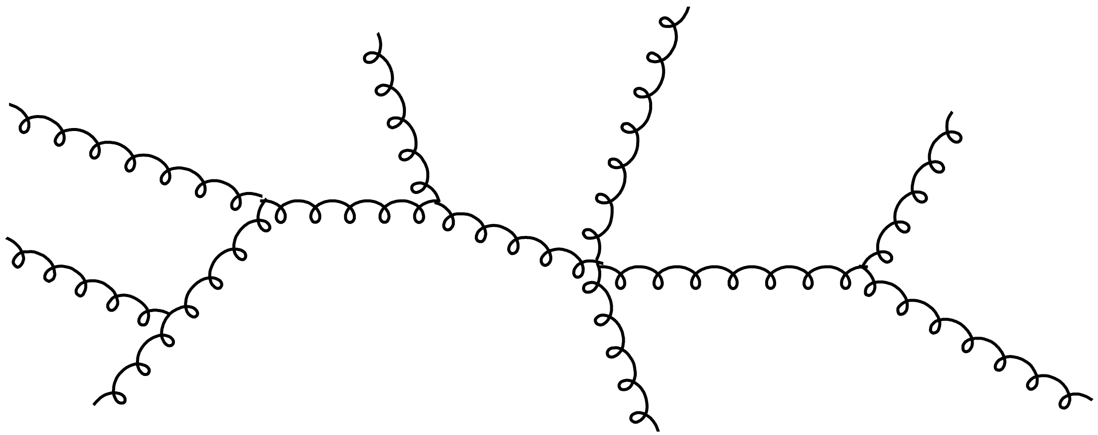
  - ✗ spectrum of anomalous dimensions

    expected integrability

    compute at weak and strong coupling (AdS/CFT)

  - ✗ integrability for other quantities in $\mathcal{N}=4$ SYM?
- ✔ Questions we want to ask:
  - ✗ can we compute tree-level amplitudes for an arbitrary number of gluons?
  - ✗ what are the symmetry properties of the amplitudes?

computing tree-level amplitudes



Tree-level scattering amplitudes

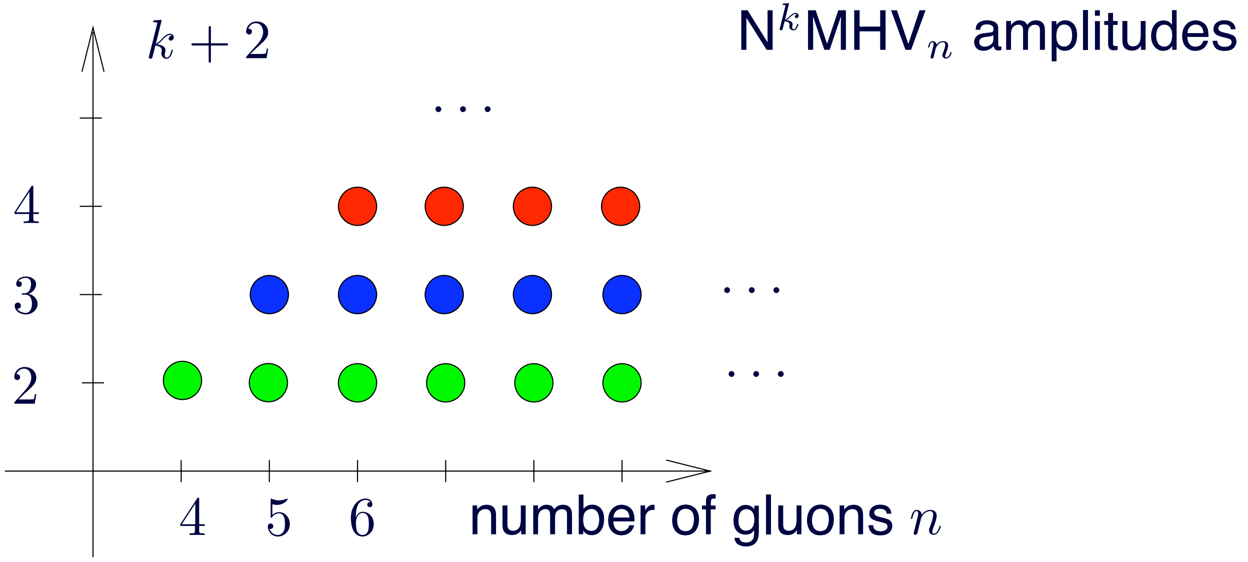
✔ maximally helicity violating (MHV) amplitudes [Parke, Taylor 1986]

$$A_n^{\rm MHV}(i,j) = \delta^4(p)\,\frac{\langle i\,j\rangle^4}{\langle 1\,2\rangle \ldots \langle n\,1\rangle} \qquad p_i^{\alpha\dot\alpha} = \lambda_i^\alpha \tilde\lambda_i^{\dot\alpha}\,, \qquad \langle i\,j\rangle = \lambda_i^\alpha \lambda_j^\beta \epsilon_{\alpha\beta}$$

✔ MHV amplitudes in $\mathcal{N}=4$ on-shell superspace [Nair 1988]

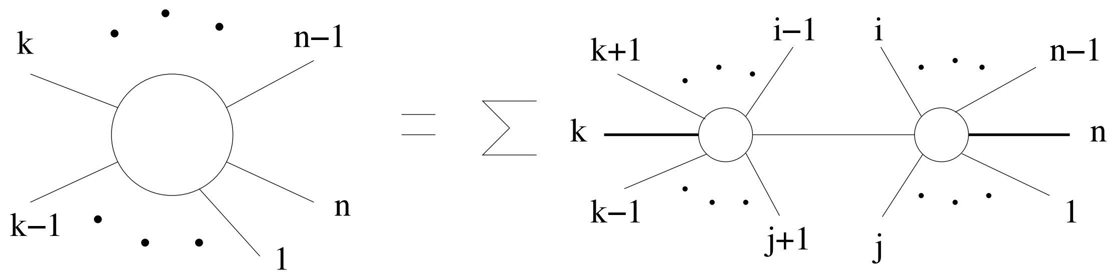
$$\Phi = g^+ + \eta^A f_A + \ldots + 1/4!\epsilon_{ABCD}\eta^A\eta^B\eta^C\eta^D g^-\,, \qquad q_i^{\alpha A} = \lambda_i^\alpha \eta_i^A$$

$$\mathcal{A}_n^{\rm MHV} = \delta^4(p)\,\delta^8(q)\,\frac{1}{\langle 1\,2\rangle \ldots \langle n\,1\rangle}$$



On-shell recursion relations

✔ on-shell recursion relations [Britto, Cachazo, Feng + Witten, 2004]



$$A = \sum A_L \frac{1}{P^2} A_R$$

- ✗ follow from analytic properties of the amplitudes (Feynman diagrams)
- ✗ n-point amplitudes are obtained recursively from lower-point amplitudes
- ✗ all amplitudes are on-shell

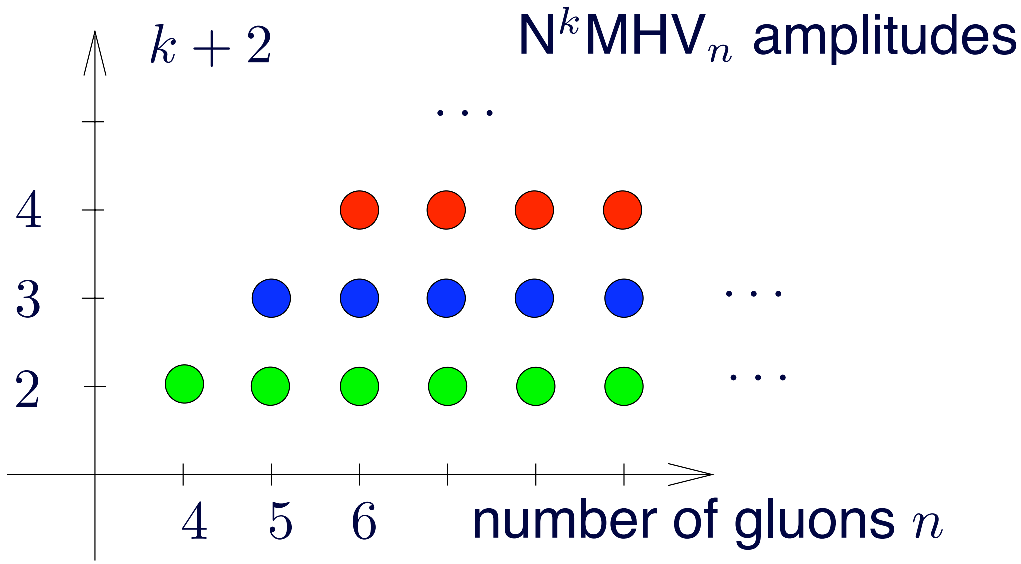
✔ $\mathcal{N} = 4$ supersymmetric version [Arkani-Hamed, Cachazo, Kaplan 2008]

All tree-level amplitudes in $\mathcal{N} = 4$ SYM

supersymmetry implies

$$\mathcal{A}_n = \delta^4(p)\delta^8(q)\frac{1}{\langle 1\,2\rangle \ldots \langle n\,1\rangle}\,\mathcal{P}_n$$

✔ $\mathcal{P}_n^{\rm MHV} = 1$ [Nair 1988]



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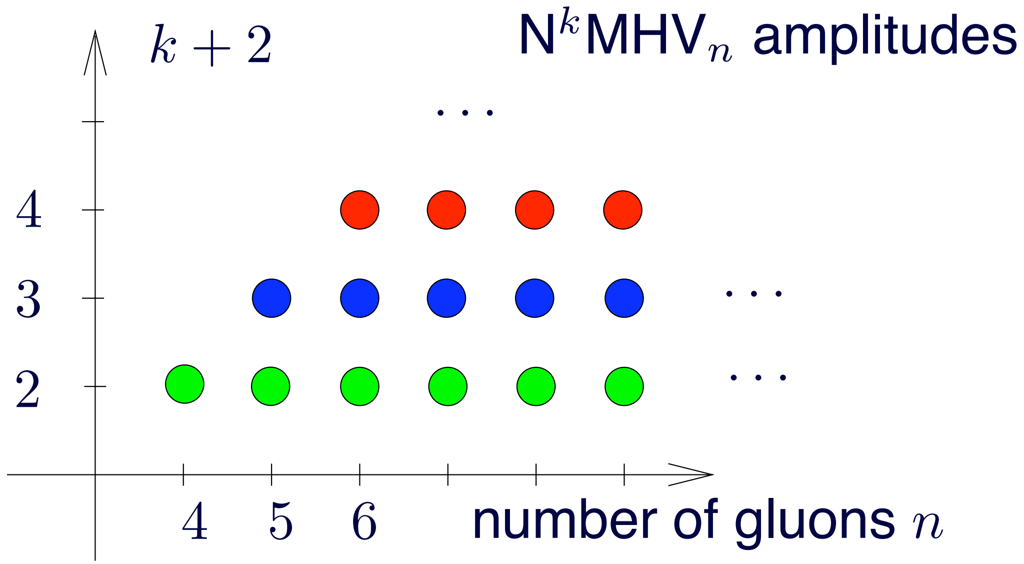
✔ $\mathcal{P}_n^{\mathrm{MHV}} = 1$ [Nair 1988]

✔ $\mathcal{P}_n^{\mathrm{NMHV}} = \sum_{2\le i,j\le n-1} R_{n;i,j}$

[Drummond, J.M.H., Korchemsky, Sokatchev 2008]



with

$$R_{n;i,j} = \frac{< i\; i-1 >< j\; j-1 > \delta^4(\Xi_{n;ij})}{x_{ij}^2 < n|x_{ni}x_{ij}|j >< n|x_{ni}x_{ij}|j-1 >< n|x_{nj}x_{ji}|i >< n|x_{nj}x_{ji}|i-1 >}$$

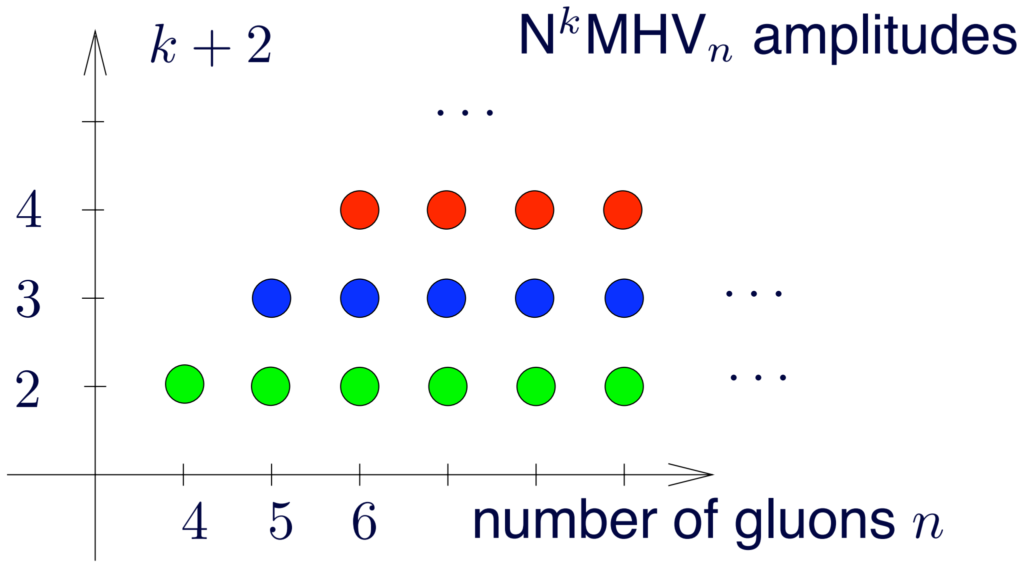
where

$$\Xi_{n;ij} = < n|x_{ni}x_{ij}|\theta_{jn} > + < n|x_{nj}x_{ji}|\theta_{in} >$$

and

$$p_i = x_i - x_{i+1}\,, \qquad q_i = \theta_i - \theta_{i+1}$$

All tree-level amplitudes in $\mathcal{N} = 4$ SYM



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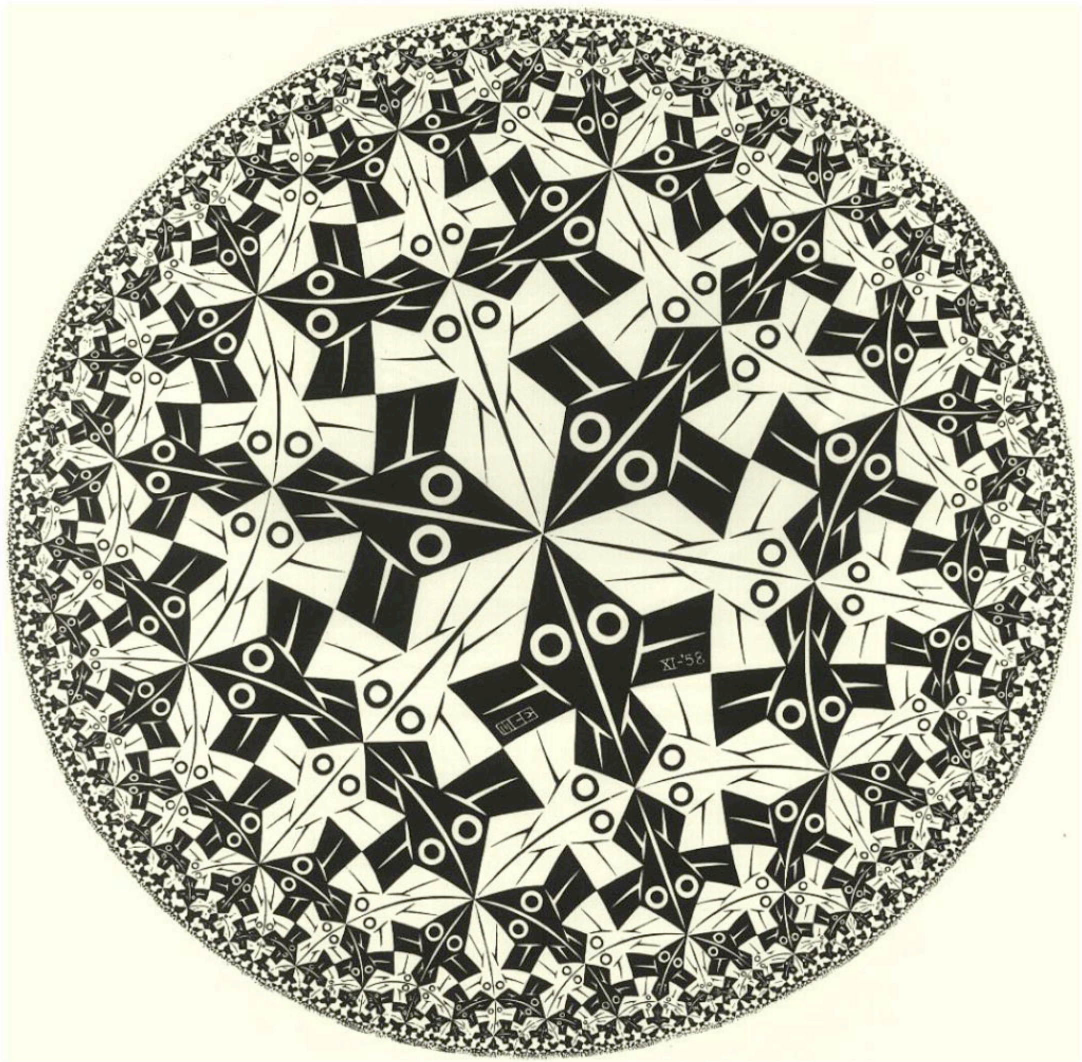
✔ $\mathcal{P}_n^{\text{NMHV}} = \sum_{2\leq i,j\leq n-1} R_{n;i,j}$ [Drummond, J.M.H., Korchemsky, Sokatchev 2008]

✔ generic case [Drummond, J.M.H. 2008]

$$\mathcal{P}_n^{\text{N}^k\text{MHV}} = \underbrace{\sum \cdots \sum}_{2k-\text{fold nested sums}} \underbrace{R \times \cdots \times R}_{k-\text{fold product}}$$

Second part - symmetries



[M.C. Escher]

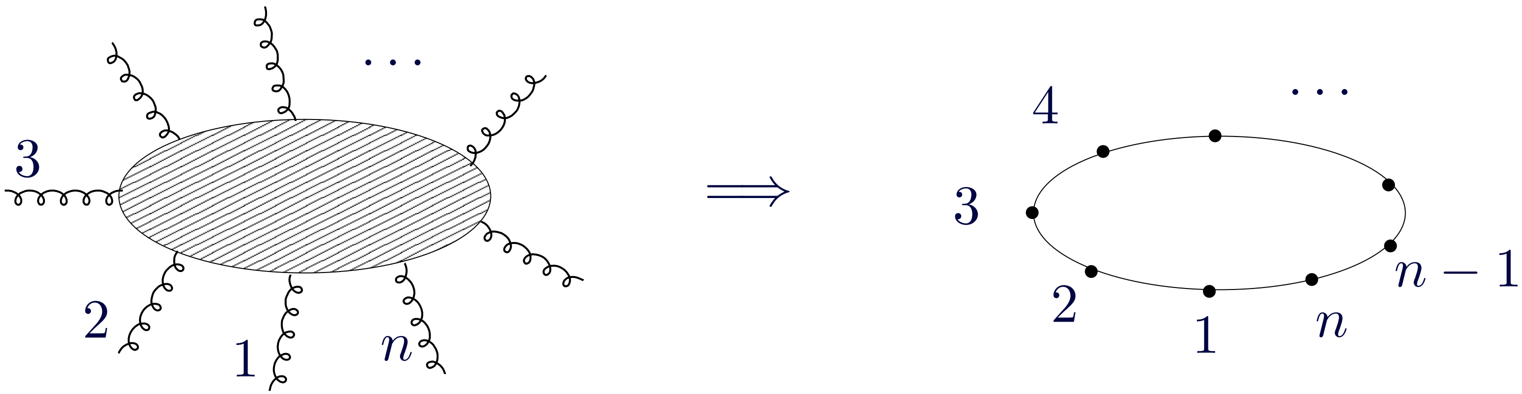
Expected symmetries of the amplitudes

✔ $psu(2,2|4)$ superconformal generators in on-shell superspace [Witten 2003]

$$[J_a, J_b\} = f_{ab}{}^{c} J_c\,, \qquad J_a = \sum_{i=1}^{n} J_{ia}$$

$$p^{\dot\alpha\alpha} = \sum_i \tilde\lambda_i^{\dot\alpha}\lambda_i^{\alpha}\,, \qquad k_{\alpha\dot\alpha} = \sum_i \frac{\partial}{\partial\lambda_i^{\alpha}}\frac{\partial}{\partial\tilde\lambda_i^{\dot\alpha}}\,,$$

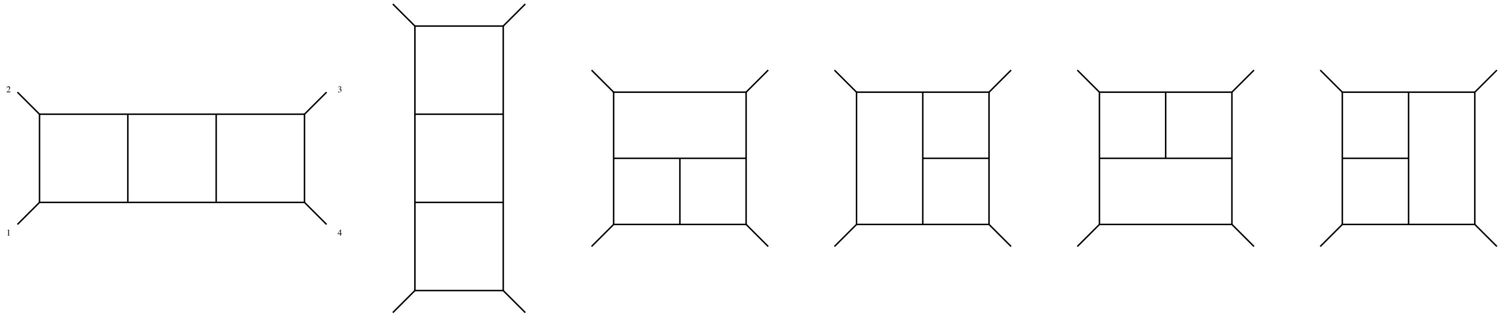
$$d = \sum_i [\tfrac{1}{2}\lambda_i^{\alpha}\frac{\partial}{\partial\lambda_i^{\alpha}} + \tfrac{1}{2}\tilde\lambda_i^{\dot\alpha}\frac{\partial}{\partial\tilde\lambda_i^{\dot\alpha}} + 1],$$



✔ invariance under $psu(2,2|4)$

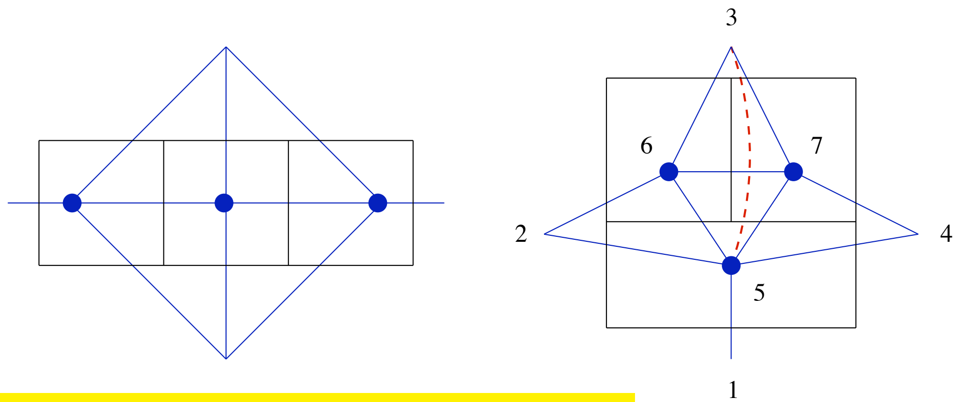
$$\{p, k, m, \bar m, d, c, r, q, \bar q, s, \bar s\}\,\mathcal{A} = 0$$

- ✔ loop corrections to four-gluon (MHV) amplitude [Bern, Dixon, Smirnov 2005]



- ✔ hints for a new symmetry [Drummond, J.M.H., Smirnov, Sokatchev 2006]

introduce dual variables $x_i - x_{i+1} = p_i$



conformal symmetry in dual space! (broken by infrared divergences)

- ✔ same variables also appeared in AdS dual [Alday, Maldacena 2007]

Dual superconformal symmetry

- ✔ amplitudes have additional symmetries!

  [Drummond, J.M.H., Korchemsky, Sokatchev '08]

$$x_i^{\alpha\dot\alpha} - x_{i+1}^{\alpha\dot\alpha} = p_i^{\alpha\dot\alpha} = \lambda_i^{\alpha}\tilde\lambda_i^{\dot\alpha}$$

- ✔ dual conformal generator

$$K_{\alpha\dot\alpha} = \sum_i \left[x_{i\alpha}{}^{\dot\beta} x_{i\dot\alpha}{}^{\beta} \frac{\partial}{\partial x_i^{\beta\dot\beta}} \right]$$

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- supersymmetric generalisation

$$\theta_i^A{}_{\alpha} - \theta_{i+1\,\alpha}^A = q_i^A{}_{\alpha} = \eta_i^A \lambda_{i\alpha}$$

- final expression for dual conformal generator

$$K^{\alpha\dot\alpha} = \sum_i \left[x_i^{\alpha\dot\beta} x_i^{\dot\alpha\beta}\frac{\partial}{\partial x_i^{\beta\dot\beta}} + x_i^{\dot\alpha\beta}\lambda_i^{\alpha}\frac{\partial}{\partial\lambda_i^{\beta}} + x_{i+1}^{\alpha\dot\beta}\tilde\lambda_i^{\dot\alpha}\frac{\partial}{\partial\tilde\lambda_i^{\dot\beta}} + x_i^{\dot\alpha\beta}\theta_i^{\alpha B}\frac{\partial}{\partial\theta_i^{\beta B}} + \tilde\lambda_i^{\dot\alpha}\theta_{i+1}^{\alpha B}\frac{\partial}{\partial\eta_i^B} \right]$$

Dual superconformal symmetry

- ✔ act on variables $\{\lambda_i, \tilde{\lambda}_i, \eta_i, x_i, \theta_i\}$
- ✔ constraints $x_i - x_{i+1} = \lambda_i \tilde{\lambda}_i \,, \qquad \theta_i - \theta_{i+1} = \eta_i \lambda_i$

$$P_{\alpha\dot{\alpha}} = \sum_i \frac{\partial}{\partial x_i^{\alpha\dot{\alpha}}} \,, \qquad Q_{\alpha A} = \sum_i \frac{\partial}{\partial \theta_i^{\alpha A}} \,, \qquad \overline{Q}_{\dot{\alpha}}^{A} = \sum_i [\theta_i^{\alpha A} \frac{\partial}{\partial x_i^{\alpha\dot{\alpha}}} + \eta_i^A \frac{\partial}{\partial \eta_i^{\dot{\alpha}}}],$$

$$K^{\alpha\dot{\alpha}} = \sum_i \left[x_i^{\alpha\dot{\beta}} x_i^{\dot{\alpha}\beta} \frac{\partial}{\partial x_i^{\beta\dot{\beta}}} + x_i^{\dot{\alpha}\beta} \lambda_i^{\alpha} \frac{\partial}{\partial \lambda_i^{\beta}} + x_{i+1}^{\alpha\dot{\beta}} \tilde{\lambda}_i^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\beta}}} + x_i^{\dot{\alpha}\beta} \theta_i^{\alpha B} \frac{\partial}{\partial \theta_i^{\beta B}} + \tilde{\lambda}_i^{\dot{\alpha}} \theta_{i+1}^{\alpha B} \frac{\partial}{\partial \eta_i^{B}} \right]$$

Similarly, $M_{\alpha\beta}, \overline{M}_{\dot{\alpha}\dot{\beta}}, R^A{}_B, D, C, S_\alpha^A, \overline{S}_{\dot{\alpha}A}$.

tree amplitudes are covariant under dual superconformal transformations

$$K^{\alpha\dot{\alpha}} \mathcal{A}_n = - \sum_i x_i^{\alpha\dot{\alpha}} \mathcal{A}_n$$

[Drummond, J.M.H., Korchemsky, Sokatchev ’08]

[Brandhuber, Heslop, Travaglini ’08], [Drummond, J.M.H. ’08]

Conventional and dual superconformal symmetry

[Drummond, J.M.H., Korchemsky, Sokatchev '08]

$$p \qquad\qquad K$$

$$q \qquad \bar{q} = \bar{S} \qquad S$$

$$s \qquad \bar{s} = \bar{Q} \qquad Q$$

$$k \qquad\qquad P$$

- ✔ also observed in the AdS dual (fermionic T-duality)
  [Maldacena, Berkovits '08] [Beisert, Ricci, Tseytlin, Wolf '08]
- ✔ closure of the algebra?

Commuting the two algebras

[Drummond, J.M.H., Plefka 2009]

- ✔ technical steps
  - ✗ reformulate covariance as an invariance

$$(K_{\alpha\dot{\alpha}} + \sum_i x_{i\,\alpha\dot{\alpha}})\,\mathcal{A}_n = 0$$

  - ✗ remove all x, θ dependence (use P and Q to set $x_1 = 0$ and $\theta_1 = 0$)
  - ✗ subtract 'trivial' terms like $p\,d$
- ✔ we find $\boxed{{K'}_{\alpha\dot{\alpha}}\mathcal{A}_n = 0}$ with

$$K_{\alpha\dot{\alpha}} \to {K'}_{\alpha\dot{\alpha}} = \sum_{i>j}\left[\left(m_{i\alpha}^{\gamma}\delta_{\dot{\alpha}}^{\dot{\gamma}} + \bar{m}_{i\dot{\alpha}}^{\dot{\gamma}}\delta_{\alpha}^{\gamma} - d_i\delta_{\alpha}^{\gamma}\delta_{\dot{\alpha}}^{\dot{\gamma}}\right)p_{j\gamma\dot{\gamma}} + \bar{q}_{i\dot{\alpha}C}q_{j\alpha}^{C} - (i \leftrightarrow j)\right]$$

- ✔ notice

$$\boxed{{K'}_{\alpha\dot{\alpha}} = f_{(p_{\alpha\dot{\alpha}})}{}^{bc}\textstyle\sum_{i<j} J_{ib}\,J_{jc}}$$

where

$$[J_a, J_b\} = f_{ab}{}^{c}J_c\,, \qquad J_a = \sum_{i=1}^{n} J_{ia}$$

- ✔ superconformal symmetry $psu(2,2|4)$ [Witten 2003]

  $psu(2,2|4)$ algebra:

$$[J_a, J_b\} = f_{ab}{}^{c} J_c\,, \qquad J_a = \sum_{i=1}^{n} J_{ia}$$

- ✔ dual superconformal symmetry

  [Drummond, J.M.H., Korchemsky, Sokatchev 2008]

- ✔ closure of algebra gives Yangian $Y(psu(2,2|4))$

  [Drummond, J.M.H., Plefka 2009]

  - ✗ level-one Yangian generators

$$Q_a = f_a{}^{cb} \textstyle\sum_{1\leq i<j\leq n} J_{ib} J_{jc}\,, \qquad [Q_a, J_b\} = f_{ab}{}^{c} Q_c$$

  - ✗ interesting observation: form of Q_a consistent with cyclicity for certain superalgebras only

- ✔ Yangian symmetry of scattering amplitudes also expected in string theory [Beisert 2009]

Summary

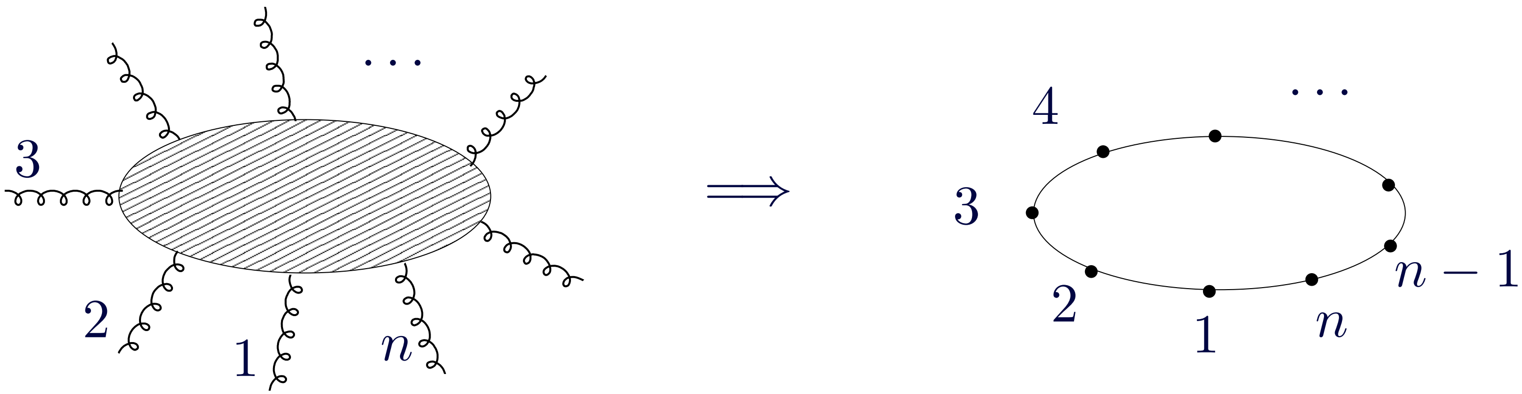
- ✔ solved problem of computing tree-level amplitudes in $\mathcal{N}=4$ SYM

applications:

- ✗ easy to extract gluon amplitudes
- ✗ useful for loop computations via generalised unitarity
- ✗ same technique of solving the recursion relations applicable to $\mathcal{N}=8$ supergravity

- ✔ new symmetries of tree-level amplitudes

- ✗ conventional and dual superconformal symmetry combine to Yangian
- ✗ sign of integrability?



Outlook and open questions

- ✔ divergences appear at loop level
- ✔ do loop-level amplitudes also have a Yangian symmetry?
  - ✗ breaking of dual conformal symmetry controlled by Ward identity

    what about conventional superconformal symmetry?
  - ✗ connection to Wilson loops? Wilson loops in dual chiral superspace $(x_i^\mu, \theta_i^{\alpha A})$?
  - ✗ insights from AdS/CFT?