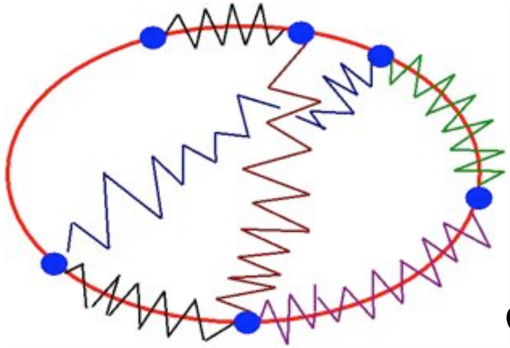
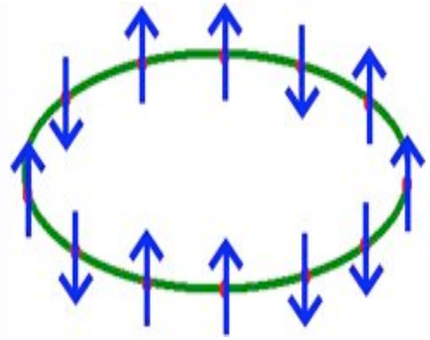
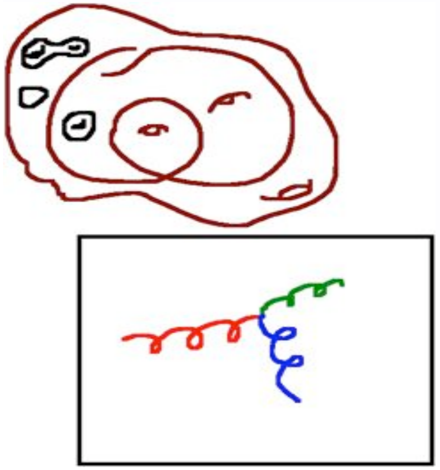


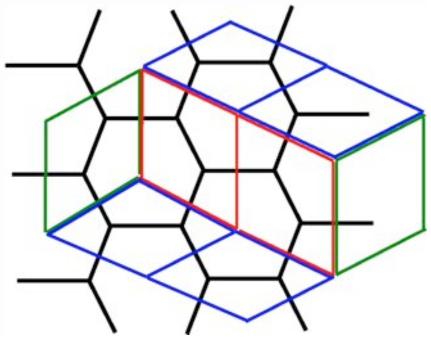


Supersymmetric Gauge Theories and Quantization of Integrable Systems



Nikita Nekrasov

Strings' 2009



Based on

NN, S.Shatashvili,

arXiv:0901.4744, arXiv:0901.4748 + to appear ;

Earlier work

G.Moore, NN, S.Shatashvili.,

arXiv:hep-th/9712241 ;

A.Gerasimov, S.Shatashvili.

arXiv:0711.1472, arXiv:hep-th/0609024

In the past few years
a connection
between two seemingly
different subjects
was found:

The supersymmetric gauge (and perhaps more exotic) theories

With as little as 4 supersymmetries

($\mathcal{N}=1$ $d=4$)

on the one hand

and

Quantum integrable systems

on the other hand

and

Quantum integrable systems

soluble by Bethe Ansatz

on the other hand

The more precise slogan is:

The **supersymmetric vacua** of

gauge theory

(*possibly, in finite volume*)

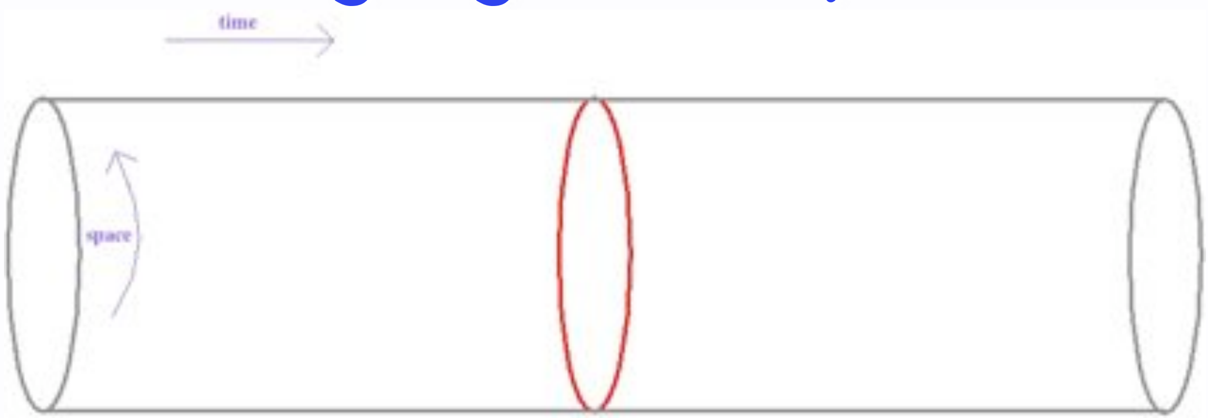
are **the eigenstates** of a

quantum integrable system

The more precise slogan is:

The **supersymmetric vacua** of

gauge theory



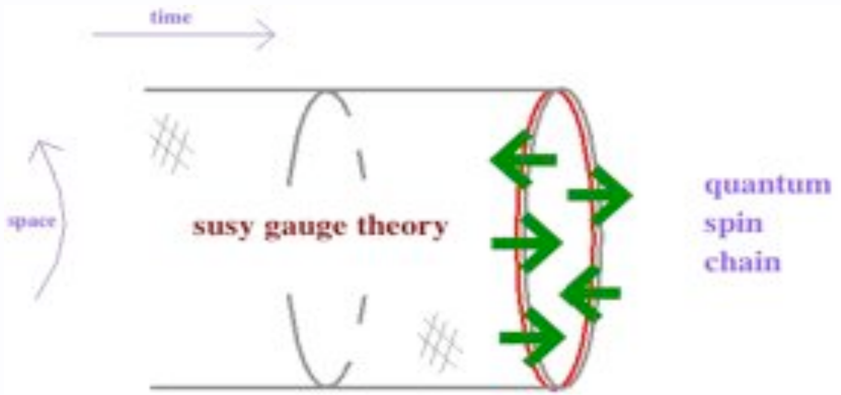
are **the eigenstates** of a

quantum integrable system

The correspondence

The **supersymmetric vacua** of

gauge theory



are **the eigenstates** of a

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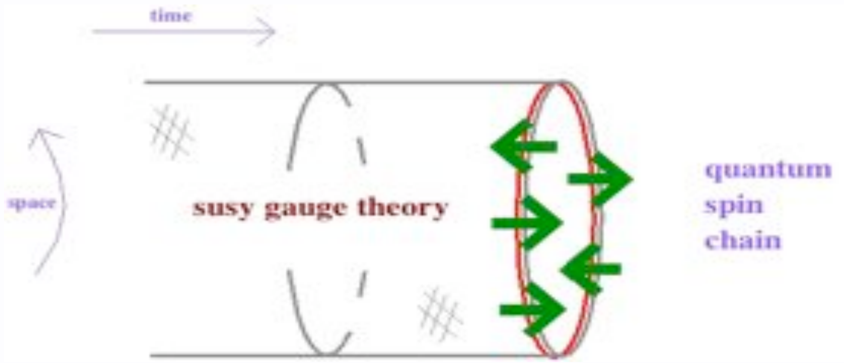
The correspondence

The **supersymmetric vacua** of

gauge theory



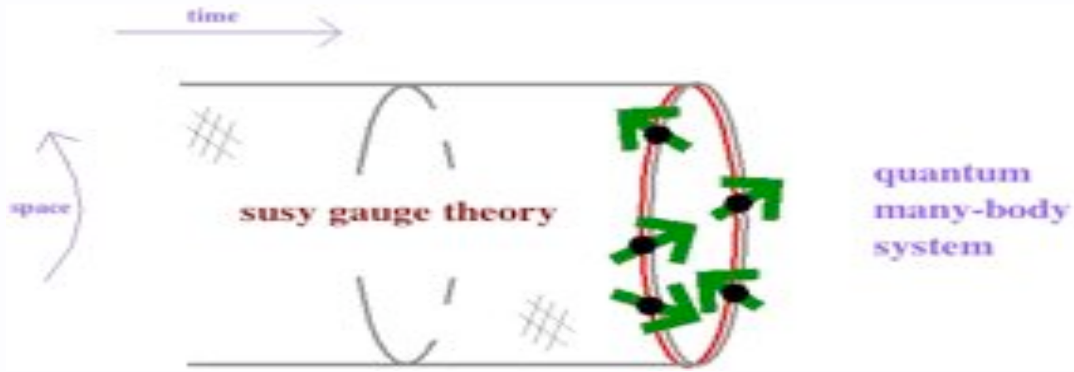
are **the eigenstates** of a

quantum integrable system

The correspondence

The **supersymmetric vacua** of

gauge theory



are **the eigenstates** of a

quantum integrable system

The correspondence

The « **twisted chiral ring** » operators

$\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \ldots, \mathcal{O}_n$

map to **quantum Hamiltonians**

$H_1, H_2, H_3, \ldots, H_n$

More precise correspondence

The vacuum expectation values of the **twisted chiral ring** operators

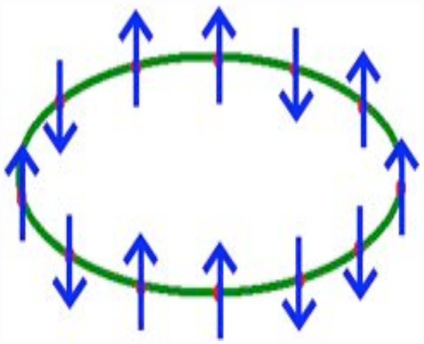
$$E_k(\lambda) = \langle \lambda | \mathcal{O}_k | \lambda \rangle$$

map to the **energies** and other **eigenvalues** on the integrable side

$$H_k \Psi_\lambda = E_k(\lambda) \Psi_\lambda$$

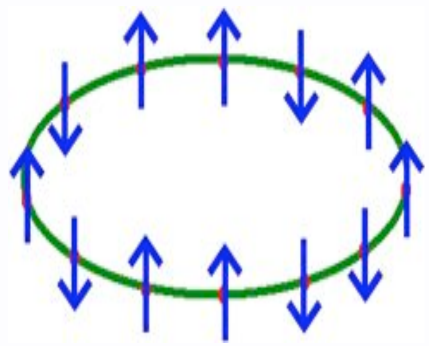
For example,

the XXX Heisenberg magnet:
the spin 1/2 $SU(2)$
isotropic spin chain of
length L



For example,

the XXX Heisenberg magnet in the sector with N spins up



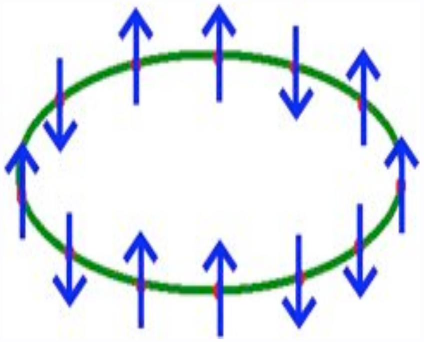
Maps to the

d=2 U(N)

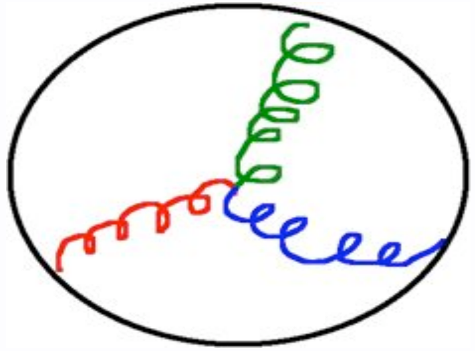
N=4 gauge theory with L

fundamental

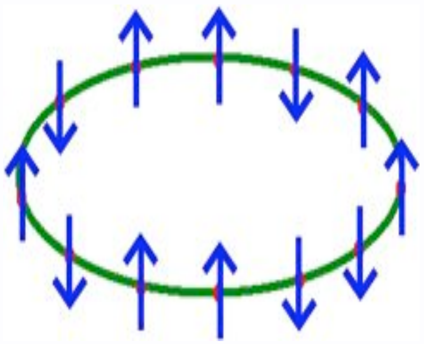
hypermultiplets



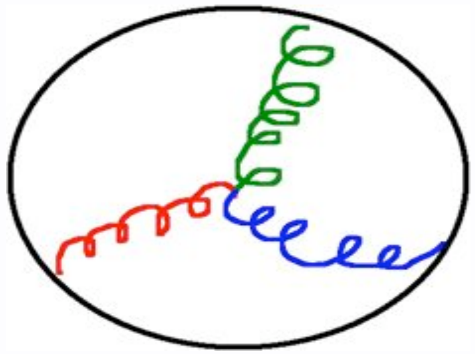
$\Longleftrightarrow$



With the $\mathcal{N}=4$ susy softly broken down to $\mathcal{N}=2$ by the generic **twisted mass** terms



$\Longleftrightarrow$



$N=2$ supersymmetry in two dimensions: Basic multiplets

Vector – gauge field,
adjoint complex scalar;

Chiral – (charged) complex scalar;

Twisted chiral – complex scalar,
gauge field strength;

Reminder on
N=2 supersymmetry in two dimensions: Lagrangians

There are two types of special Potential terms:
The F-terms
And the Twisted F-terms

Reminder on N=2 supersymmetry in two dimensions: Lagrangians

Fayet-Iliopoulos and theta terms

$$\tau = ir + \theta/2\pi.$$

Give an example of
the twisted superpotential coupling

Reminder on N=2 supersymmetry in two dimensions: Lagrangians

Twisted superpotential terms

$$= \int \mathrm{d}^2 y \left(\sqrt{2} \widetilde{W}'(\sigma)(D - i v_{01}) + 2\widetilde{W}''(\sigma)\overline{\lambda}_+ \lambda_- \right) + h.c.$$

Reminder on
N=2 supersymmetry in two dimensions: Lagrangians

Superpotential terms (F-terms)

$$L_W = -\int \mathrm{d}^2 y \left(F_i \frac{\partial W}{\partial \phi_i} + \frac{\partial^2 W}{\partial \phi_i \partial \phi_j} \psi_{-,i} \psi_{+,j} \right) - h.c.$$

Reminder on N=2 supersymmetry in two dimensions: Lagrangians

Twisted mass terms

$$\mathcal{L}_{\widetilde{\text{mass}}} = \int \mathrm{d}^4\theta \ \operatorname{tr}\left(\boldsymbol{\Phi}^\dagger e^{\widetilde{V}} \otimes \mathrm{Id}_{\text{color space}} \boldsymbol{\Phi}\right)$$

Reminder on N=2 supersymmetry in two dimensions: Lagrangians

Twisted mass terms

$$\mathcal{L}_{\widetilde{\text{mass}}} = \int \mathrm{d}^4\theta \;\; \mathrm{tr}\left(\mathbf{\Phi}^{\dagger} e^{\widetilde{V}} \otimes \mathrm{Id}_{\text{color space}} \mathbf{\Phi} \right)$$

where $\widetilde{V} = \widetilde{m}\,\theta_+\overline{\theta}_-$, $\widetilde{m}$ acts in a flavour space, and, to preserve susy:

$$[\widetilde{m}, \widetilde{m}^*] = 0$$

Reminder on N=2 supersymmetry in two dimensions: Lagrangians

Twisted mass terms

$$\mathcal{L}_{\widetilde{\text{mass}}} = \int \mathrm{d}^4\theta \ \mathrm{tr}\left(\mathbf{\Phi}^\dagger e^{\widetilde{V}} \otimes \mathrm{Id}_{\text{color space}} \mathbf{\Phi}\right)$$

$$\widetilde{V} = \widetilde{m}\,\theta_+\bar{\theta}_-$$

- Background vector field for global symmetry

Reminder on N=2 supersymmetry in two dimensions: Lagrangians

Cf. the ordinary mass terms

$$\mathcal{L}_{mass} = \sum_{i,\tilde{j}} \int \mathrm{d}^2\theta \; m_i^{\tilde{j}} \tilde{Q}_{\tilde{j}} Q^i + h.c. \, ,$$

Which are just the superpotential terms

Unbroken global symmetries

Allow the deformations by the

Twisted masses

Unbroken global symmetries

allow the deformations by the

Twisted masses

Alvarez-Gaume, Freedman
Gates, Hull, Rocek
Hanany-Hori

Unbroken global symmetries, not the R-symmetry

allow the susy-preserving deformations by the

Twisted masses

General strategy

Take an $\mathcal{N}=2$ $d=2$ gauge theory with matter,

in some representations $\mathbf{R_f}$

of the gauge group $\mathbf{G}$

integrate out the massive matter fields

Study the effective theory

General strategy

compute
the effective
twisted super-potential
on the Coulomb branch

$$W_{\text{eff}} = \sum_f \text{Tr}_{R_f} \, (\sigma + m_f)\left(\log\left(\sigma + m_f\right) - 1\right) + \; 2\pi i \tau \, \text{Tr}\sigma$$

General strategy

compute

the *effective*

twisted super-potential

on the Coulomb branch

$$W_{\rm eff} = \sum_f {\rm Tr}_{R_f}\,(\sigma + m_f)\left(\log\left(\sigma + m_f\right) - 1\right) + \;2\pi i\tau\,{\rm Tr}\sigma$$

Plus the non-perturbative corrections

General strategy

compute

the effective

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on the Coulomb branch

$$W_{\text{eff}} = \sum_f \text{Tr}_{R_f}\,(\sigma + m_f)\,(\log(\sigma + m_f) - 1) + \;2\pi i\tau\,\text{Tr}\sigma$$

Plus the non-perturbative corrections

If the theory is secretly **d>2**

Vacua of the gauge theory

For **G = U(N)**

$$\sigma \to \mathrm{diag}\left(\sigma_1, \ldots, \sigma_N\right)$$

$$\frac{\partial W_{\mathrm{eff}}}{\partial \sigma_i} = \lambda_i - i + \tfrac{1}{2}\left(N+1\right)$$

Due to quantization of the gauge flux

$$\lambda_i \in \mathbf{Z}$$

Familiar example:
CP^N model

Field content:

(**N**+1) chiral multiplet of charge +1

Q^i i=1, … , **N**+1

U(1) gauge group

σ is a scalar

Familiar example:
CP^N model

Effective twisted superpotential

(d'Adda, A.Luscher, di Vecchia)

$$(\mathbf{N}+1)\sigma(\log\sigma - 1) + 2\pi i\tau\,\sigma$$

N+1 vacuum

$$\sigma^{\mathbf{N}+1} = e^{2\pi i\tau}$$

Quantum cohomology

More interesting example

Field content

Gauge group: **G=U(N)** $\sigma \to \mathrm{diag}\,(\sigma_1, \ldots, \sigma_N)$

Matter chiral multiplets:

1	Adjoint	twisted mass	M
N_f	fundamentals	... mass	m_f
$N_{\bar{f}}$	anti-fundamentals	... mass	$m_{\bar{f}}$

More interesting example

Effective superpotential:

$$N_f\left(\sigma_i + m_f\right)\left(\log\left(\sigma_i + m_f\right) - 1\right) +$$

$$N_{\bar{f}}(-\sigma_i + m_{\bar{f}})\left(\log(-\sigma_i + m_{\bar{f}}) - 1\right) +$$

$$\sum_{i,j}\left(\sigma_i - \sigma_j + M\right)\left(\log\left(\sigma_i - \sigma_j + M\right) - 1\right)$$

$$+(N_{\bar{f}} - N_f)\log\Lambda\sum_i \sigma_i$$

More interesting example

Equations for vacua:

$$\Lambda^{N_{\bar{f}}-N_f}\frac{\left(\sigma_i+m_f\right)^{N_f}}{\left(-\sigma_i+m_{\bar{f}}\right)^{N_{\bar{f}}}}=\prod_{j\neq i}\frac{\sigma_i-\sigma_j+M}{\sigma_i-\sigma_j+M}$$

More interesting example

Non-anomalous, UV finite case.

$$N_f = N_{\overline{f}} = L$$

More interesting example

Non-anomalous, UV finite case.

$$N_f = N_{\overline{f}} = L$$

Redefine:

$$\sigma_j = \tfrac{1}{2}\left(m_{\overline{f}} - m_f\right) - iM\lambda_j$$

$$\frac{1}{2}\left(m_{\overline{f}} + m_f\right) = Ms$$

Vacua of gauge theory

$$\left(\frac{\lambda_i + is}{\lambda_i - is}\right)^L = \prod_{j\neq i} \frac{\lambda_i - \lambda_j + i}{\lambda_i - \lambda_j - i}$$

Vacua of gauge theory

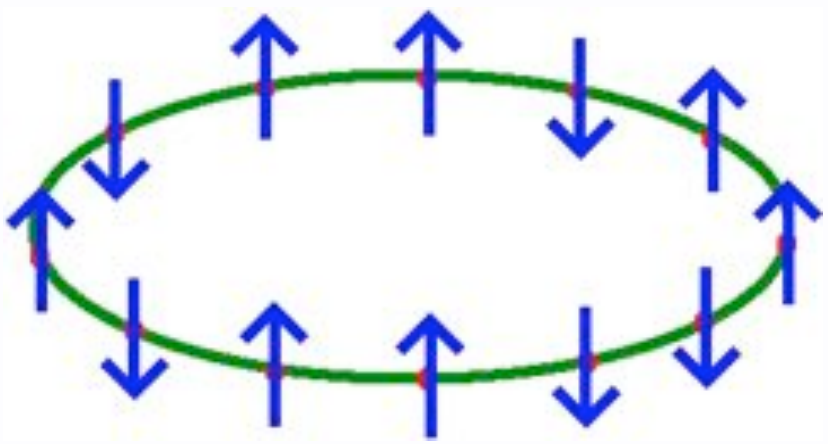
$$\left(\frac{\lambda_i + is}{\lambda_i - is}\right)^L = \prod_{j\neq i} \frac{\lambda_i - \lambda_j + i}{\lambda_i - \lambda_j - i} \times e^t$$

$$t = r + i\vartheta$$

Gauge theory - spin chain

Identical to the Bethe equations for spin s XXX magnet

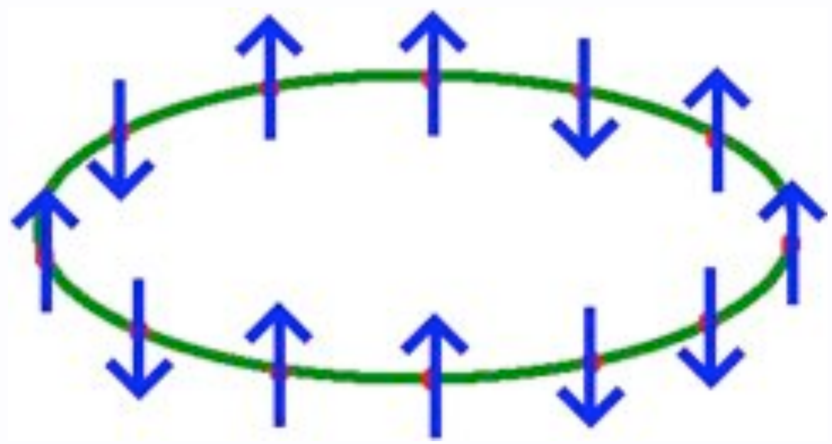
$$\left(\frac{\lambda_i + is}{\lambda_i - is}\right)^L = \prod_{j \neq i} \frac{\lambda_i - \lambda_j + i}{\lambda_i - \lambda_j - i}$$



Gauge theory – spin chain

Identical to the Bethe equations for spin s XXX magnet With twisted boundary conditions

$$\left(\frac{\lambda_i + is}{\lambda_i - is}\right)^L = \prod_{j \neq i} \frac{\lambda_i - \lambda_j + i}{\lambda_i - \lambda_j - i} \times e^t$$



Gauge theory - spin chain

Gauge theory vacua - eigenstates of the spin Hamiltonian (transfer-matrix)

$$H = \sum_{n=1}^{L} \sigma_n^a \otimes \sigma_{n+1}^a$$

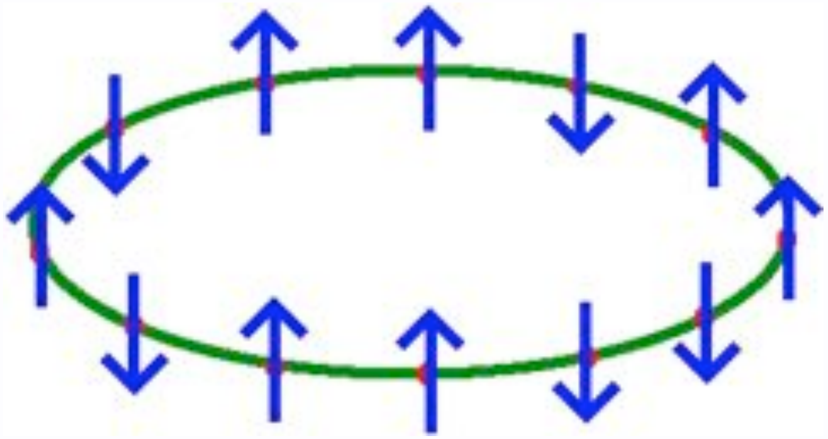


Table of dualities

XXX spin chain
SU(2)
L spins
N excitations

$$\left(\frac{\lambda_i + is}{\lambda_i - is}\right)^L = \prod_{j\neq i} \frac{\lambda_i - \lambda_j + i}{\lambda_i - \lambda_j - i}$$

U(N) d=2 N=2
Chiral
multiplets:
1 adjoint
L fundamentals
L anti-fund.

Higher dimensional theories

Consider a $d=2+k$ gauge theory with $\mathcal{N}=2$ $d=2$ supersymmetry

For example, $\mathcal{N}=1,2$ $d=4$ would do

Higher dimensional theories

Viewed as the $N=2$ $d=2$ theory it has a global symmetry group:

Poincare(k)

Higher dimensional theories

One has several possibilities for
the twisted masses,
corresponding to the

abelian subalgebras in Poincare(k)

Higher dimensional theories

One has several possibilities for the twisted masses,

abelian subalgebras in Poincare(k)

Translations in $\mathbf{R}^l$

Rotations in R^{k-l}

Higher dimensional theories

The resulting theories are:

Translations in $\mathbf{R}^l$: **compactification on** $\mathbf{T}^l$

Rotations in R^{k-l} **:**

the theory in the Ω-background

Table of dualities

XXZ spin chain

SU(2)

L spins

N excitations

$$\left(\frac{\sinh(\lambda_i + is\gamma)}{\sinh(\lambda_i - is\gamma)}\right)^L = \prod_{j\neq i}\frac{\sinh(\lambda_i - \lambda_j + i\gamma)}{\sinh(\lambda_i - \lambda_j - i\gamma)}$$

U(N) d=3 N=2

Compactified on a circle

Chiral multiplets:

1 adjoint

L fundamentals

L anti-fund.

Table of dualities

XYZ spin chain

SU(2), L = 2N spins

N excitations

$$H = \sum_{n=1}^{L} J_x \sigma_n^x \otimes \sigma_{n+1}^x + J_y \sigma_n^y \otimes \sigma_{n+1}^y + J_z \sigma_n^z \otimes \sigma_{n+1}^z$$

U(N) d=4 $\mathcal{N}$=1

Compactified on a 2-torus = elliptic curve E

Chiral multiplets:

1 adjoint

L = 2N fundamentals

L = 2N anti-fund.

Masses = wilson loops of the flavour group

Table of dualities

XYZ spin chain

SU(2), L = 2N spins

N excitations

$$H = \sum_{n=1}^{L} J_x \sigma_n^x \otimes \sigma_{n+1}^x + J_y \sigma_n^y \otimes \sigma_{n+1}^y + J_z \sigma_n^z \otimes \sigma_{n+1}^z$$

U(N) d=4 N=2

Compactified on a

2-torus = elliptic curve E

L = 2N fundamental
hypermultiplets

Softly broken down to N=1 by the wilson loops of the global symmetry group = flavour group U(L) X U(1) = points on the Jacobian of E

Table of dualities

It is remarkable that the spin chain has
precisely those generalizations:
rational (XXX), trigonometric (XXZ) and elliptic (XYZ)
that can be matched to the 2, 3, and 4 dim cases.

$$H = \sum_{n=1}^{L} J_x \sigma_n^x \otimes \sigma_{n+1}^x + J_y \sigma_n^y \otimes \sigma_{n+1}^y + J_z \sigma_n^z \otimes \sigma_{n+1}^z$$

$$J_x = 1 + k\,\mathrm{sn}^2\, 2\eta, \quad J_y = 1 - k\,\mathrm{sn}^2\, 2\eta, \quad J_z = \mathrm{cn}\, 2\eta\, \mathrm{dn}\, 2\eta$$

Table of dualities

Yang-Yang counting function =
effective twisted superpotential

Table of dualities

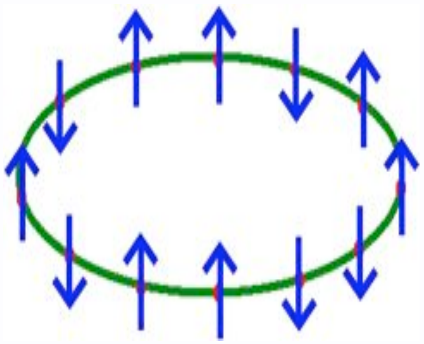
Commuting hamiltonians (expansion of transfer matrix) =

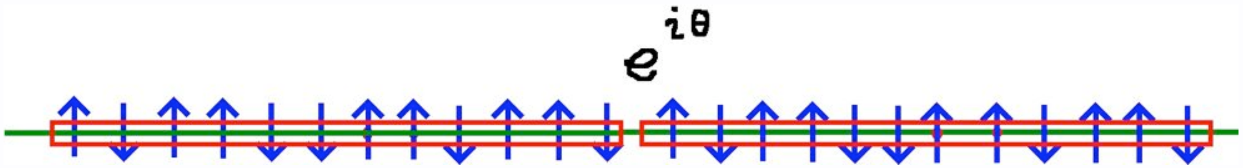
the twisted chiral ring generators, e.g.

$$\mathcal{O}_M = \mathrm{Tr}\, \sigma^M$$

Table of dualities

Gauge theory theta angle (complexified)
is mapped to the spin chain theta angle
(twisted boundary conditions)





Algebraic Bethe Ansatz

Faddeev et al.

The spin chains are solved algebraically using certain operators,

$$A(\lambda), B(\lambda), C(\lambda), D(\lambda)$$

obeying exchange commutation relations

Algebraic Bethe Ansatz

Faddeev, Takhtajan,
Reshetikhin,
Jimbo-Miwa,
Drinfeld,
Sklyanin
Lusztig

$$A(\lambda), B(\lambda), C(\lambda), D(\lambda)$$

Yangian, quantum affine $U_q(sl_2)$,
Elliptic quantum group, ...

Algebraic Bethe Ansatz

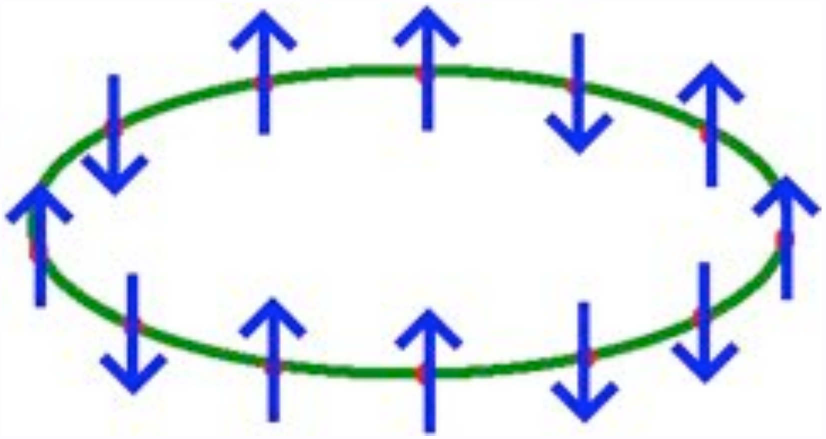
The eigenvectors, Bethe vectors, are obtained by applying these operators to the (pseudo)vacuum.

$$\Psi_{\vec{\lambda}} = B(\lambda_1)B(\lambda_2)\ldots B(\lambda_N)\Omega$$

Algebraic Bethe Ansatz vs GAUGE THEORY

For the spin chain it is natural to fix L = total number of spins

and consider various N = excitation levels



In the gauge theory context N is fixed.

Algebraic Bethe Ansatz
vs STRING THEORY

However, if the theory is embedded into string theory via brane realization

then changing N is easy: bring in an extra brane.

One might use the constructions of Witten'96, Hanany-Hori'02

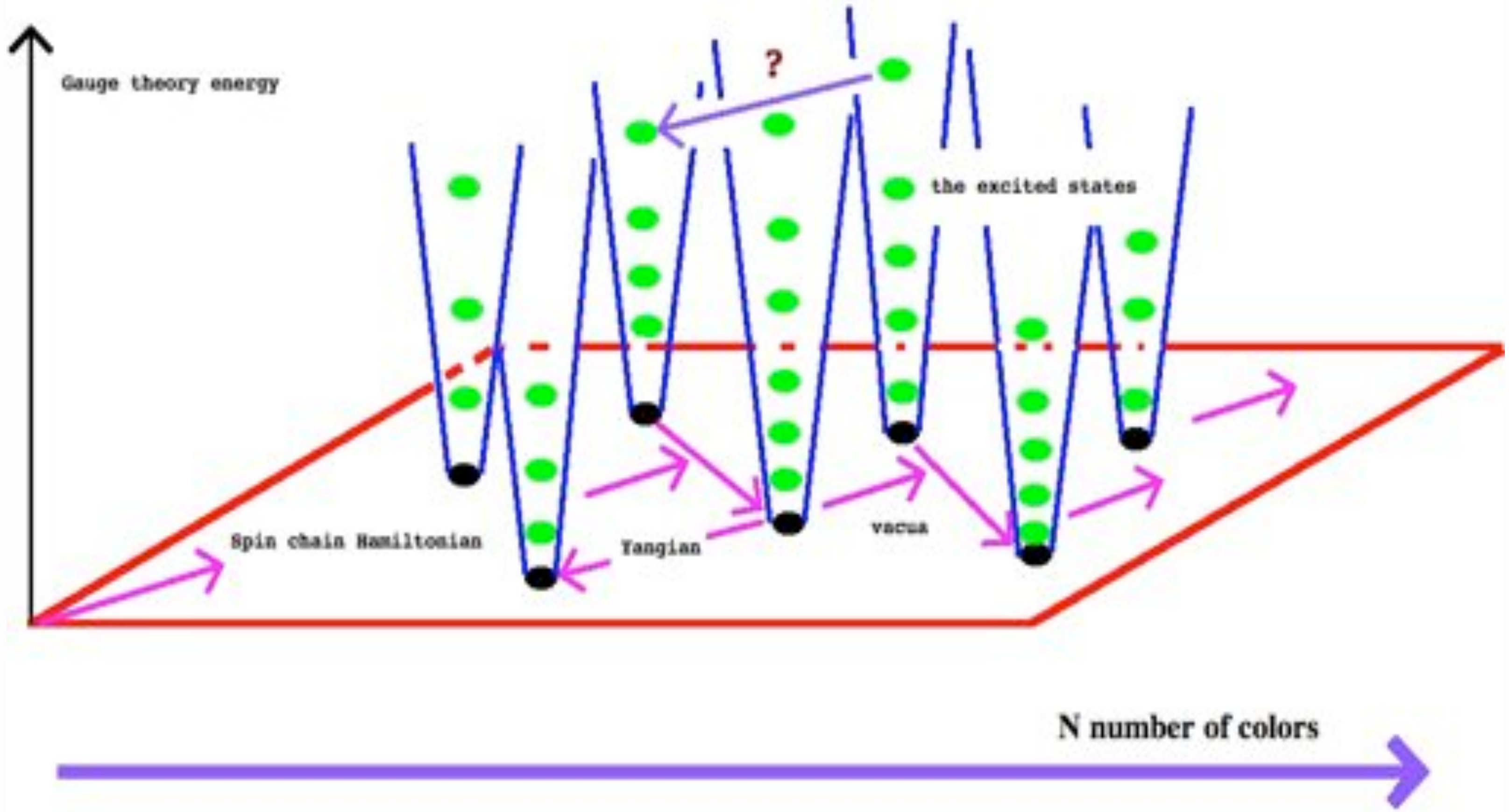
Algebraic Bethe Ansatz
vs STRING THEORY

THUS:

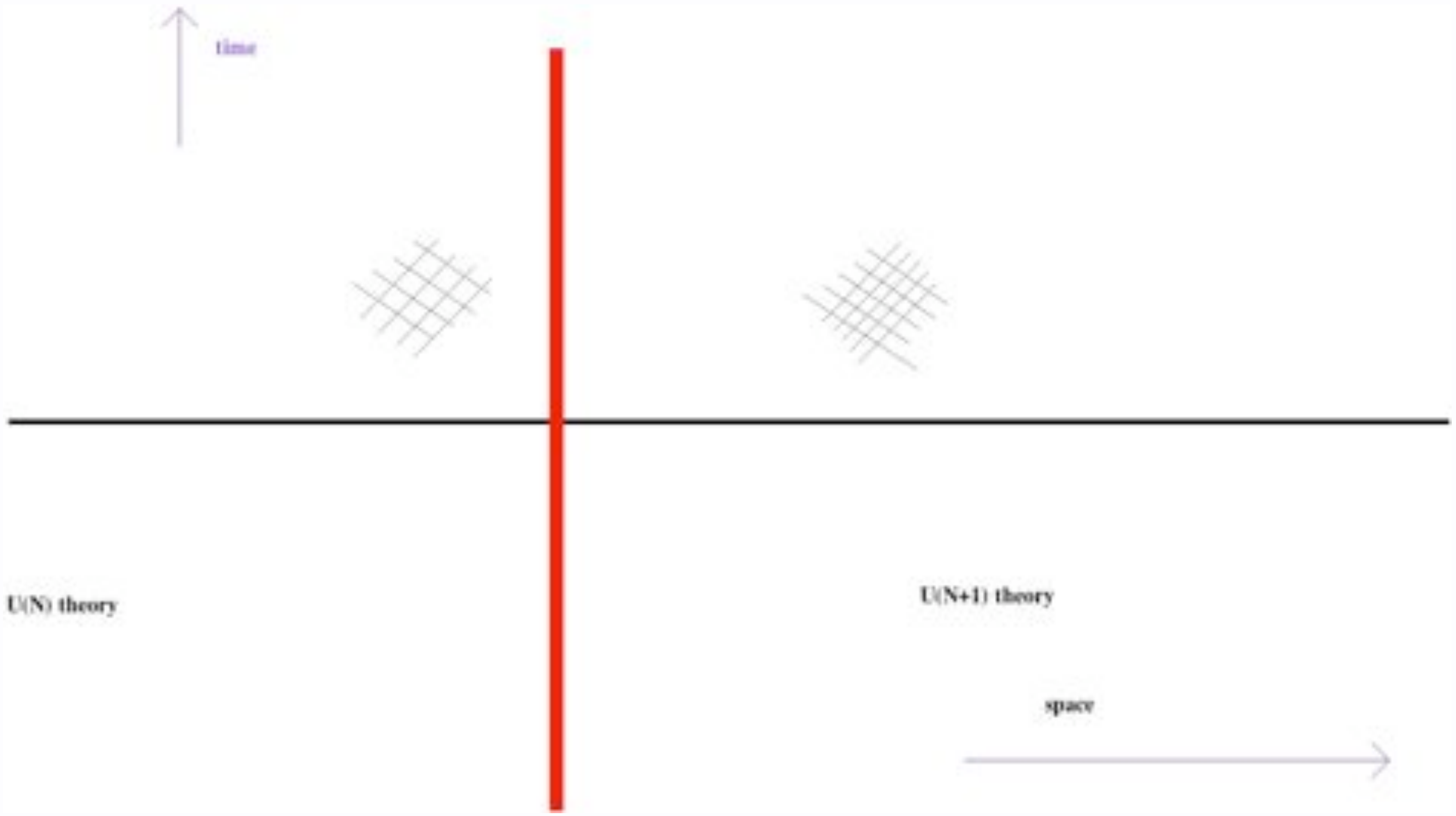
$B(\lambda)$ is for BRANE!

λ is for location!

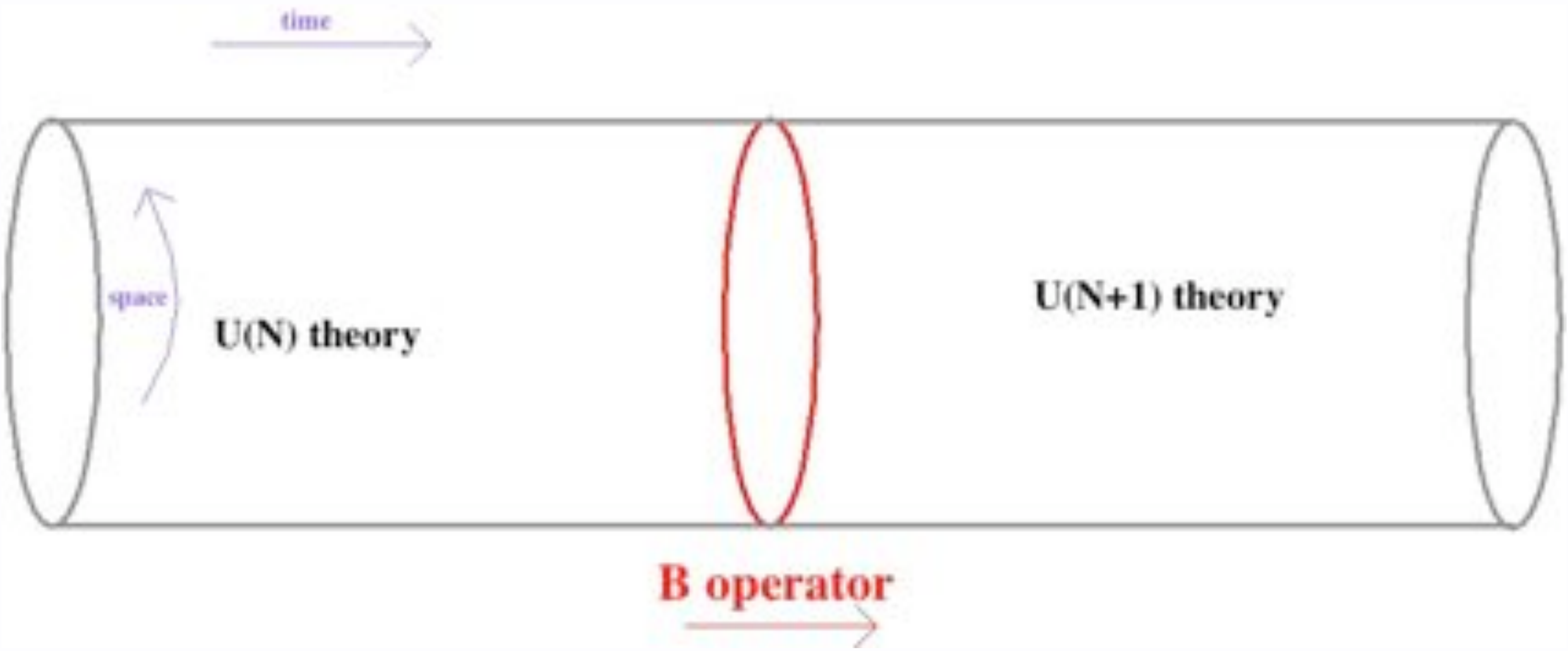
Gauge theoretically this is a miracle



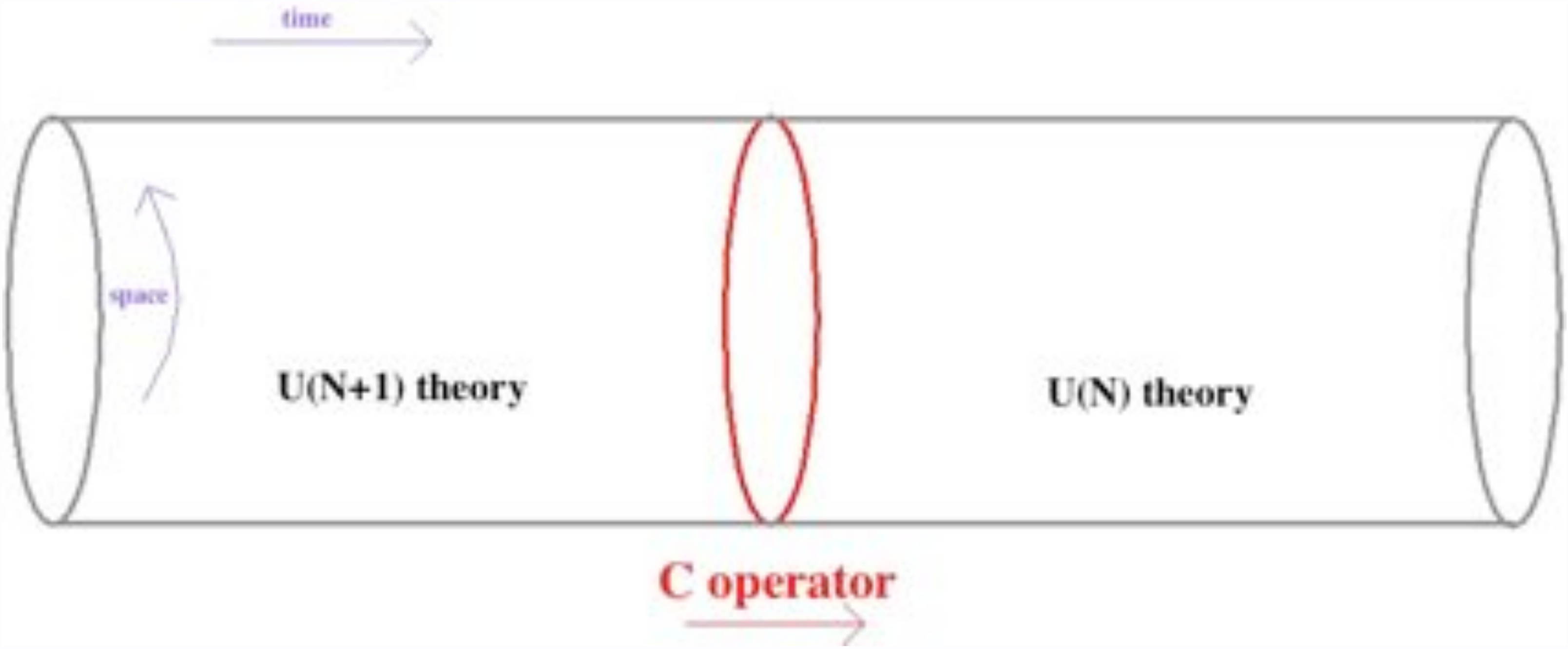
Non-local operators from domain walls



Non-local operators from domain walls



Non-local operators from domain walls

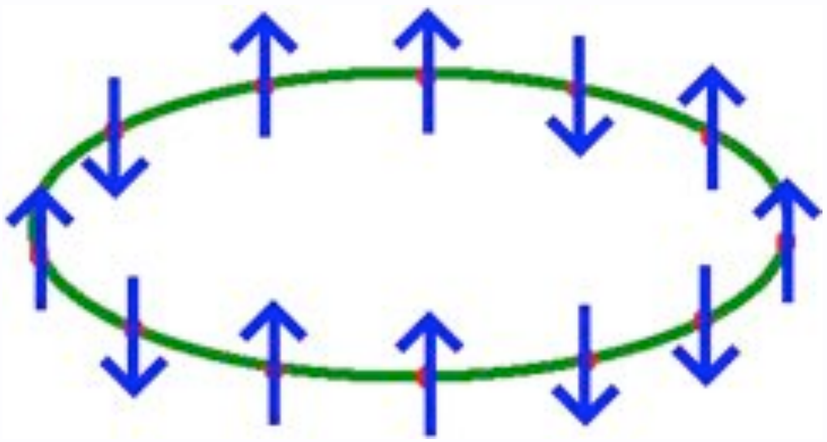


Are these models too special, or the gauge theory/integegable lattice model correspondence is more general?

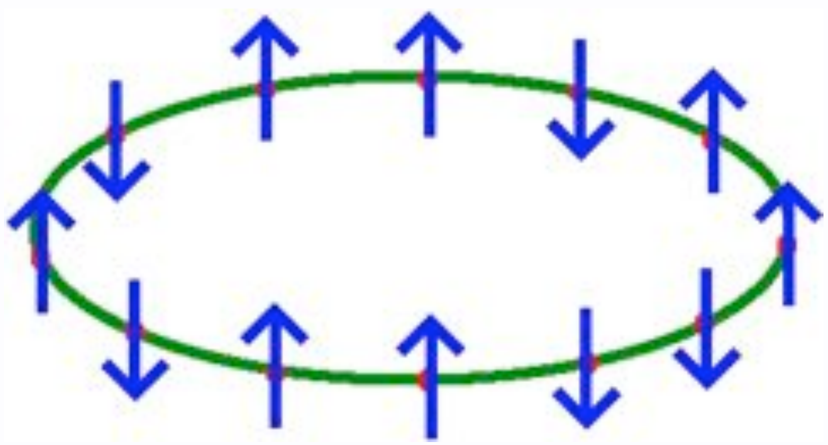
Extends to every spin group/supergroup, spin representations, inhomogeneity, anisotropy....

So far we were dealing with «very» quantum systems:

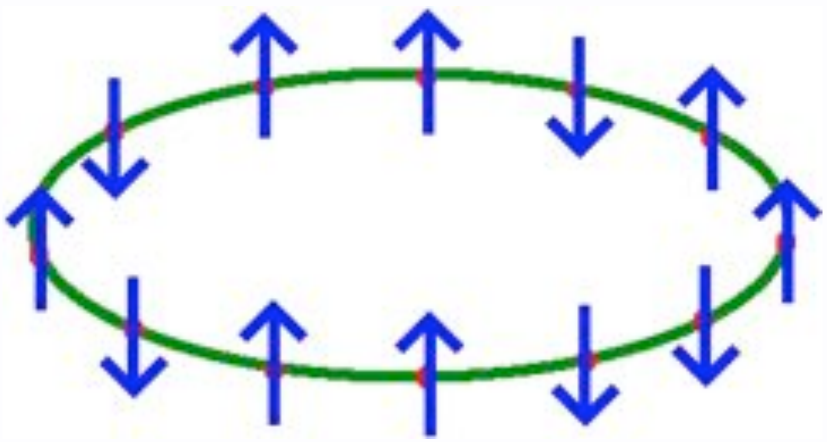
the Planck constant is quantized



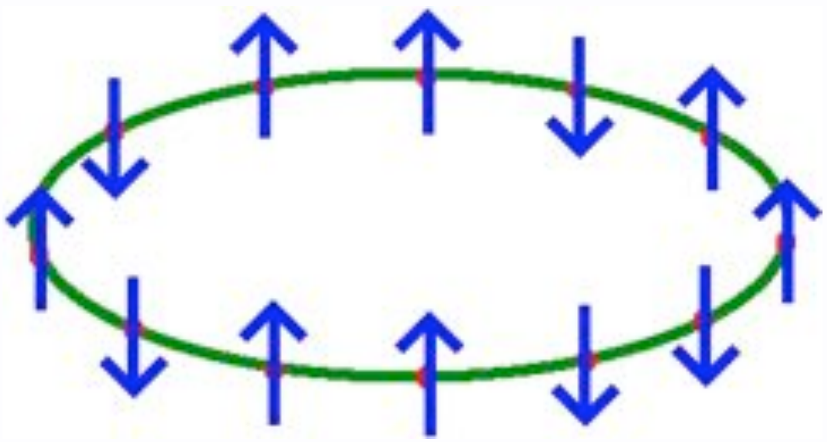
Spins were half-integers



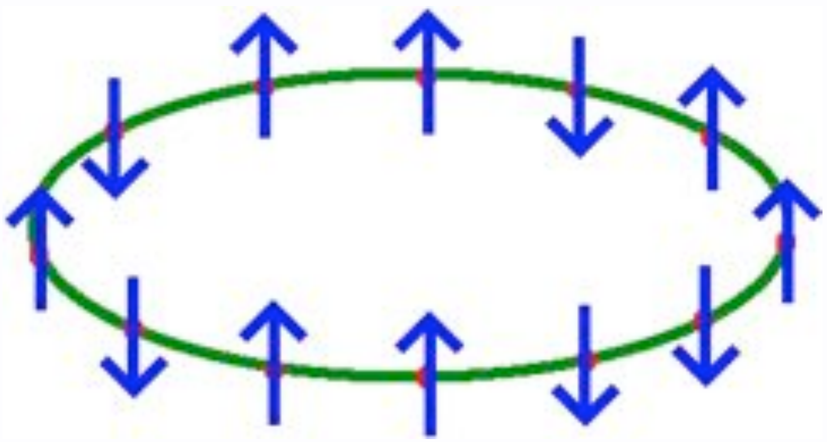
The classical limit unnatural ?



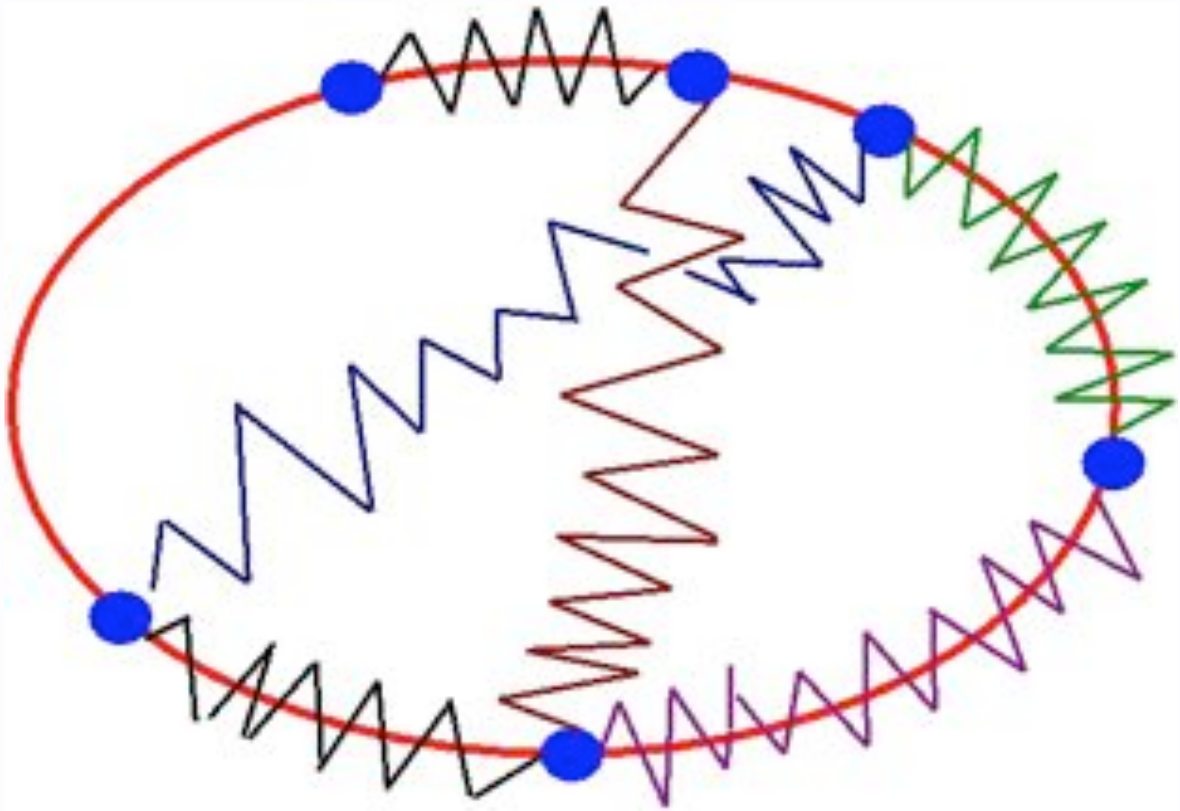
More formally,
we study the finite-dimensional representations of spin algebras (compact spin groups)



Can we cross over to the infinite-dimensional representation theory? (non-compact spin groups?)



Can we study the Many-body systems (continuous degrees of freedom?)



Can we study the Many-body systems (continuous degrees of freedom?)

WHEN IN ROME.....

Many-body systems
The Roman style

Calogero system

$$H_{\mathrm{eCM}} = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \mathrm{g}^2 \sum_{i<j} U(x_i - x_j; \mathrm{q})$$

$$p_k = -i\hbar \frac{\partial}{\partial x_k}$$

Many-body systems
The Roman style

Calogero-Moser-Sutherland system

$$H_{\mathrm{eCM}} = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \mathrm{g}^2 \sum_{i<j} U(x_i - x_j; \mathrm{q})$$

$$p_k = -i\hbar \frac{\partial}{\partial x_k}$$

Many-body systems

Elliptic Calogero-Moser system

$$H_{\mathrm{eCM}} = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \mathrm{g}^2 \sum_{i<j} U(x_i - x_j; \mathrm{q})$$

$$U(x; \mathrm{q}) = U(-x; \mathrm{q}) = \sum_{n\in\mathbf{Z}} \frac{1}{\sinh^2 (x + 2\pi n\beta)}$$

Francesco Calogero, ‘69-70

Many-body systems

Calogero-Moser system

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Francesco Calogero, circa ‘69-70

(University of Rome)

Many-body systems

Calogero-Moser system

$$H_{\mathrm{eCM}} = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \mathrm{g}^2 \sum_{i<j} U(x_i - x_j; \mathrm{q})$$

Francesco Calogero, circa ‘69-70

(on sabbatical at ITEP, Moscow at that time)

Many-body systems

The elliptic Calogero-Moser system describes a system of identical particles on a circle of radius

β

subject to the two-body interaction potential, given by the elliptic (double-periodic) function

$$U(x;\mathrm{q}) = U(-x;\mathrm{q}) = \sum_{n\in\mathbf{Z}} \frac{1}{\sinh^2(x+2\pi n\beta)}$$

Many-body systems

One is interested in the β-periodic symmetric, L^2-normalizable wavefunctions

$$\Psi(x_1, \ldots, x_N)$$

Many-body systems

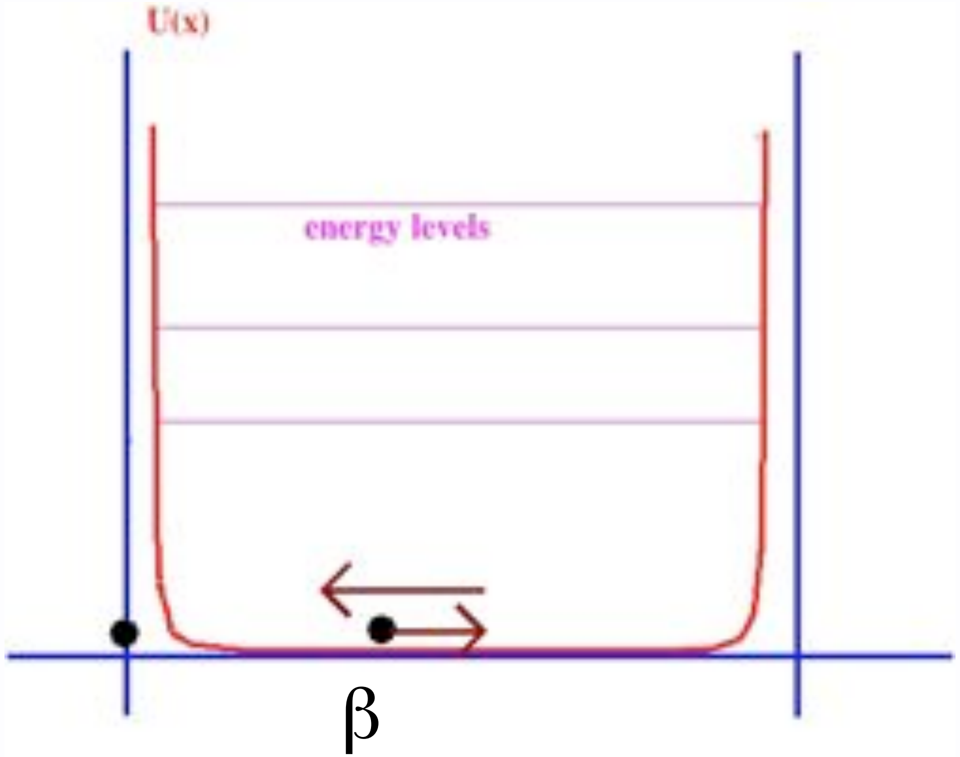
β-periodic, symmetric, L^2-normalizable wavefunctions

$$\Psi(x_1, \ldots, x_N)$$

It is clear that one should get an infinite discrete spectrum

Many-body systems

It is clear that one should
get an
infinite discrete spectrum



Many-body systems

Our main claim:
the
infinite discrete spectrum
of
the integrable many-body system
=
The vacua of the $N=2$ $d=2$ theory

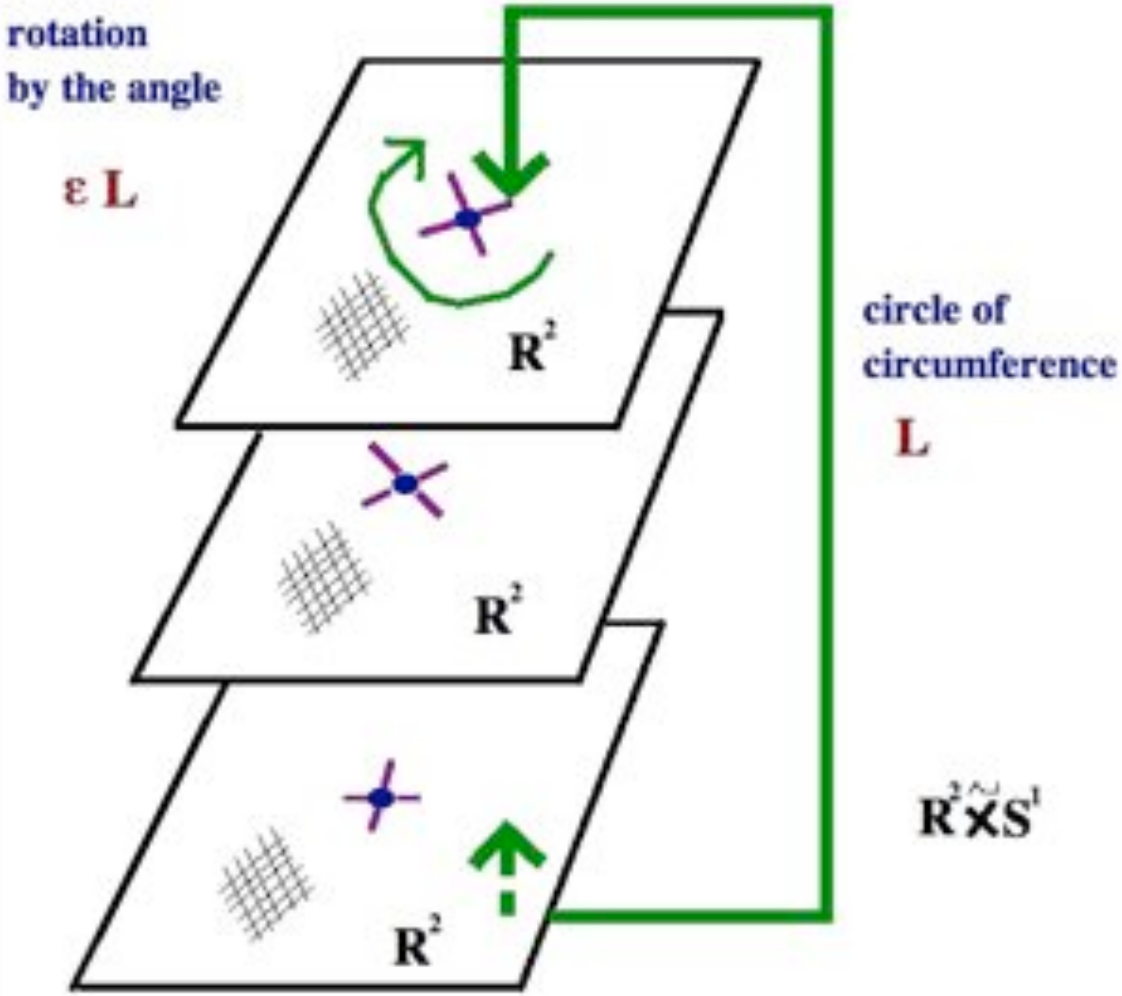
Many-body systems

The vacua of the $N=2$ $d=2$ theory,
Obtained by subjecting the $N=2$ $d=4$
Theory to an Ω-background in $\mathbf{R}^2$

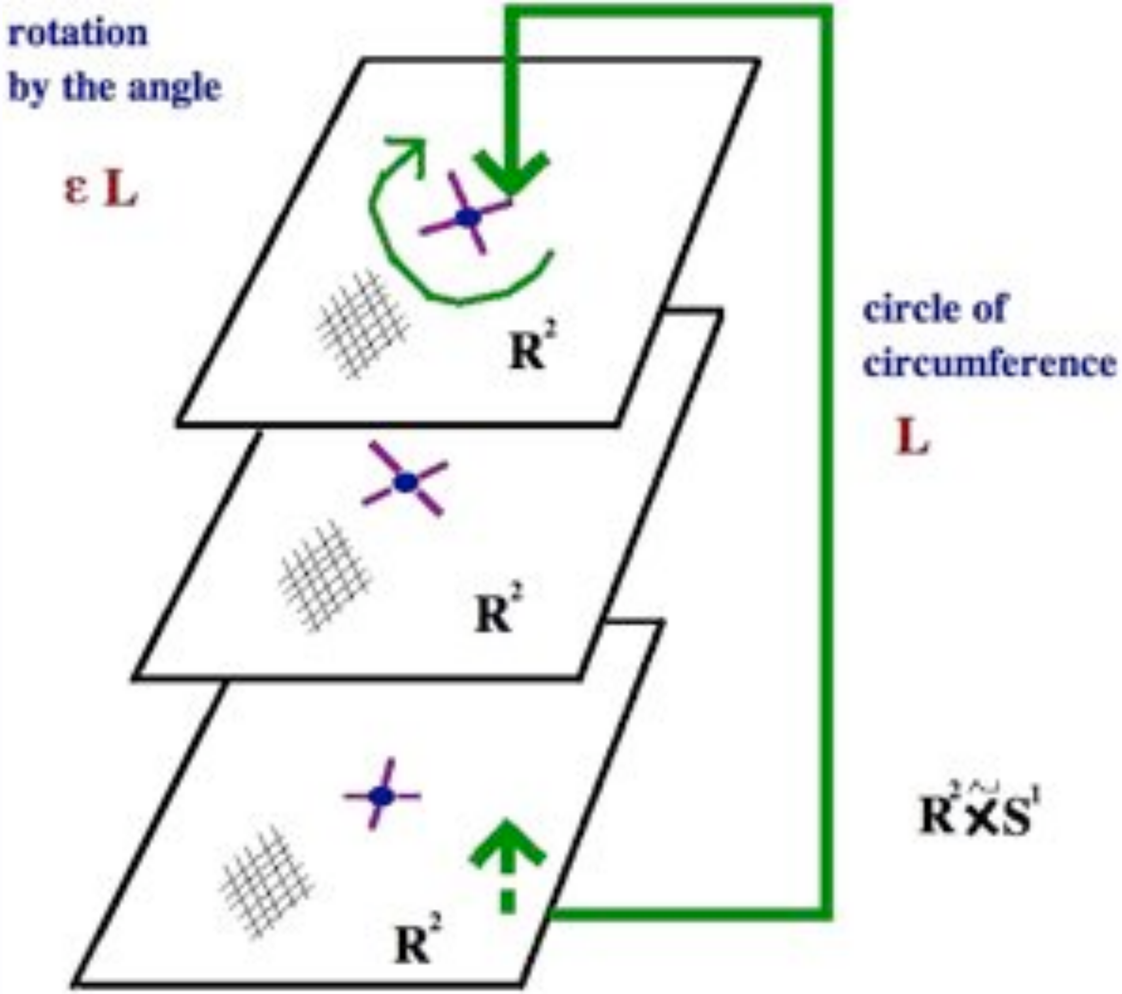
The Ω-background

The $\mathcal{N}=2$ $d=4$
theory in the Ω-background in $\mathbf{R}^2$

The Ω–background



The Ω–background



Rotation
in space
is accompanied
by an **R**-symmetry
rotation

The Ω–background: Lagrangian of the theory

L -> 0 limit gives back the 4d theory

$$\mathrm{tr}\,\|F_A\|^2 + \mathrm{tr}\,(D_A\phi - \iota_V F_A)\star(D_A\overline{\phi} - \iota_{\overline{V}} F_A) +$$

$$+\,\mathrm{tr}\,\|[\phi,\overline{\phi}] + \iota_V D_A\overline{\phi} - \iota_{\overline{V}} D_A\phi + \iota_V\iota_{\overline{V}} F_A\|^2$$

For pure N=2

The Ω–background: Lagrangian of the theory

$$\mathrm{tr}\,\|F_A\|^2 + \mathrm{tr}\,(D_A\phi - \iota_V F_A)\star(D_A\overline{\phi} - \iota_{\overline{V}} F_A) +$$

$$+\,\mathrm{tr}\,\|[\phi,\overline{\phi}] + \iota_V D_A\overline{\phi} - \iota_{\overline{V}} D_A\phi + \iota_V \iota_{\overline{V}} F_A\|^2$$

$$V = \varepsilon\left[x^2\partial_{x^3} - x^3\partial_{x^2}\right]$$

$$\overline{V} = \overline{\varepsilon}\left[x^2\partial_{x^3} - x^3\partial_{x^2}\right]$$

The Ω–background: Lagrangian of the theory

$$V = \varepsilon \left[x^2 \partial_{x^3} - x^3 \partial_{x^2} \right]$$

$$\overline{V} = \bar{\varepsilon} \left[x^2 \partial_{x^3} - x^3 \partial_{x^2} \right]$$

Rotation in two dimensions

The theory in the Ω-background

Looks two dimensional
in the infrared:
localization at
the «cosmic string»

$x^2 = x^3 = 0$

The theory in the Ω-background

The two dimensional theory
Has $N=2$ $d=2$ susy
It has an
effective twisted
superpotential

The theory in the Ω–background

The effective twisted superpotential

$$\widetilde{W}^{\text{eff}}(a_1,\ldots,a_N;\varepsilon;\tau,m)$$

The theory in the Ω–background

The effective twisted superpotential

$$\widetilde{W}^{\mathrm{eff}}(a_1, \ldots, a_N; \varepsilon; \tau, m)$$

$$\sigma_i = a_i$$

The special coordinates on the moduli space of vacua of the original four dimensional *N=2* theory

The effective twisted superpotential

$$\widetilde{W}^{\text{eff}}(a_1,\ldots,a_N;\varepsilon;\tau,m)$$

Computed by the instanton Partition function

$$Z(a,\varepsilon_1,\varepsilon_2;m,\tau)$$

The effective twisted superpotential

$$Z(a,\varepsilon_1,\varepsilon_2;m,\tau) \sim e^{\frac{1}{\varepsilon_2}\widetilde{W}^{\mathrm{eff}}(a_1,\ldots,a_N;\varepsilon_1;\tau,m)+\ldots}$$

as $\varepsilon_2 \to 0$

The effective twisted superpotential leads to the vacuum equations

$$\exp \frac{\partial \widetilde{W}(a)}{\partial a_i} = \mathbf{1}$$

The effective twisted superpotential leads to the vacuum equations

$$\exp \frac{\partial \widetilde{W}(a)}{\partial a_i} = \mathbf{1}$$

Lift the 4d vacuum Degeneracy!

The effective twisted superpotential leads to the vacuum equations

$$\exp \frac{\partial \widetilde{W}(a)}{\partial a_i} = \mathbf{1}$$

The solutions are isolated: Discrete spectrum

The effective twisted superpotential leads to the vacuum equations

$$\exp \frac{\partial \widetilde{W}(a)}{\partial a_i} = \mathbf{1}$$

«Attractor SW curves»

An interesting possibility for the brane world scenarios

The effective twisted superpotential has one-loop perturbative and all-order instanton corrections

$$\widetilde{W}^{\text{eff}}(a;\tau) = \widetilde{W}^{\text{pert}}(a) + \sum_{n=1}^{\infty} q^n \widetilde{W}_{n-\text{inst}}(a)$$

For example, for the N=2* theory
(adjoint hypermultiplet with mass m)

$$\exp \frac{\partial \widetilde{W}^{\text{pert}}(a)}{\partial a_i} =$$

$$e^{\frac{\pi i \tau a_i}{\varepsilon}} \prod_{j \neq i} S(a_i - a_j)$$

$$S(x) = \frac{\Gamma\left(\frac{-m+x}{\varepsilon}\right)}{\Gamma\left(\frac{-m-x}{\varepsilon}\right)} \frac{\Gamma\left(1-\frac{x}{\varepsilon}\right)}{\Gamma\left(1+\frac{x}{\varepsilon}\right)}$$

Bethe equations
Factorized S-matrix

$$S(x) = \frac{\Gamma\left(\frac{-m+x}{\varepsilon}\right)}{\Gamma\left(\frac{-m-x}{\varepsilon}\right)} \frac{\Gamma\left(1-\frac{x}{\varepsilon}\right)}{\Gamma\left(1+\frac{x}{\varepsilon}\right)}$$

This is the two-body scattering
In hyperbolic Calogero-Sutherland

Bethe equations
Factorized S-matrix

$$S(x) = \frac{\Gamma\left(\frac{-m+x}{\varepsilon}\right)}{\Gamma\left(\frac{-m-x}{\varepsilon}\right)} \frac{\Gamma\left(1-\frac{x}{\varepsilon}\right)}{\Gamma\left(1+\frac{x}{\varepsilon}\right)}$$

$$U_0(x) = \frac{1}{\sinh^2(x)}$$

Two-body potential

Bethe equations
Factorized S-matrix

$$S(x) = \frac{\Gamma\left(\frac{-m+x}{\varepsilon}\right)}{\Gamma\left(\frac{-m-x}{\varepsilon}\right)} \frac{\Gamma\left(1-\frac{x}{\varepsilon}\right)}{\Gamma\left(1+\frac{x}{\varepsilon}\right)}$$

Harish-Chandra, Gindikin-Karpelevich,
Olshanetsky-Perelomov, Heckmann,
final result: Opdam

The full superpotential of N=2* theory leads to the vacuum equations

Momentum phase shift

$$e^{\frac{\pi i \tau a_i}{\varepsilon}} \prod_{j \neq i} S(a_i - a_j) \times \left[1 + q \sum_{k \neq i} \prod_{l \neq k} \text{rational}(a_i, a_l, a_k, m(m+\varepsilon), \varepsilon) + \ldots \right]$$

Two-body scattering

The finite size corrections

$$q = \exp(-N\beta)$$

The main slogan this year

Four dimensional gauge theories give rise to the Instanton corrected Bethe Ansatz equations

The main slogan this year

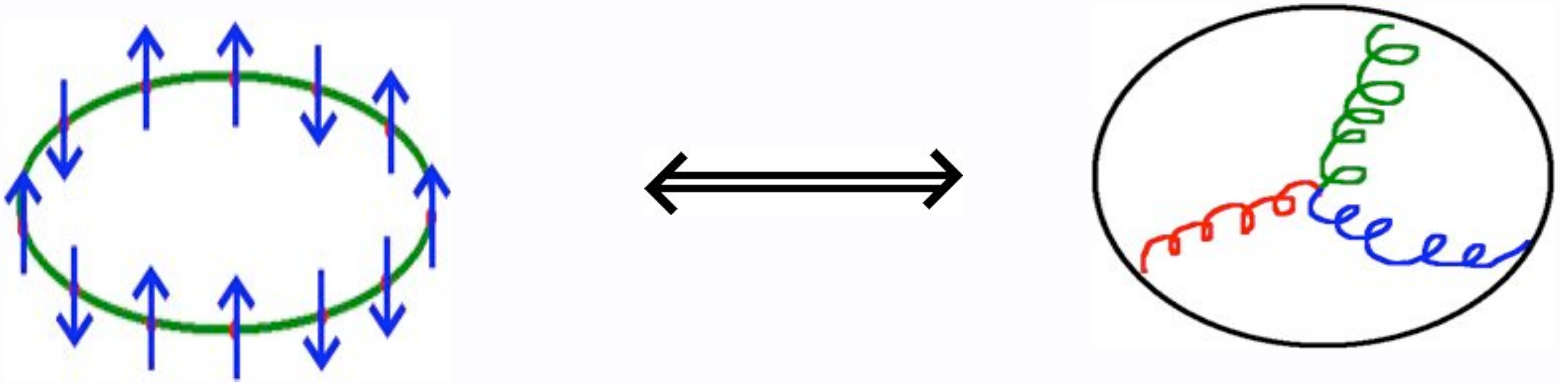
aka the

Bethe Ansatz

with

the finite size effects

Dictionary



Dictionary

Elliptic CM system $\Longleftrightarrow$ $\mathcal{N}=2^*$ theory

Dictionary

classical Elliptic CM system $\Longleftrightarrow$ 4d $\mathcal{N}=2^*$ theory

Dictionary

classical
Elliptic
CM
system

$\Longleftrightarrow$

4d $\mathcal{N}=2^*$
theory

Donagi-Witten, Martinec-Warner,
Gorsky-Nekrasov

Dictionary

quantum
Elliptic
CM
system

$\Longleftrightarrow$

4d N=2*
Theory
In the
Ω-background

NN-Shatashvili

Dictionary

The (complexified) system size β $\Longleftrightarrow$ The gauge coupling τ

Dictionary

The Planck constant $\Longleftrightarrow$ The Ω-background parameter ε

Dictionary

The correspondence
Extends to other
integrable sytems:
Toda, relativistic Systems,
Perhaps all 1+1 iQFTs

Classical Limit: back to 4d

In the limit $\varepsilon \longrightarrow 0$

$$\widetilde{W}(a;\varepsilon) \sim \frac{\mathcal{F}(a)}{\varepsilon} + \ldots$$

The prepotential F(a) of the low-energy effective theory in 4d

Classical Limit: back to 4d

The prepotential F(a) *of the low-energy effective theory is governed by a classical (holomorphic) integrable system*

Gorsky-Krichever-Marshakov-Mironov-Morozov
Donagi-Witten'95

Classical Limit: back to 4d

Liouville tori =
Jacobians (Prims) of the
Seiberg-Witten curves

Classical Limit: back to 4d

$$a^i = \frac{1}{2\pi}\oint_{A_i} \mathbf{p}d\mathbf{q}\,,\;\; a_{D,i} = \frac{1}{2\pi}\oint_{B_i} \mathbf{p}d\mathbf{q}$$

Classical Limit: back to 4d

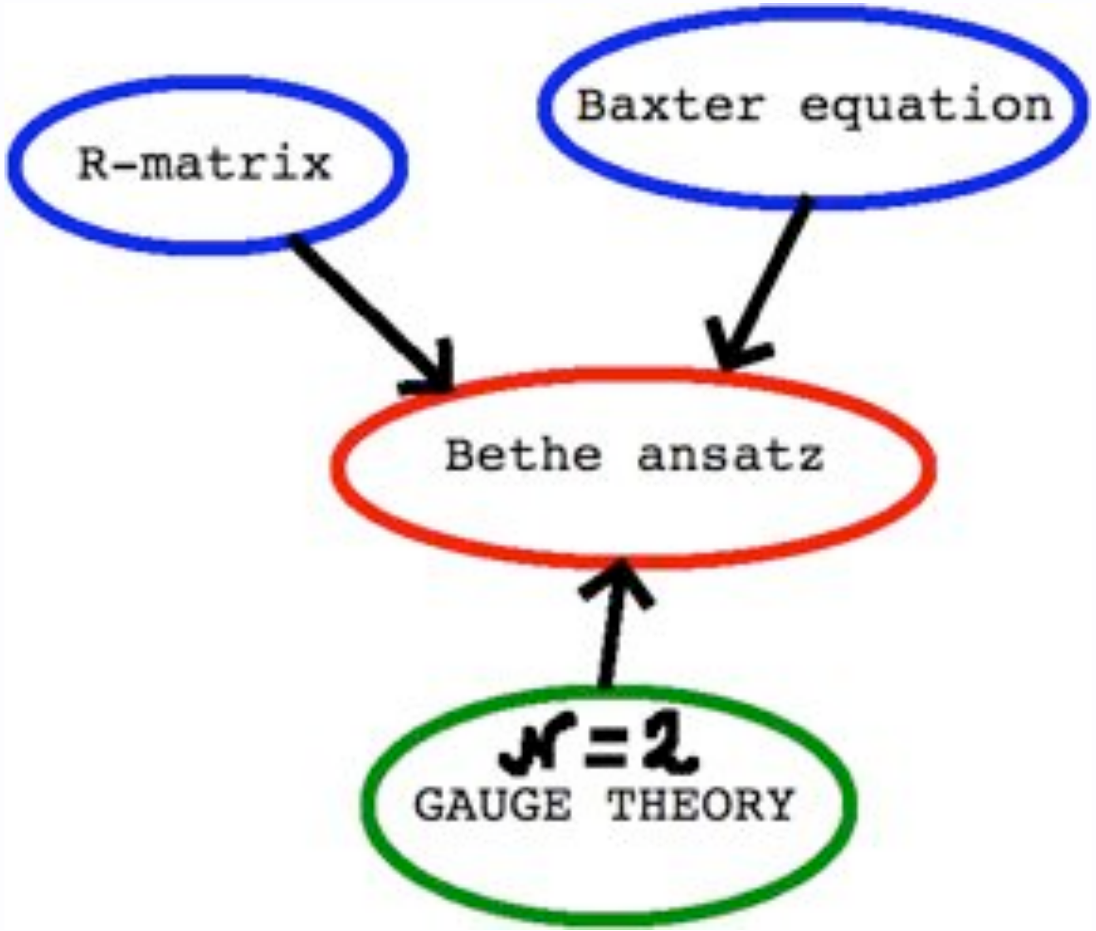
$$a_{D,i} = \frac{\partial \mathcal{F}}{\partial a^i}$$

Classical integrable system
vs
Quantum integrable system

That system is quantized when the gauge theory is subject to the Ω-background

Earlier indications:
NN'02
NN-Okounkov'03
Braverman'03
Full story:
NN-Shatashvili'09
Related developments:
Alday-Gaiotto-Tachikawa

CONCLUSIONS



CONCLUSIONS

We have found a striking connection between the vacua of the N=2 theories and the eigenstates of the quantum integrable systems, both with finite and infinite number of degrees of freedom, Finite size effects etc.

CONCLUSIONS

The connection is fruitful for both gauge theorists and Bethe people

WISHFUL CONCLUSIONS

It would be nice to lift
this connection
further to string/M theory
constructions,
e.g. M5 *branes wrapping curves*
in K3

WISHFUL CONCLUSIONS

This would shed light on the «real» quantization of Hitchin systems (cf. «complex» quantization of Beilinson-Drinfeld)

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This would shed light on the «real» quantization of Hitchin systems (cf. «complex» quantization of Beilinson-Drinfeld)

Kapustin-Witten
(N=4 with boundaries)

VERY WISHFUL CONCLUSIONS

And perhaps push the geometric
Langlands program
beyond geometry
back to the number theory....

VERY WISHFUL CONCLUSIONS

Another possibility is that
Planar N=4 super-Yang-Mills
Is the integrable system....

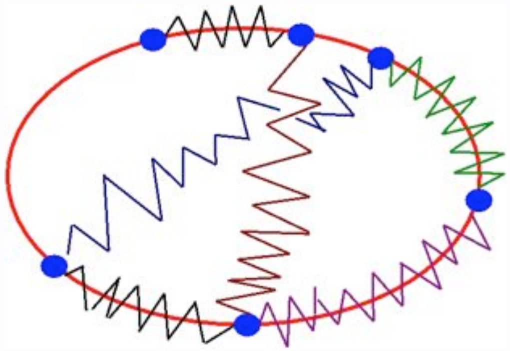
VERY WISHFUL CONCLUSIONS

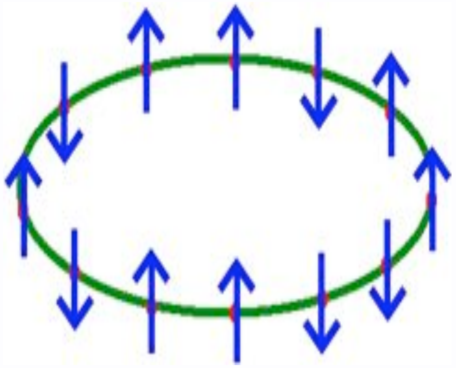
Dual to the TOTALITY of
the $N=2$ $d=4$ theories
Of a certain quiver type
In the Vacuum sector

CONCLUSIONS

These margins are
Too thin and the time
Is too short
To list them all....

The story is
beautiful,
complex,
and rich, yet
has real
applications





THANK
YOU