Integrability for the full spectrum of planar AdS/CFT

Pedro Vieira

[with N.Gromov, V.Kazakov and A.Kozak]

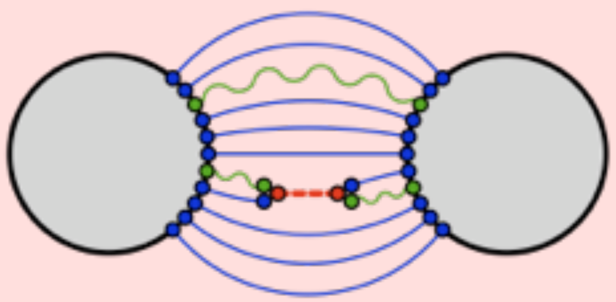
Plan

- AdS/CFT, motivation
- 2D integrable models at finite volume
- Integrability in AdS/CFT
  - ✦ Asymptotic spectrum
  - ✦ Exact planar spectrum and the Konishi operator

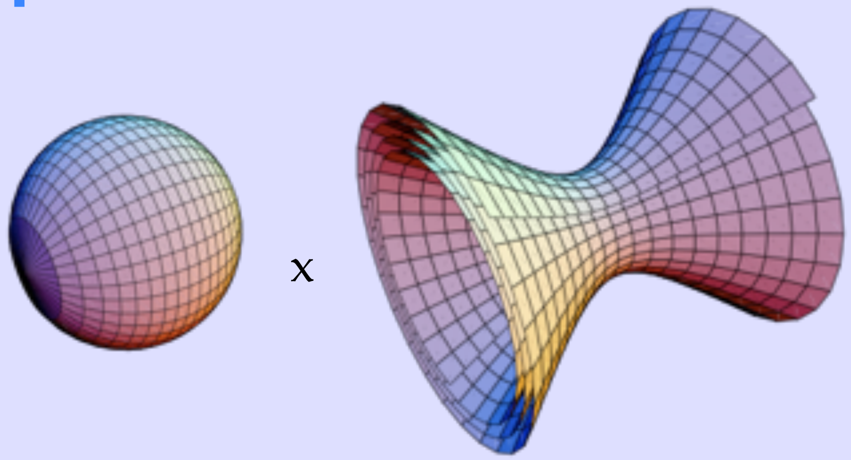
CFT

AdS

N=4 SYM in 4d



type IIB in $AdS_5 \times S^5$



$$\mathcal{L} = F^2 + (D\Phi)^2 + [\Phi, \Phi]^2 + \bar{\Psi}\mathcal{D}\Psi$$

$$\mathcal{L} = (\partial x)^2 + \Lambda(x^2 - 1) + \dots$$

$$\lambda = g_{YM}^2 N = \frac{R^4}{\alpha'^2} = g_s N$$

Planar Limit $N \to \infty$

Planar CFT

Free Strings

CFT

AdS

N=4 SYM in 4d

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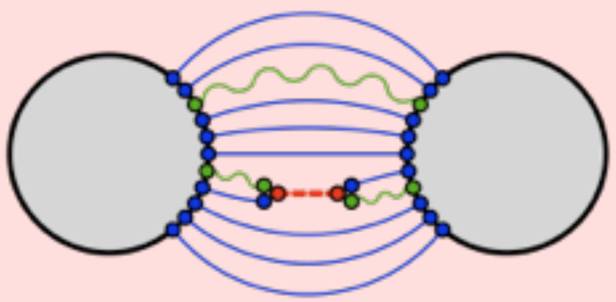

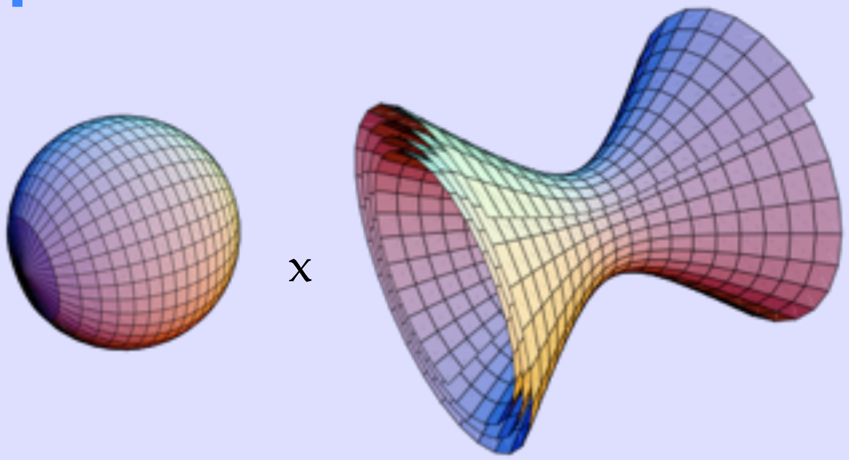

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Strongly coupled gauge theory

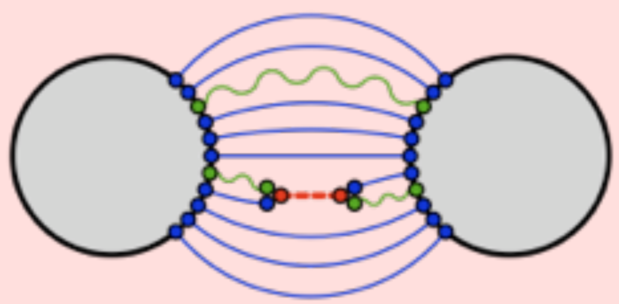
Classical strings

Perturbative gauge theory

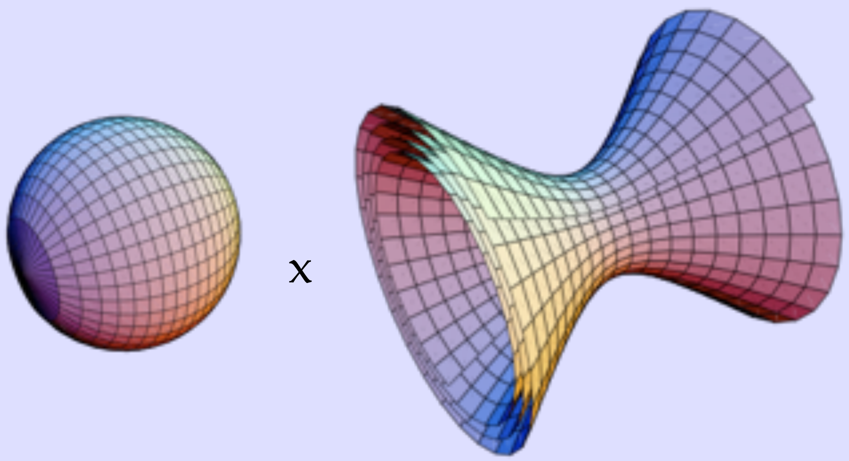
Highly quantum strings

Solve = ?

- Correlation functions:
  2 pt functions (**spectrum of anomalous dimensions**)
  3 pt functions (structure constants)
- S-matrix



anomalous dimensions



string energies

Integrability

Asymptotic Bethe equations and integrable 2D QFT

In 1+1D $\quad Q_1 = \sum p_j \ , \ Q_2 = \sum p_j^2 \ , \ \Rightarrow \ \{p_1, p_2\} = \{p'_1, p'_2\}$

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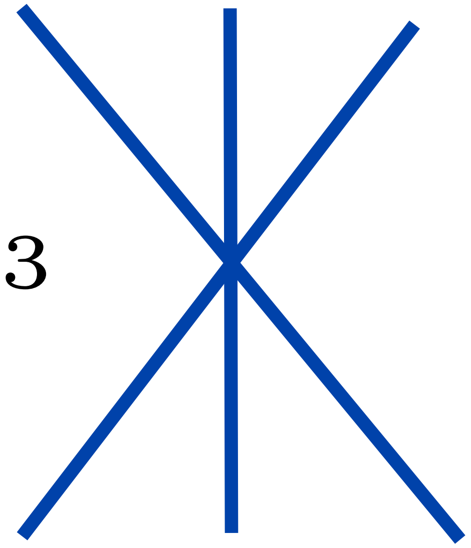
$e^{i\alpha Q_3}$



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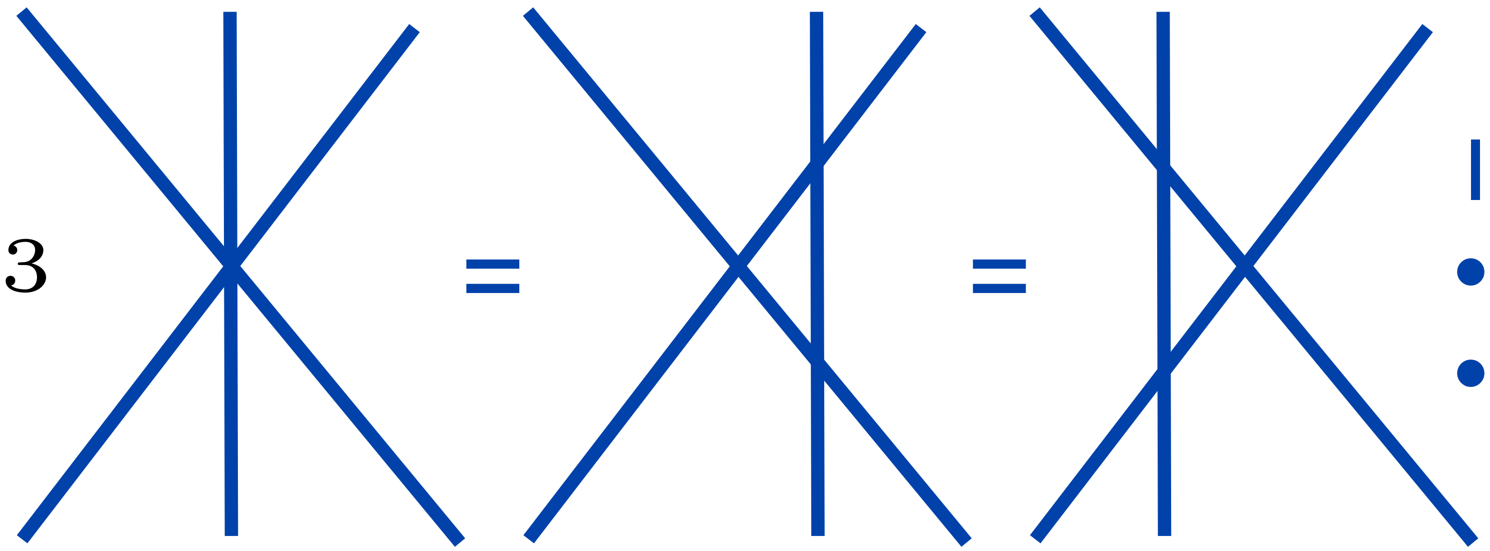

Integrable theories:
- Factorized scattering.
- S-matrices obey YB.

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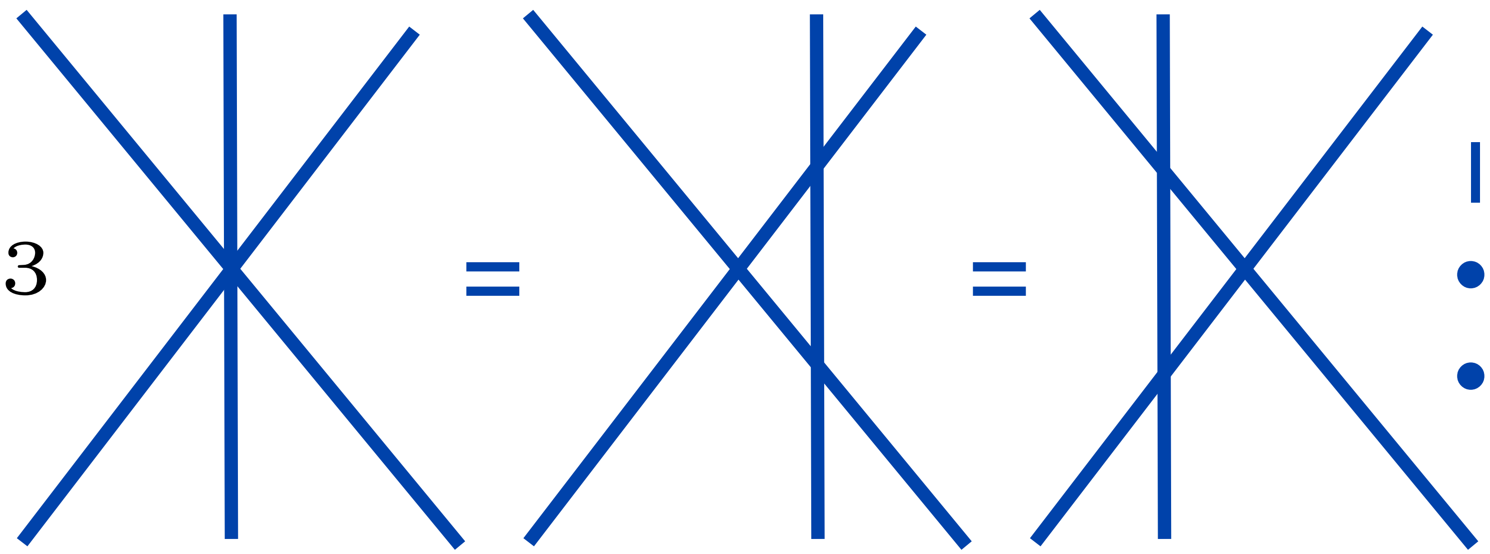

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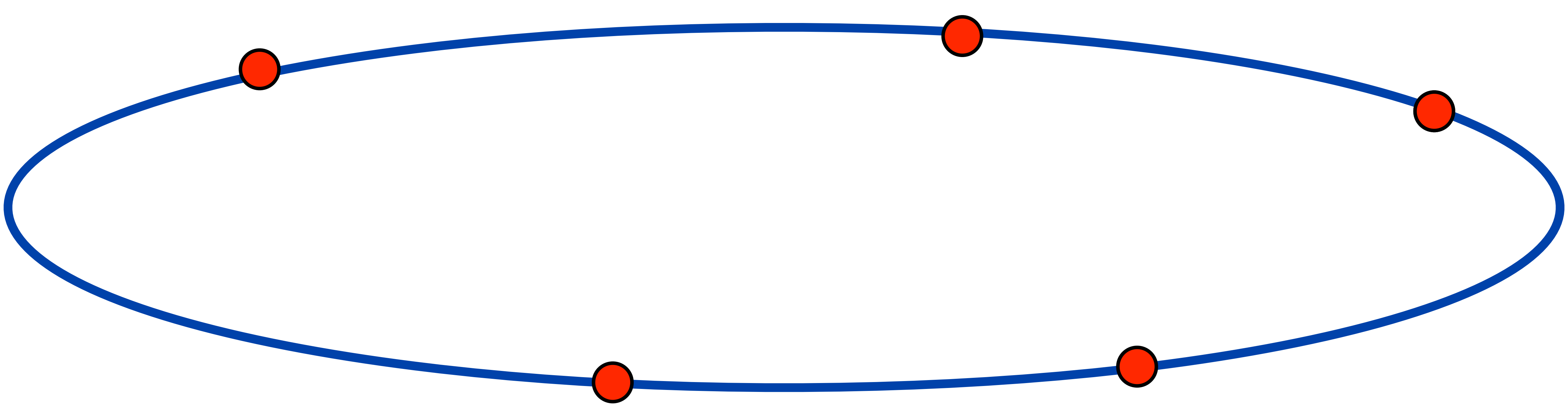

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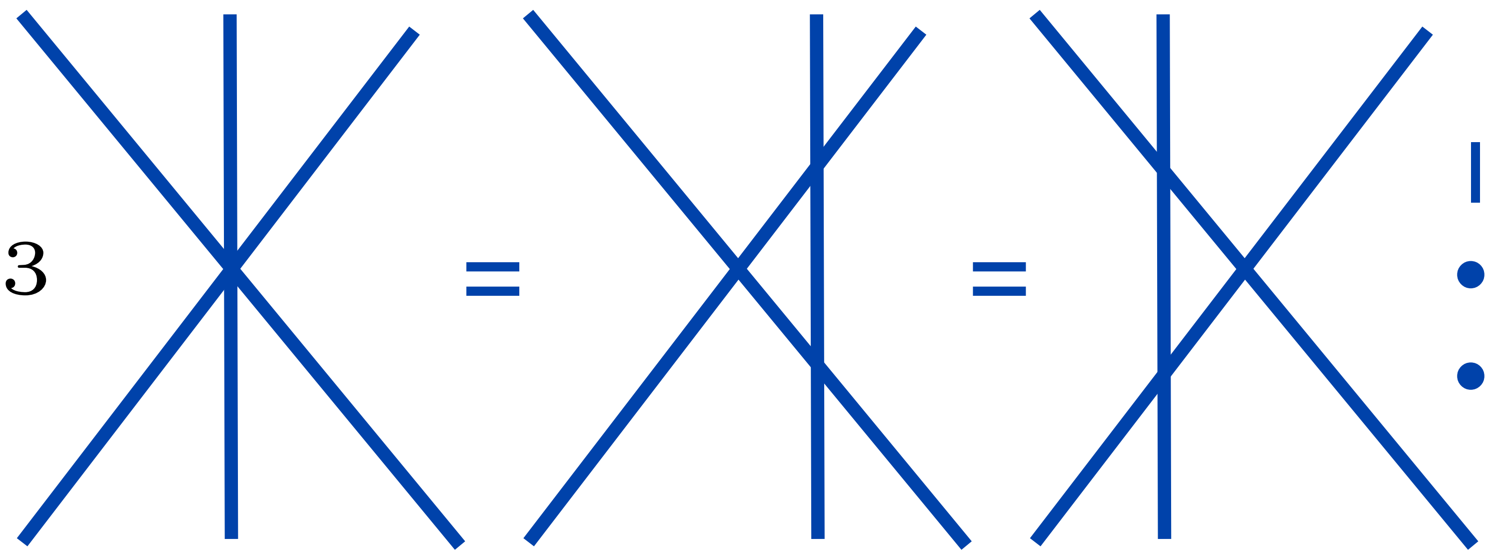

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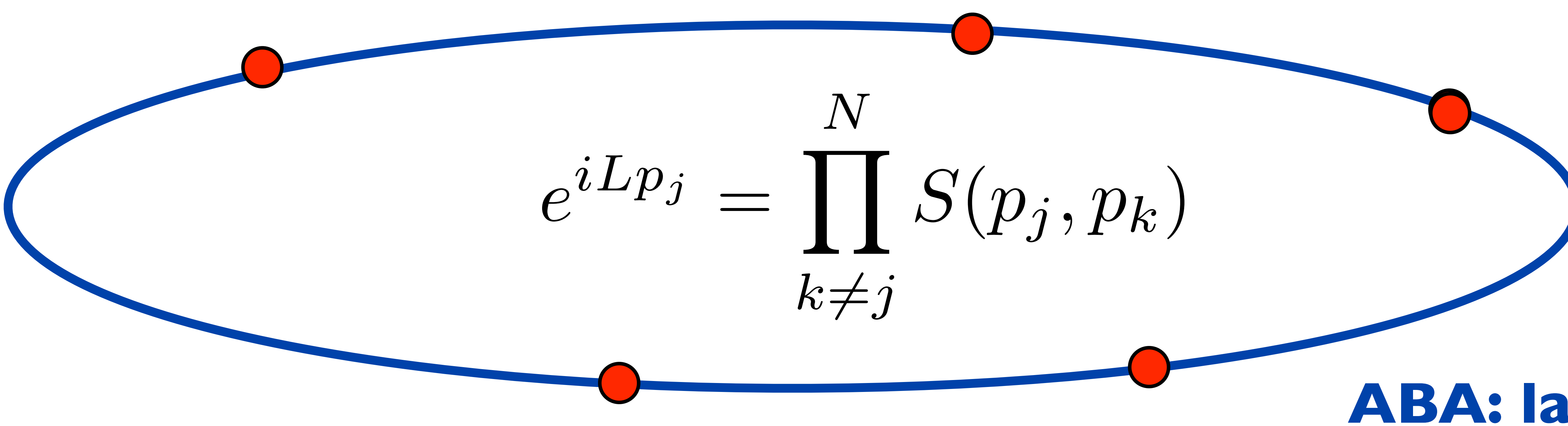

$$e^{iLp_j} = \prod_{k \neq j}^{N} S(p_j, p_k)$$

ABA: large L!

Spin chains in N=4

$$\mathrm{tr}\ \left(\Phi_1\Phi_2(D_3)^2(\Phi_1)^5\ldots\right)$$

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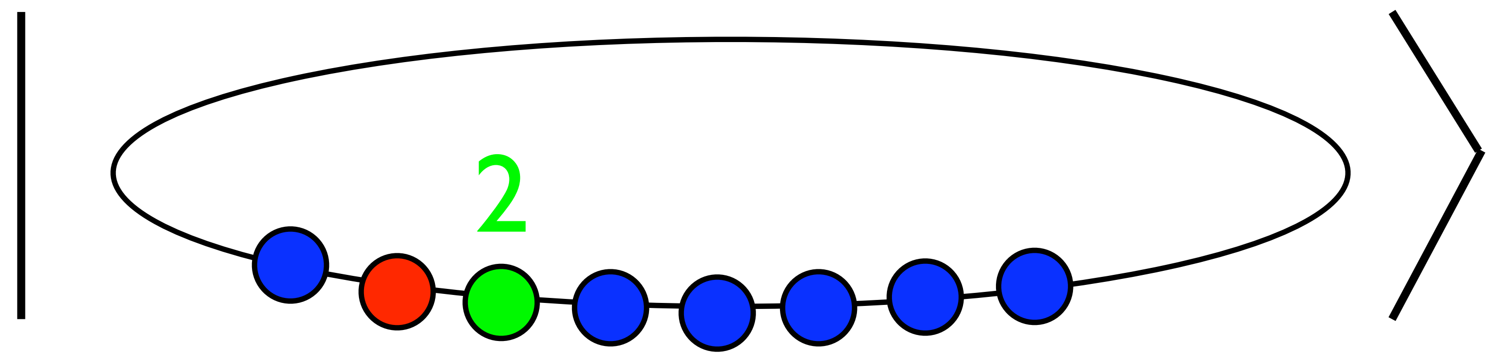

Spin chains in N=4

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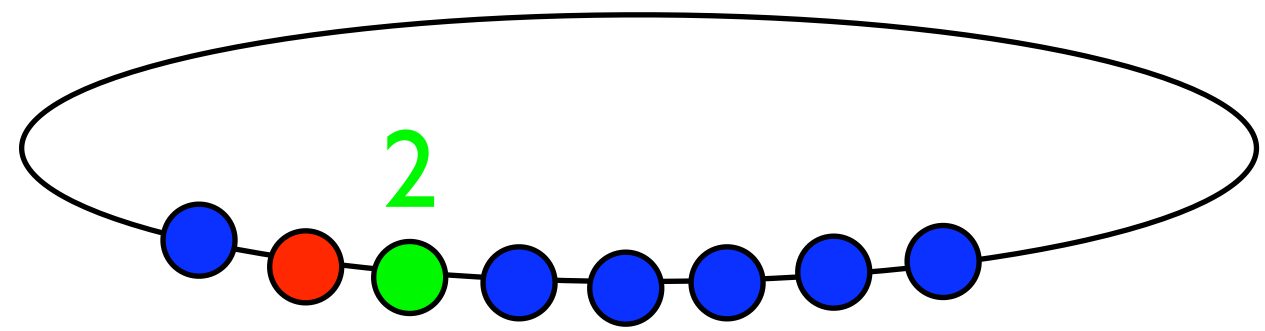
$$\Bigg|$$  $$\Bigg\rangle$$

Mixing matrix or
Dilatation operator

$$\mathcal{O}_A^{ren}(x) = (e^{\,\hat{H}\log\Lambda})_{AB}\,\mathcal{O}_B(x)$$

$$\langle \mathcal{O}_A^{ren}(x)\mathcal{O}_B^{ren}(y)\rangle = \frac{\delta_{AB}}{|x-y|^{2\Delta}}$$

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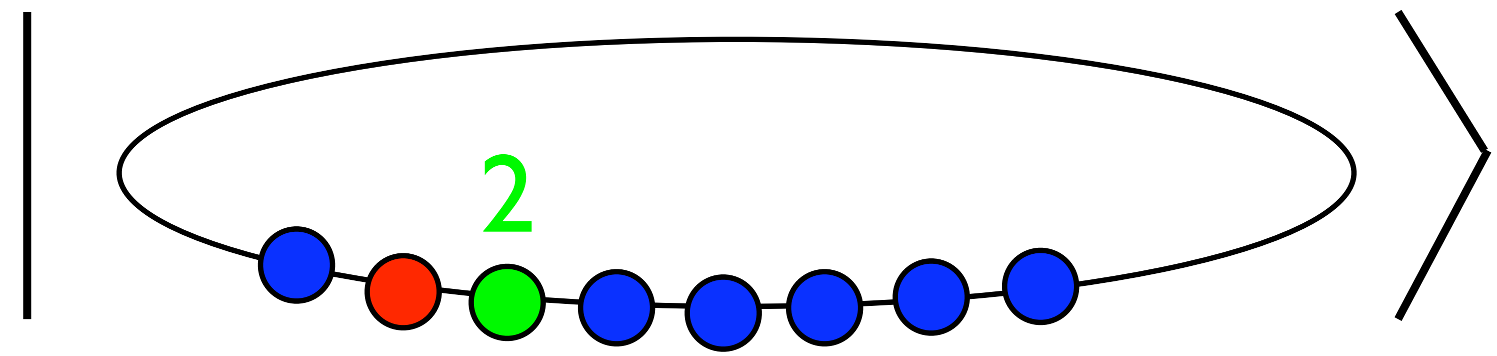

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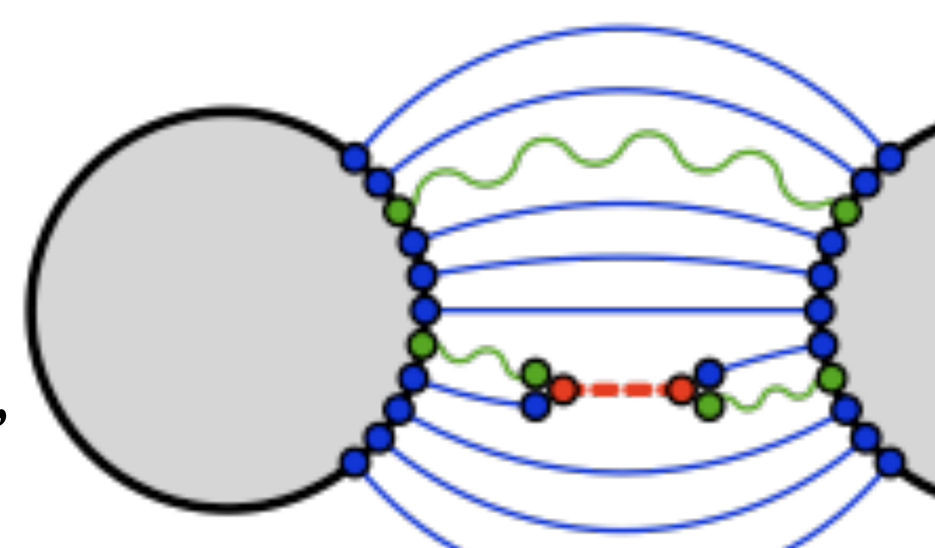
H is nearest neighbors to leading order in perturbation theory,
next to nearest neighbors at next to leading order etc...



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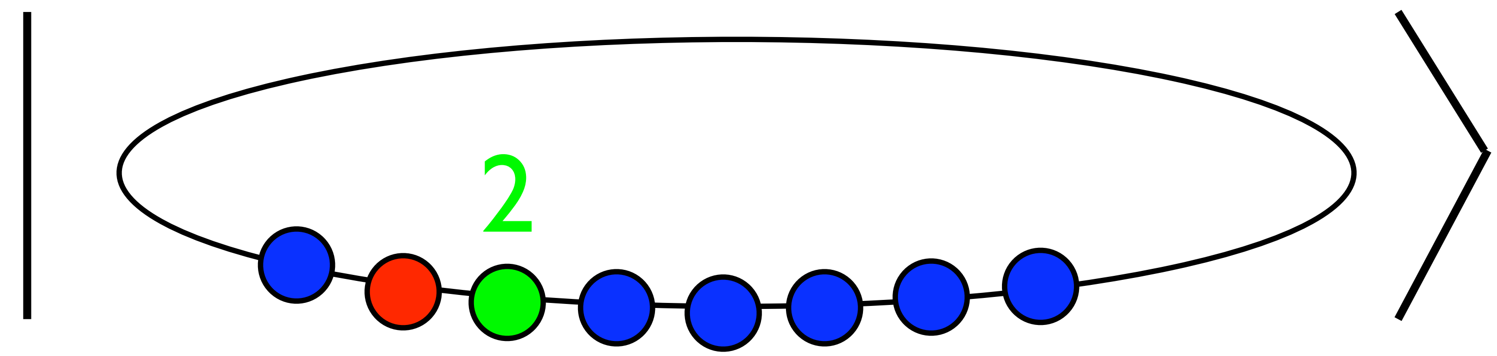

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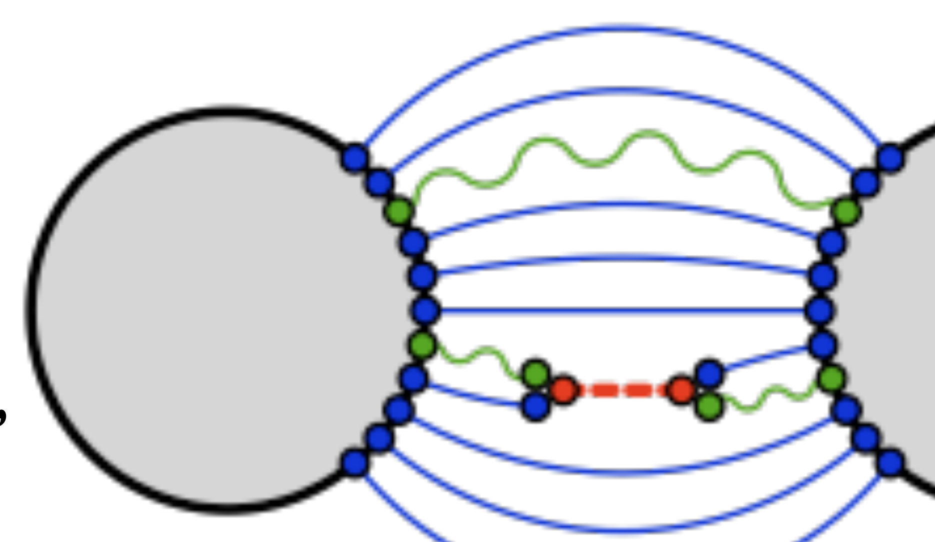
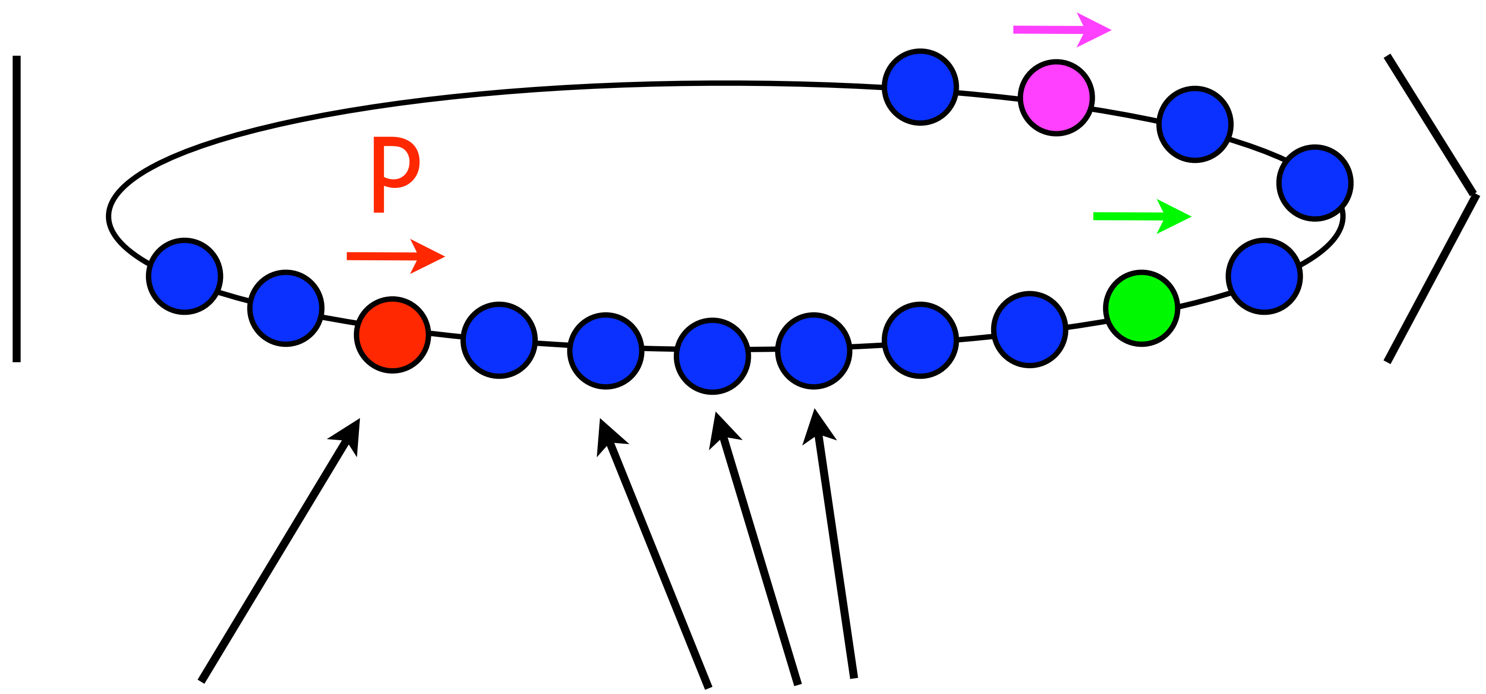
Integrable spin chain
Hamiltonian

[Minahan, Zarembo; Beisert, Staudacher]

H is nearest neighbors to leading order in perturbation theory, next to nearest neighbors at next to leading order etc...

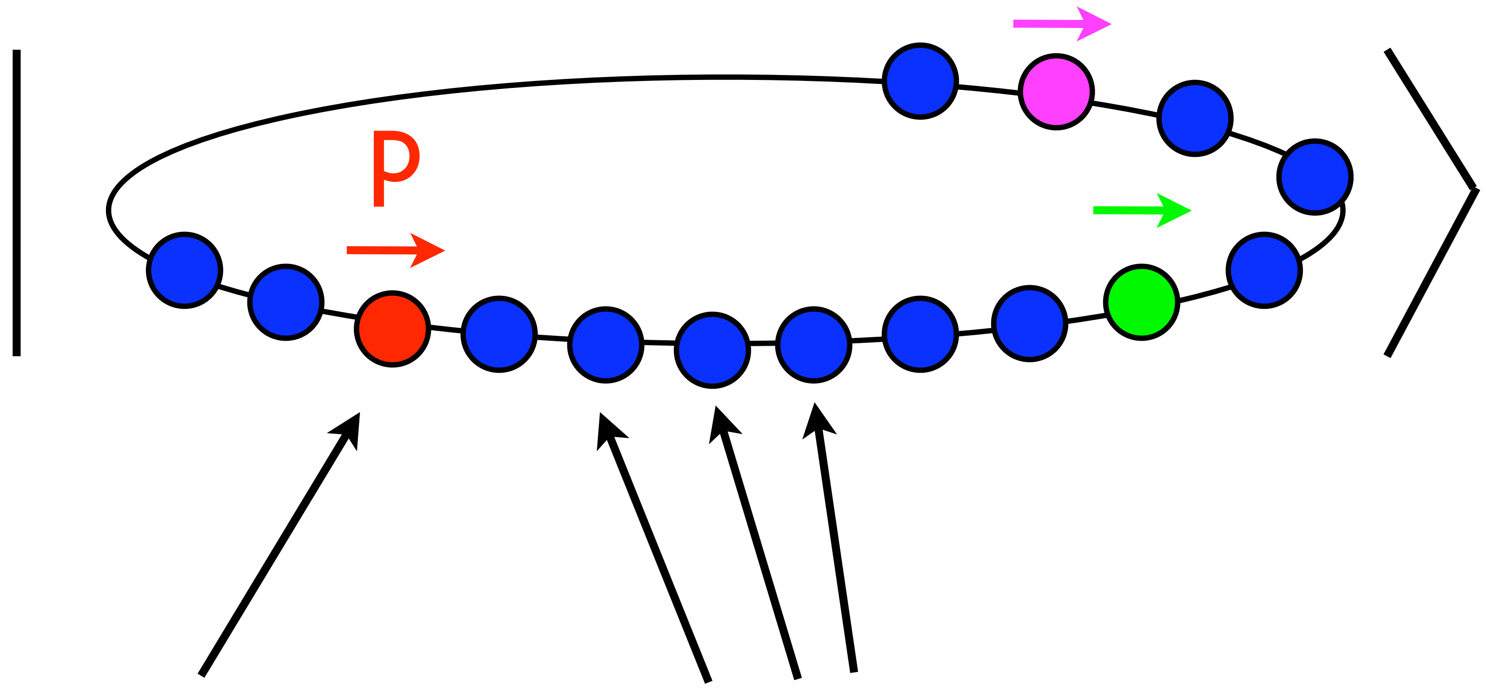


2d S-matrix in N=4

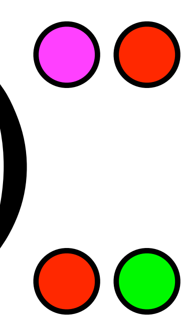


Particles (or magnons), Vacuum

2d S-matrix in N=4

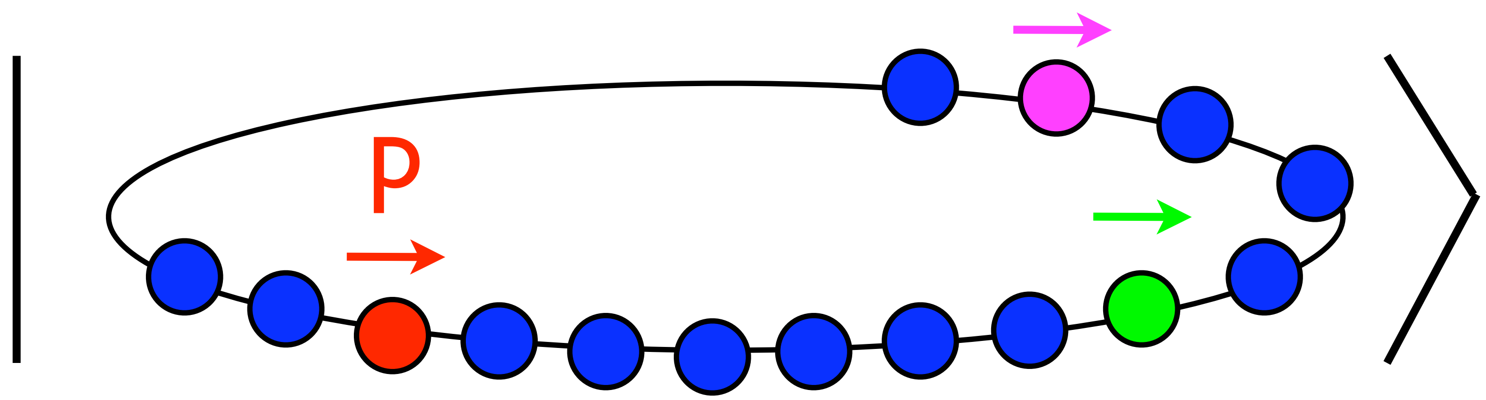


Particles (or magnons), Vacuum

Particles can scatter, e.g. $S(p,k)$  [Staudacher]

Particles transform in $PSU(2|2)^2$ extended [Beisert]

2d S-matrix in N=4



$$H \longrightarrow S(p,k)$$

$$PSU(2,2|4) \longrightarrow PSU(2|2)^2 \text{ extended}$$

S-matrix (up to a scalar factor) and magnon dispersion relation almost fixed by symmetry

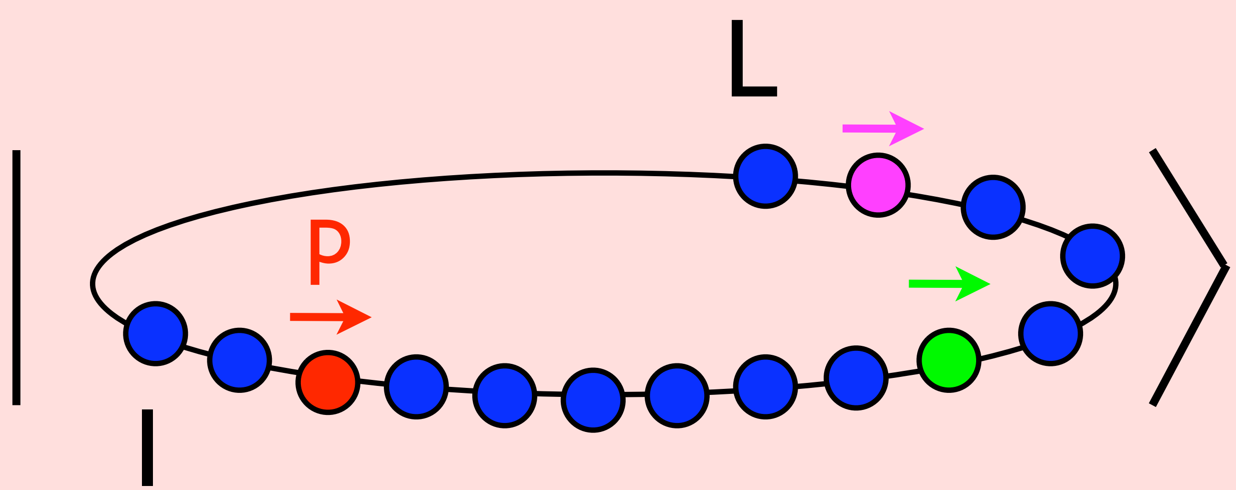
[Beisert, Aryutunov, Frolov, Zamaklar]

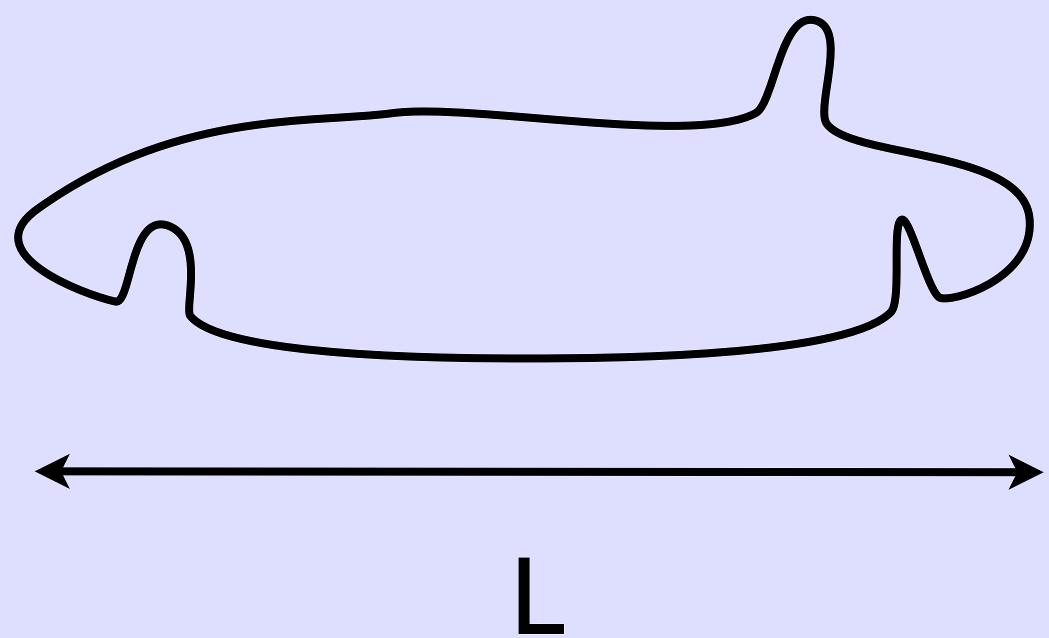
2d S-matrix in AdS/CFT

Integrability at strong coupling:
[Bena, Polchinski, Roiban],[KMMZ],...

Spin chain magnons in an operator with L fields

Worldsheet excitations in light-cone gauged string theory. 2D QFT in a circle of size L



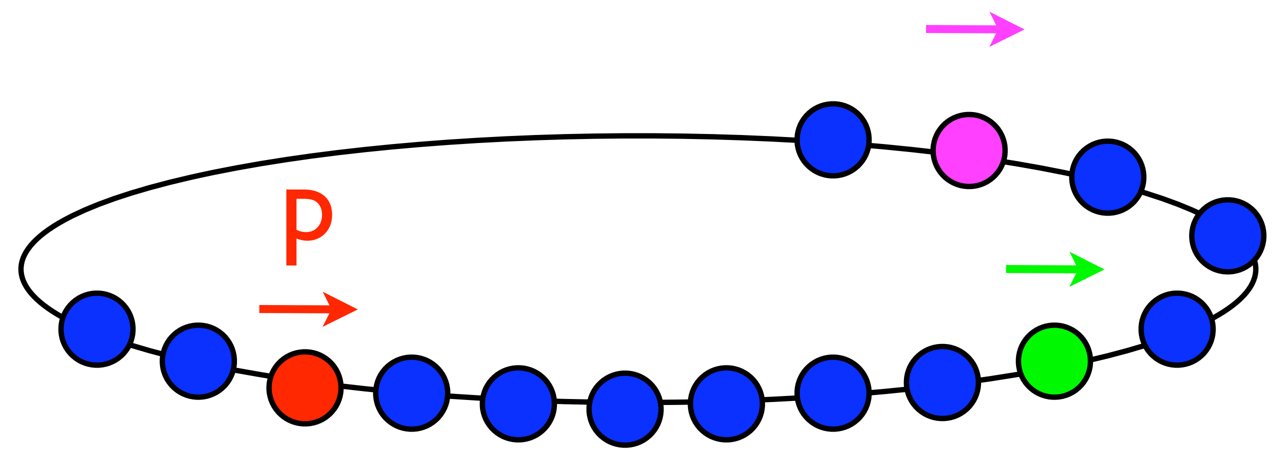


Previous arguments hold for both string and gauge theory

see also
[Aryutunov, Frolov, Plefka, Zamaklar]

Bethe Equations

$$0 = \left(e^{iLp_j} \prod_{k \neq j}^{M} \hat{S}(p_j, p_k) - 1 \right) \Bigg|$$

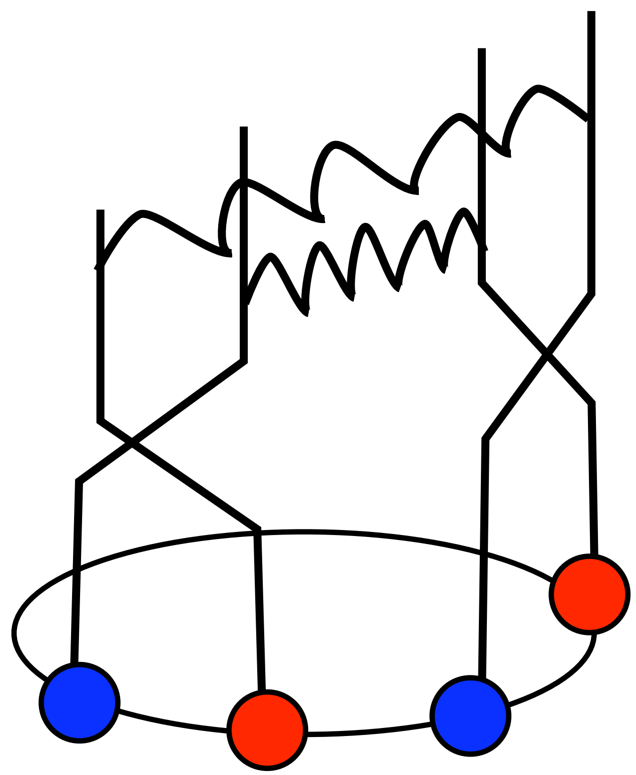


$$\Bigg\rangle$$

$$\Delta = J + \sum_{j=1}^{M} \sqrt{1 + \lambda \sin^2 \frac{p_j}{2}} + \ldots$$

[Staudacher; Beisert, Staudacher; Janik; Beisert, Eden, Staudacher; Beisert, Hernandez, Lopez; Arutyunov, Frolov, Zamaklar]

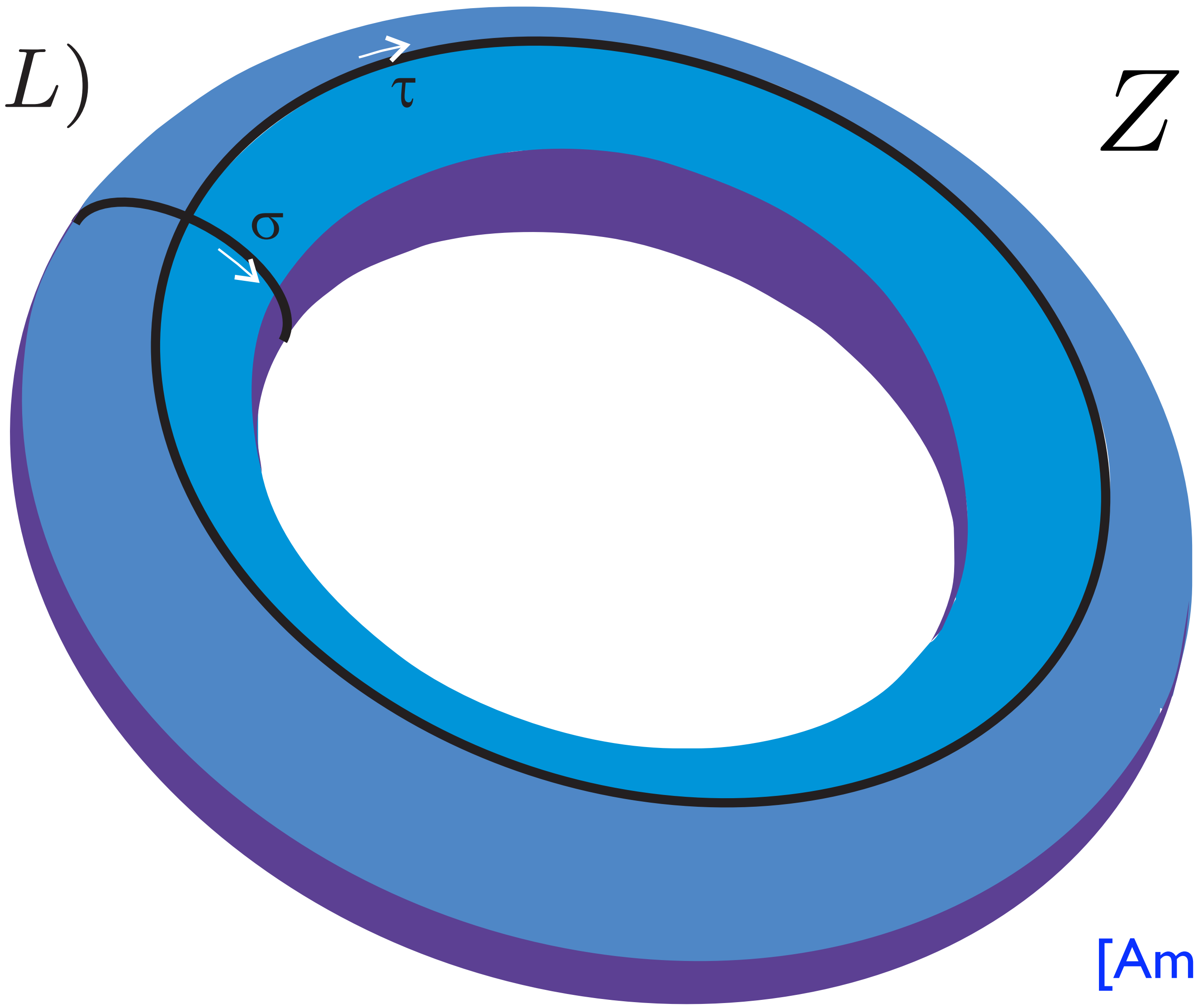
What about small operators?



[...,Matsubara, Zamolodchikov,...]

TBA Wick rotation and exact ground state energy

$$Z = e^{-RE_0(L)}$$



$$Z = e^{-Rf(L)}$$

Length: L
Temperature: 0
Asymptotic Bethe Ansatz

Length: infinity
Temperature: 1/L
Exact Bethe Ansatz

On the mirror theory in AdS/CFT:

[Ambjorn, Janik, Kristjansen]
[Arutyunov, Frolov]

Ground state energy at size L = free energy per unit length at temperature 1/L

Recipe

- Compute the particle dispersion relation
- Compute the two body S-matrix
- Write down the mirror BAE in the continuum limit (introducing one density for each kind of particle and bound state)
- Compute the entropy and energy and free energy by a saddle point computation
  - ✦ Saddle point equations: Y-system equations (in or functional form)
  - ✦ Free energy at saddle point: Exact vacuum energy
- Pick extra singularities in the free energy and in the integral equations to get the energy of excited states [Dorey and Tateo]

The Dorey-Tateo trick

Vacuum equations

$$E_0 = \int \frac{du}{2\pi i} \partial_u \epsilon(u) \log(1 + Y(u))$$

$$\log Y(u) = iLp(u) + \int \frac{dv}{2\pi i} \partial_v \log S(u,v) \log(1 + Y(v))$$

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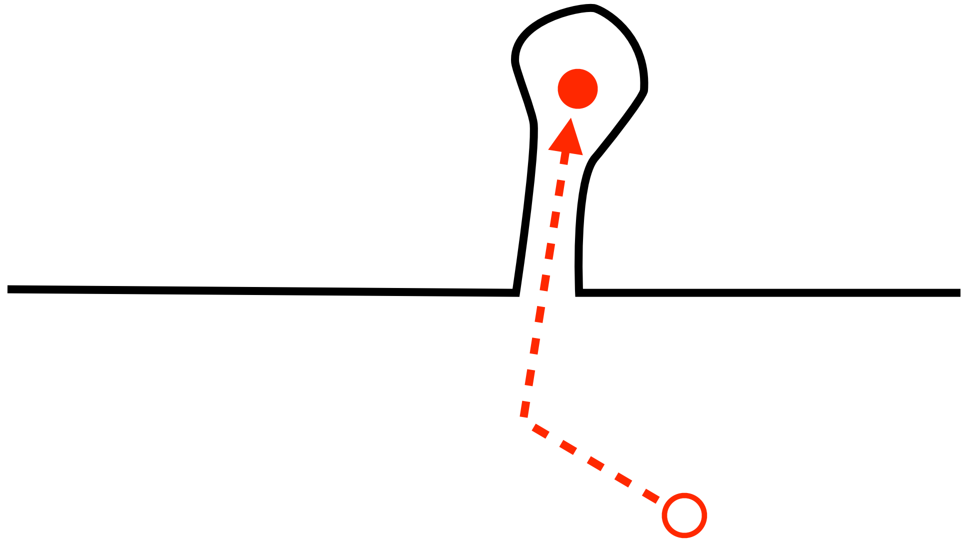

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Vacuum equations become

$$E_0 = \int \frac{du}{2\pi i} \partial_u \epsilon(u) \log(1 + Y(u)) \quad + \sum_{j=1}^{N} \epsilon(u_j)$$

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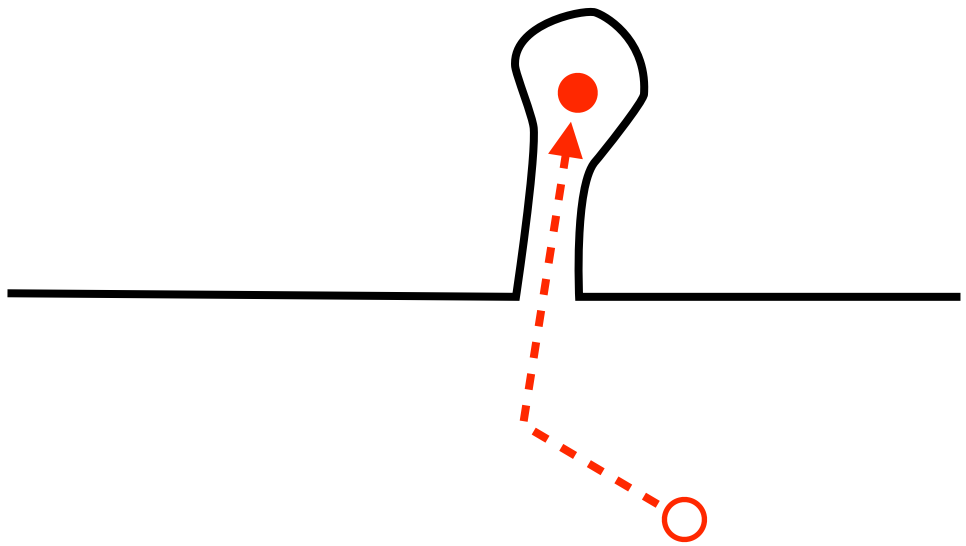

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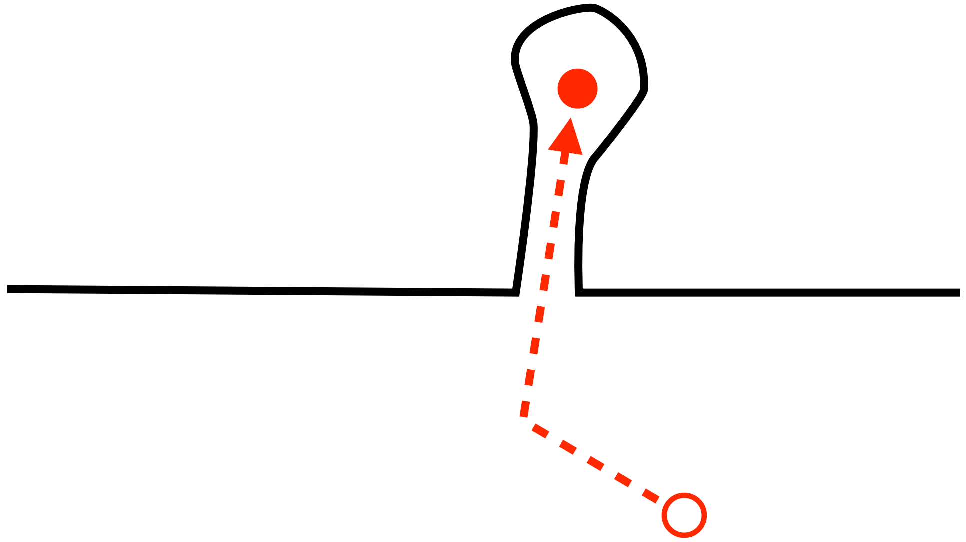

Exact Bethe Equations: $Y(u_j) = -1$

Large L

$$Y(u) \simeq e^{iLp(u)} \prod_{j=1}^{N} S(u, u_j)$$

$$E(L) \simeq \sum_{j=1}^{N} \epsilon(u_j) + \int \frac{du}{2\pi i} \partial_u \epsilon(u) e^{ip(u)L} \prod_{j=1}^{N} S(u, u_j)$$

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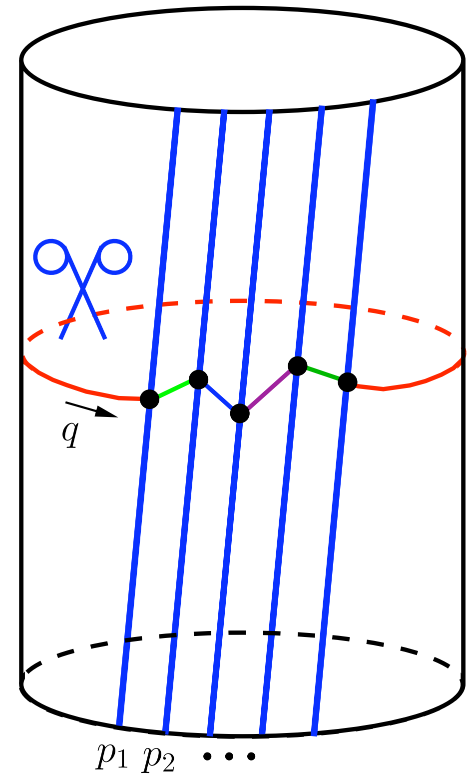

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Asymptotic Bethe Ansatz + Luscher corrections come out of large L

[Luscher; Janik, Lukowsky]

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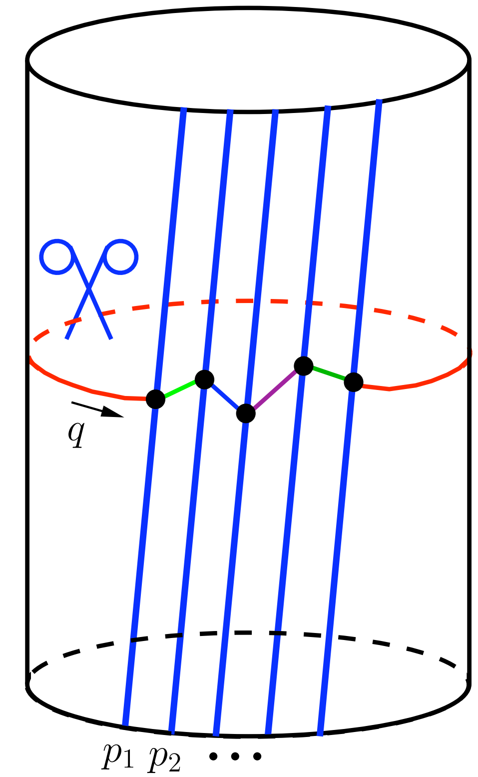
Large L



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Asymptotic Bethe Ansatz + Luscher corrections come out of large L

[Luscher; Janik, Lukowsky]

Asymptotic Equations: $Y(u_j) = -1$

At finite L we simply iterate

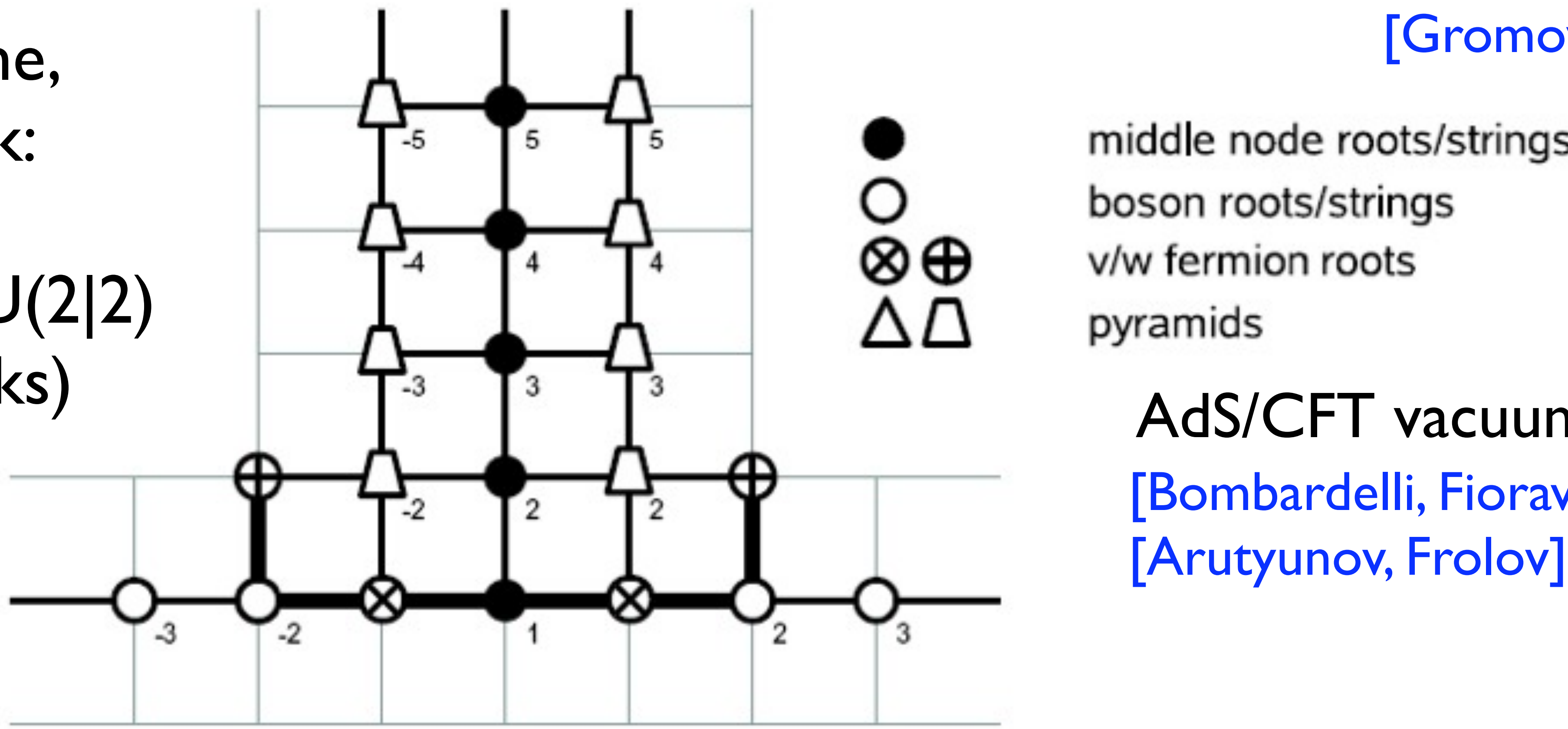
$$\log Y(u) = iLp(u) + \sum_{j=1}^{N} \log S(u, u_j) + \int \frac{dv}{2\pi i} \partial_v \log S(u, v) \log(1 + Y(v))$$

The AdS/CFT Y-system

[Gromov,Kazakov,PV]
[Gromov,Kazakov,Kozak, PV]

a,s plane,
T-hook:

(glues 2 SU(2|2) fat hooks)



AdS/CFT vacuum TBA equations:
[Bombardelli, Fioravanti, Tateo],
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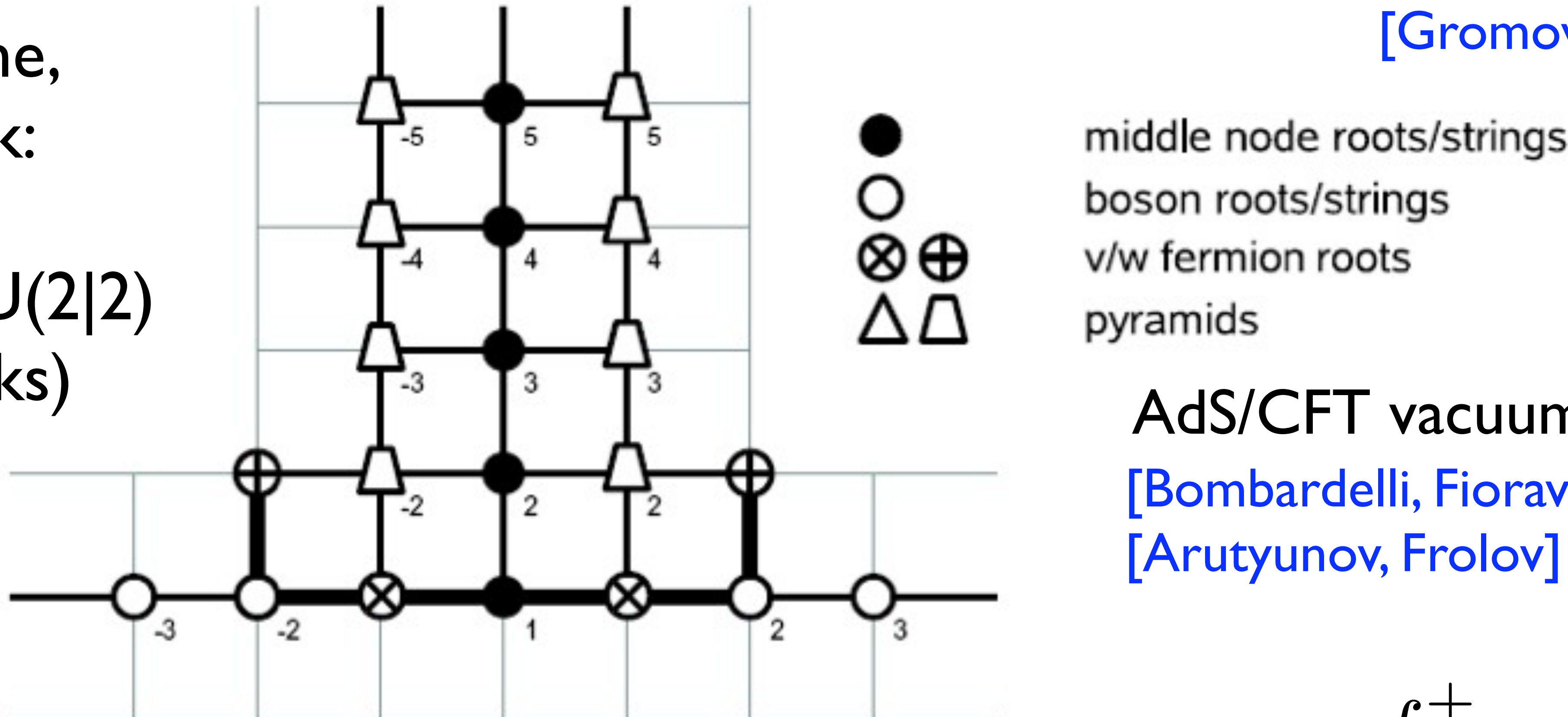

AdS/CFT vacuum TBA equations:
[Bombardelli, Fioravanti, Tateo],
[Arutyunov, Frolov]

$$f^{\pm} = f(u \pm i/2)$$

$$\frac{Y_{a,s}^{+} Y_{a,s}^{-}}{Y_{a+1,s} Y_{a-1,s}} = \frac{(1+Y_{a,s+1})(1+Y_{a,s-1})}{(1+Y_{a+1,s})(1+Y_{a-1,s})}$$

+ boundary conditions
+ analyticity

The AdS/CFT Y-system

[Gromov,Kazakov,PV]
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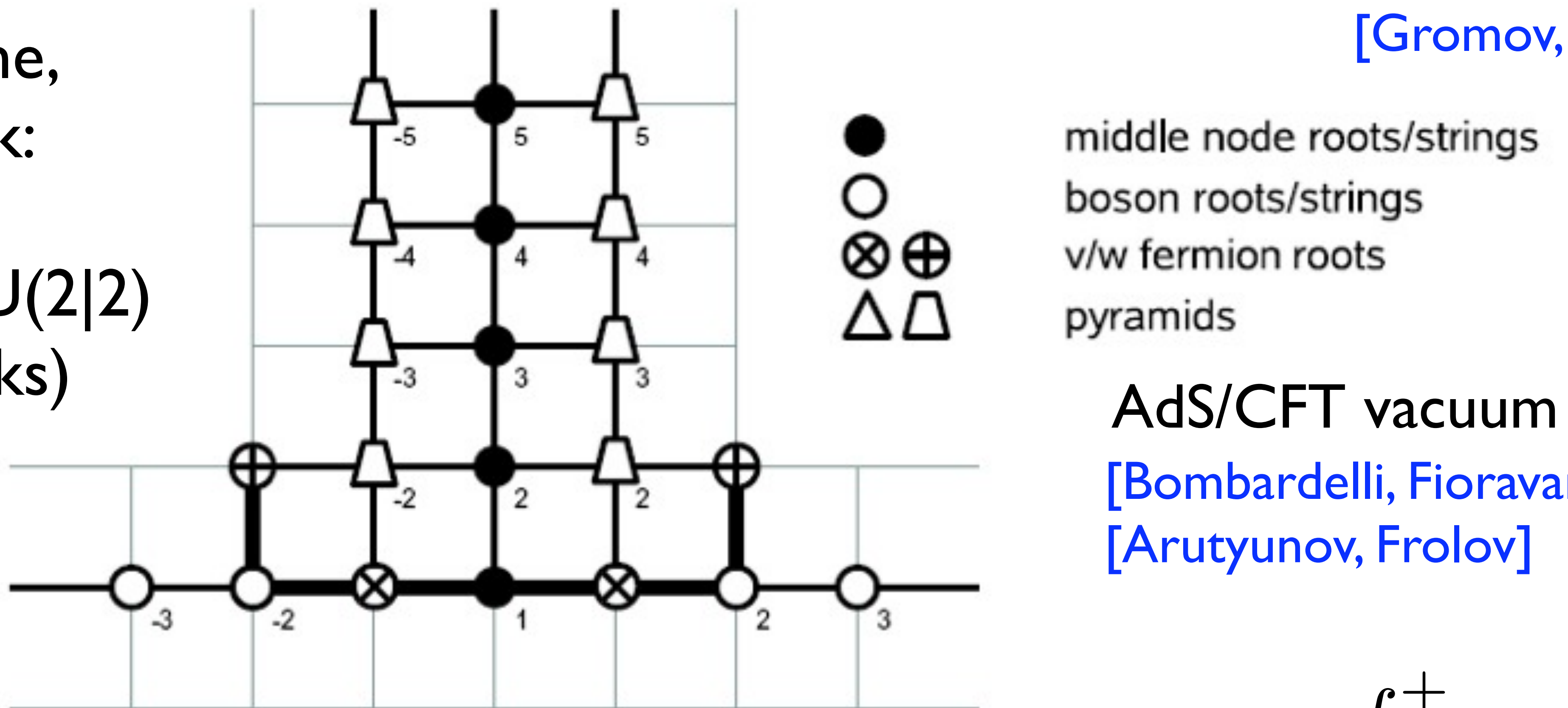

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+ boundary conditions
+ analyticity

The energy depends on the magnon dispersion relation and on the Y-functions:

$$E = \sum_j \epsilon_1(u_{4,j}) + \sum_{a=1}^{\infty} \int_{-\infty}^{\infty} \frac{du}{2\pi i} \, \frac{\partial \epsilon^*_a}{\partial u} \log\left(1 + Y^*_{a,0}(u)\right)$$

The AdS/CFT Y-system

[Gromov,Kazakov,PV]
[Gromov,Kazakov,Kozak, PV]

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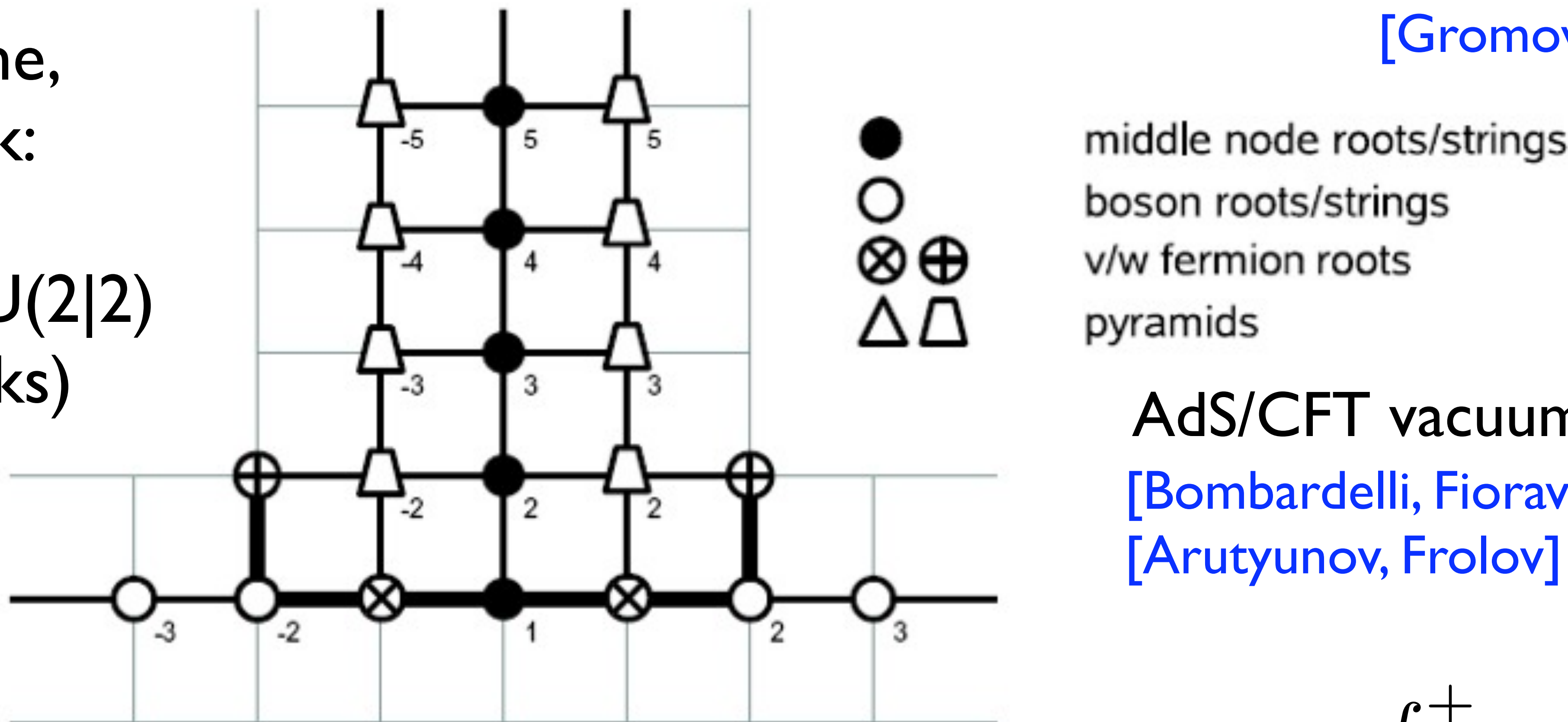
T-hook:

(glues 2 SU(2|2) fat hooks)



AdS/CFT vacuum TBA equations:
[Bombardelli, Fioravanti, Tateo],
[Arutyunov, Frolov]

$$f^{\pm} = f(u \pm i/2)$$

$$\frac{Y_{a,s}^{+} Y_{a,s}^{-}}{Y_{a+1,s} Y_{a-1,s}} = \frac{(1+Y_{a,s+1})(1+Y_{a,s-1})}{(1+Y_{a+1,s})(1+Y_{a-1,s})}$$

+ boundary conditions
+ analyticity

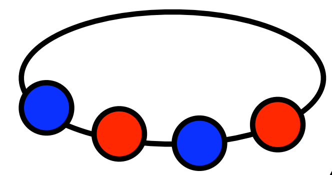
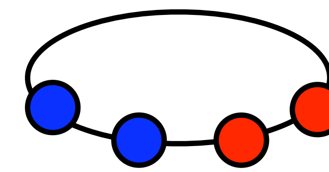
The energy depends on the magnon dispersion relation and on the Y-functions:

$$Y_{1,0}(u_{4,j}) = -1$$

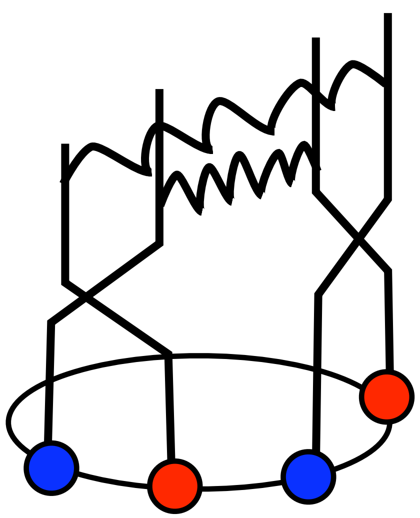
$$E = \sum_j \epsilon_1(u_{4,j}) + \sum_{a=1}^{\infty} \int_{-\infty}^{\infty} \frac{du}{2\pi i} \frac{\partial \epsilon_a^*}{\partial u} \log\left(1 + Y_{a,0}^*(u)\right)$$

Konishi operator at weak coupling

Konishi multiplet:: [Bianchi,Kovacs,Rossi,Stanev],[Eden,Jarczak,Sokatchev,Stanev]

$$Konishi = \mathrm{tr}[Z,X]^2 = \Big|$$  $$\Big\rangle \;-\; \Big|$$  $$\Big\rangle$$

$$\Delta_{Konishi} = 12g^2 - 48g^4 + 336g^6 + (-2584 + 384\zeta_3 - 1440\zeta_5)g^8$$



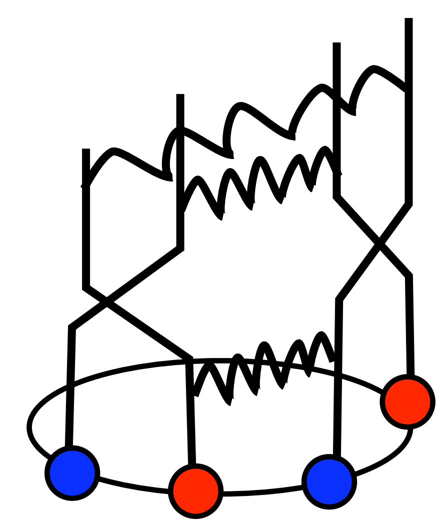
+ ....

Reproduces the YM 4 loop computation involving more than 130000 Feynman diagrams! (in components)

[Bajnok,Janik], [Gromov,Kazakov,PV]

[Fiamberti, Stambrogio, Sieg, Zanon][Velizhanin]

5 loops last Tuesday: [Bajnok,Hegedus,Janik,Lukowsky]



Konishi at *any* coupling

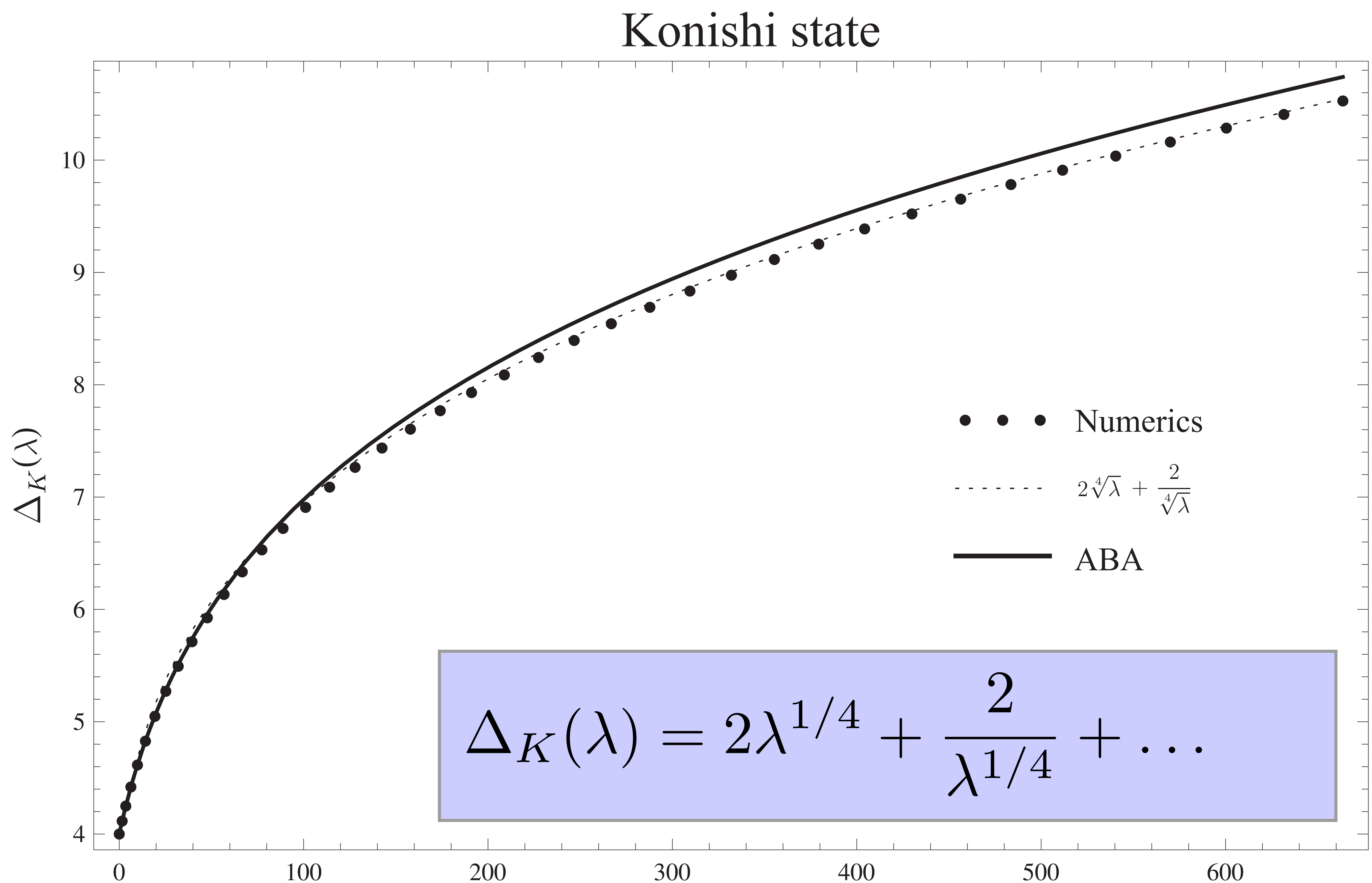
[Gromov,Kazakov,Kozak, PV]

(generalizing the Dorey-Tateo trick and using the Aryutunov-Frolov choice of branches)

- We start with large L solution predicted from the Y-system in functional form and then we "simply" iterate the integral equations:

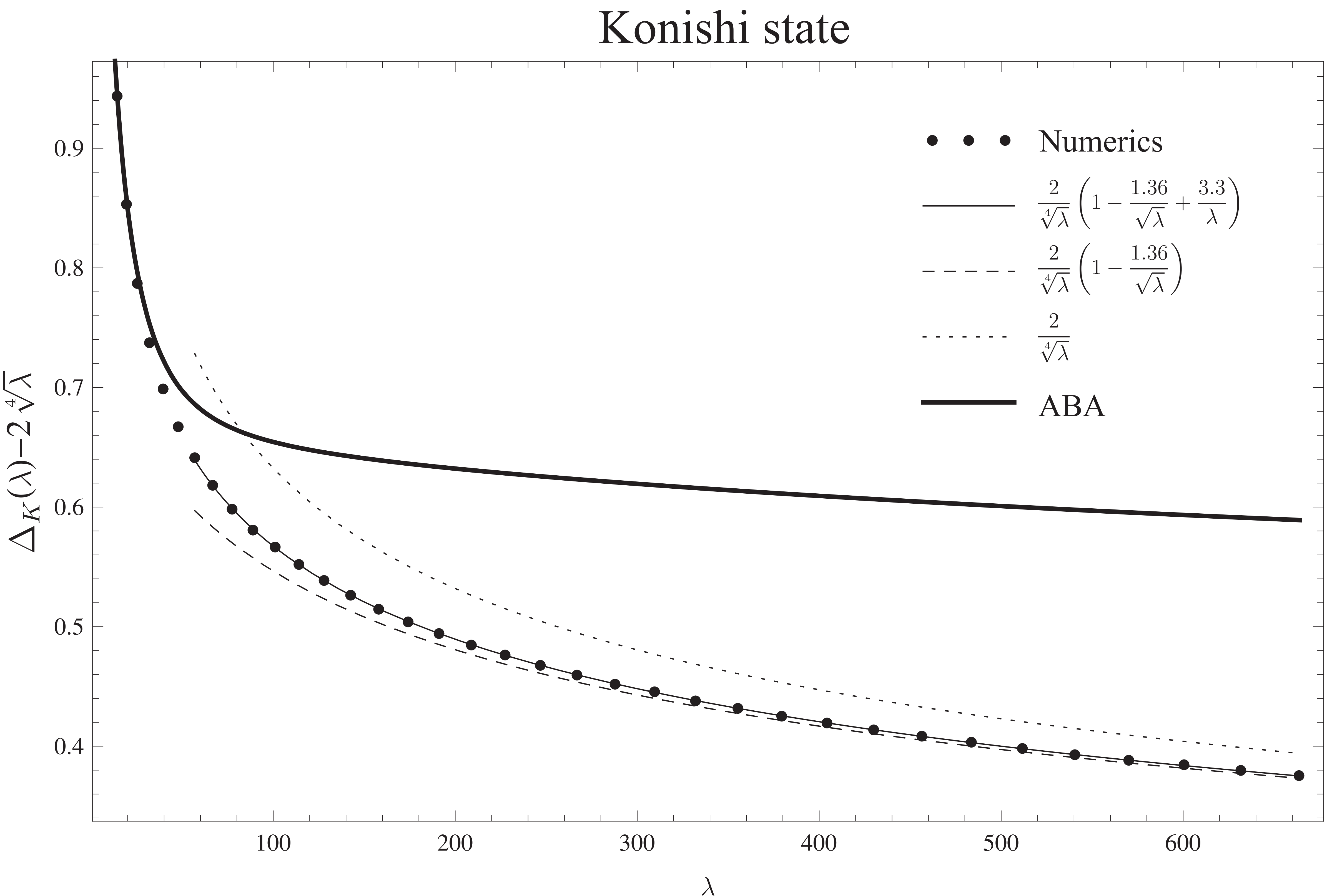
$$
\begin{aligned}
\log Y_{\otimes} &= K_{m-1} * \log(1+1/Y_{\circ_m})/(1+Y_{\triangle_m}) \\
&\quad + \mathcal{R}^{(0m)} * \log(1+Y_{\bullet_m}) + \log \frac{-R^{(+)}}{R^{(-)}} \\
\log Y_{\triangle_n} &= \mathcal{M}_{nm} * \log(1+Y_{\bullet_m}) - K_{n-1} \circledast \log(1+Y_{\otimes}) \\
&\quad - K_{n-1,m-1} * \log(1+Y_{\triangle_m}) + \log \frac{R_n^{(+)} B_{n-2}^{(+)}}{R_n^{(-)} B_{n-2}^{(-)}} \\
\log Y_{\circ_n} &= K_{n-1,m-1} * \log(1+1/Y_{\circ_m}) + K_{n-1} \circledast \log(1+Y_{\otimes}) \\
\log Y_{\bullet_n} &= \mathcal{T}_{nm} * \log(1+Y_{\bullet_m}) + 2\mathcal{R}^{(n0)} \circledast \log(1+Y_{\otimes}) \\
&\quad + \mathcal{N}_{nm} * \log(1+Y_{\triangle_m}) + i\Phi_n \ .
\end{aligned}
$$

Konishi at *any* coupling



$$\Delta_K(\lambda) = 2\lambda^{1/4} + \frac{2}{\lambda^{1/4}} + \dots$$

Konishi at *any* coupling



Konishi at *any* coupling

- "Simply" iterate = more that 1000h of computer time
- Integers are absolutely non-trivial!

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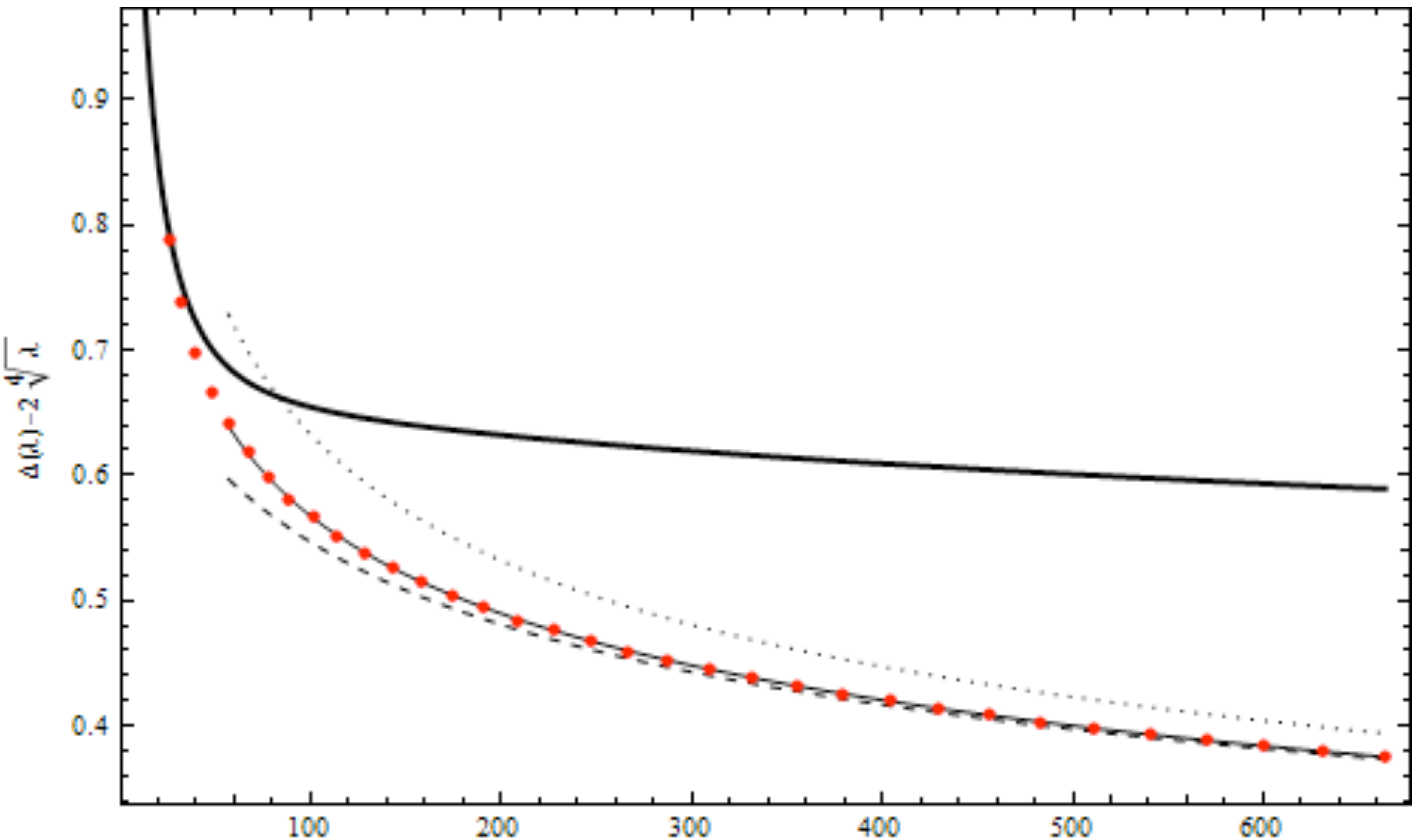


Conclusions

- **We proposed a set of Y-system (integral and functional) equations describing the exact planar spectrum of AdS/CFT.**

- We did not find any singularity from the perturbative deep into the strong coupling regime. On the contrary we found many nice cancelations ensuring nice properties such as reality of the Y-system. **Integers in predictions below are absolutely non-trivial**.

- New predictions:

5 loops: [Bajnok,Hegedus,Janik,Lukowsky]

$$\Delta_K(\lambda) = 2\lambda^{1/4} + \frac{2}{\lambda^{1/4}} + \ldots$$

Strong coupling: [Roiban,Tseytlin]

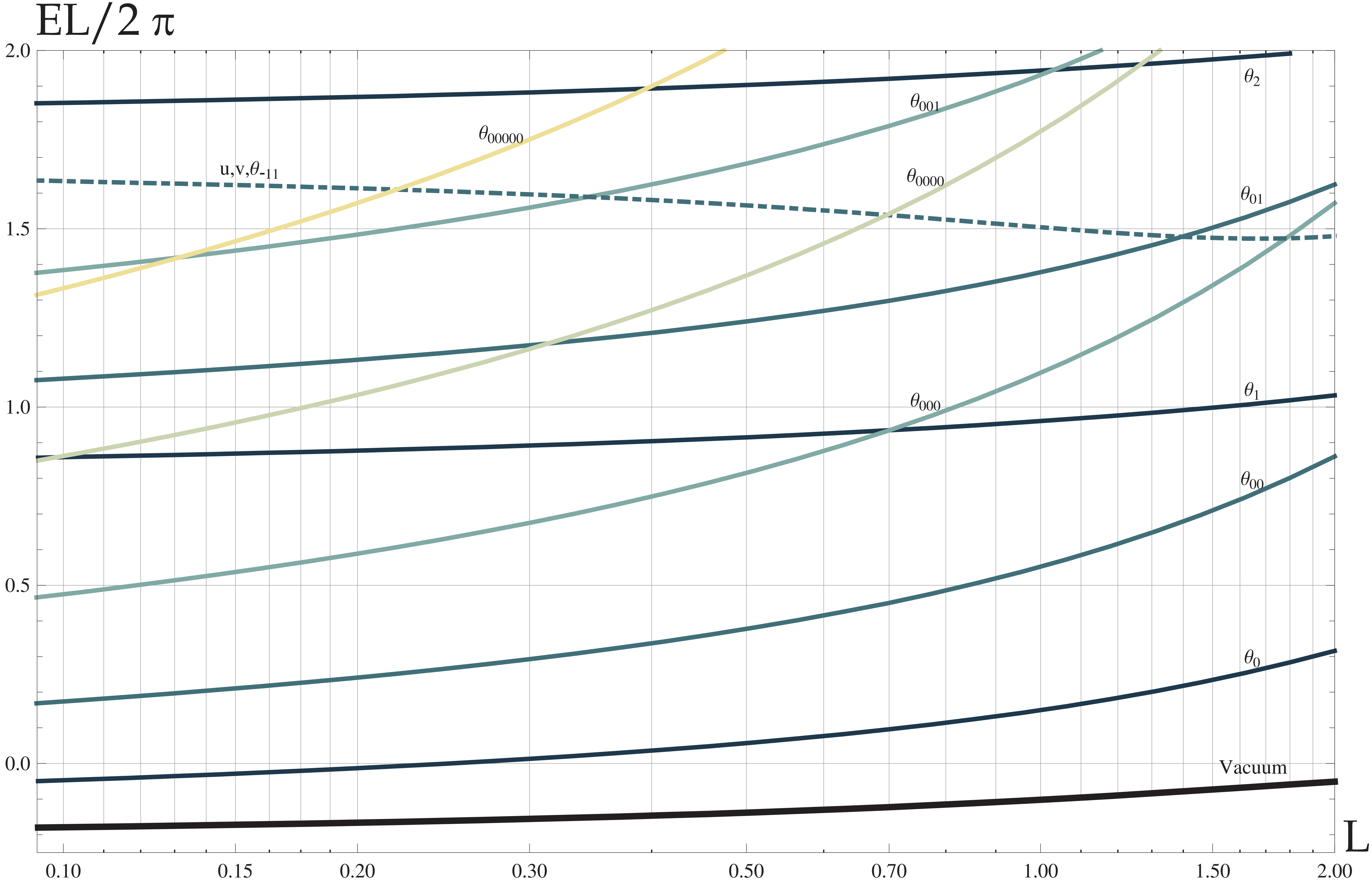
[Carlini Vallilo, Mazzucato], using pure spinor methods, in progress

$$\cdots \simeq -\frac{2.66}{\lambda^{3/4}} + \frac{6.4}{\lambda^{5/4}} + \mathcal{O}(\lambda^{-7/4})$$

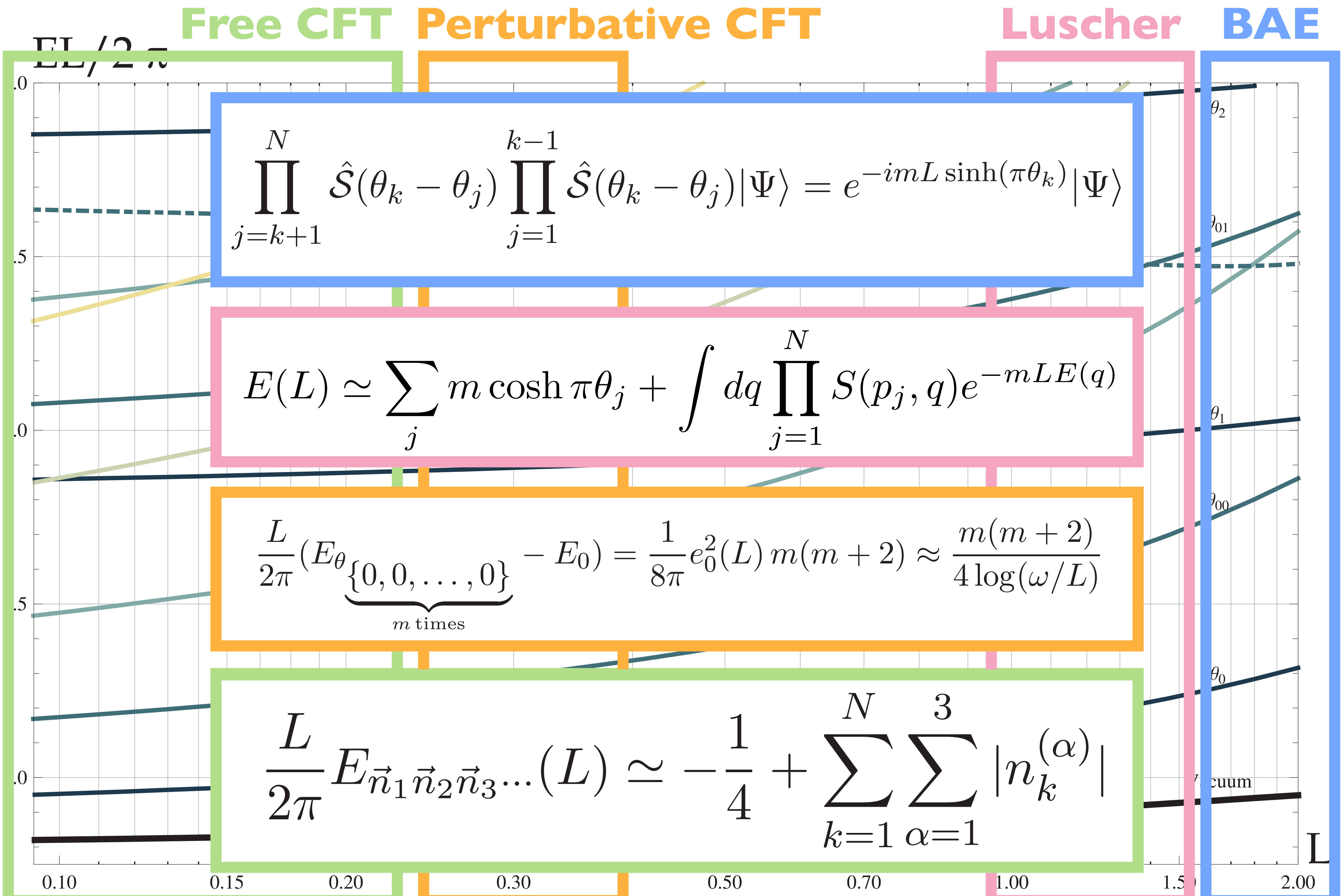
- Future: One single DdV integral equation (as for the SU(2) PCF)... ? Generalize to all states.

Extra slides

The exact spectrum of the SU(2) PCF



The exact spectrum of the SU(2) PCF



$$\prod_{j=k+1}^{N} \hat{\mathcal{S}}(\theta_k-\theta_j) \prod_{j=1}^{k-1} \hat{\mathcal{S}}(\theta_k-\theta_j)|\Psi\rangle = e^{-imL\sinh(\pi\theta_k)}|\Psi\rangle$$

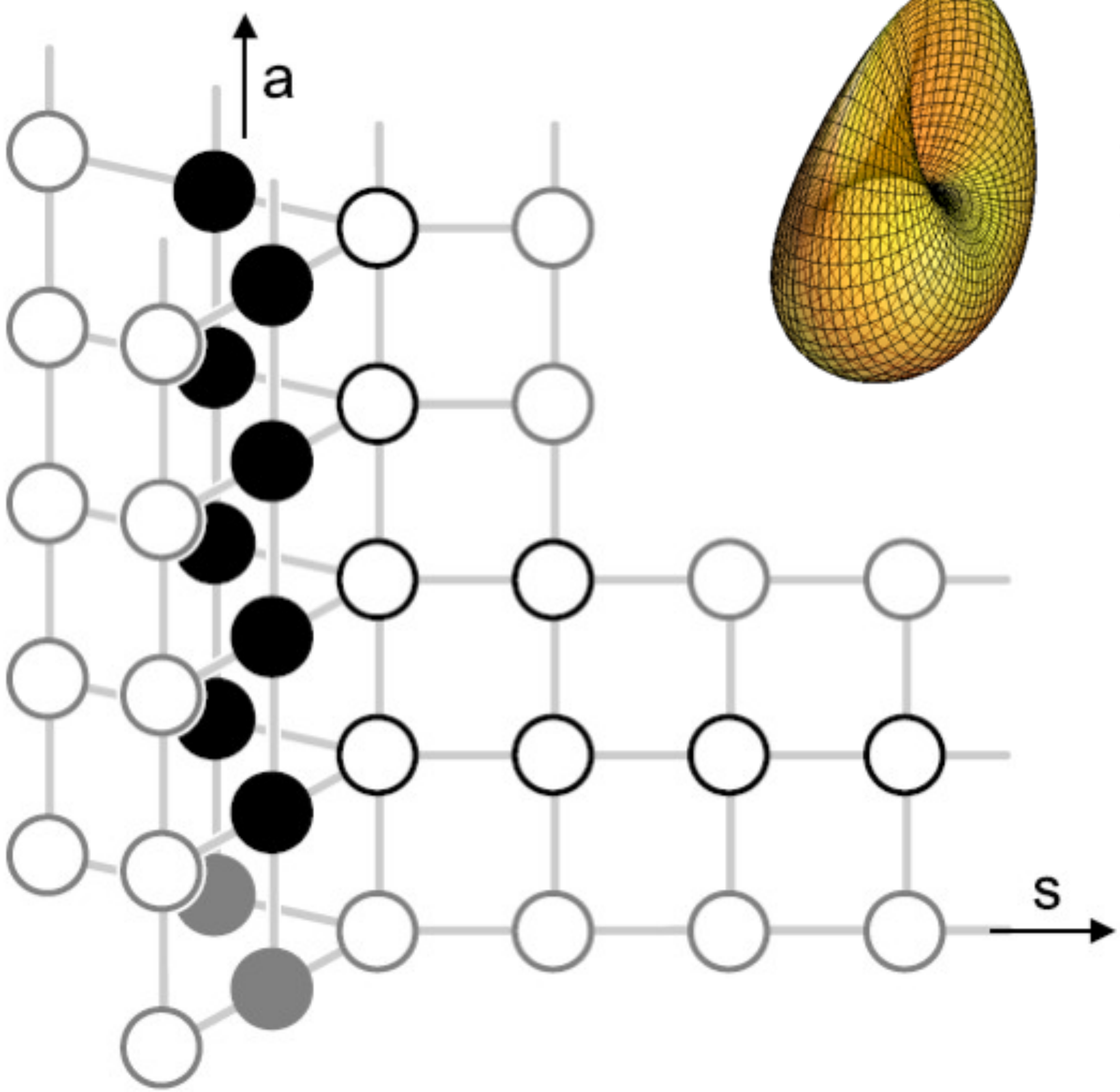
$$E(L) \simeq \sum_j m\cosh\pi\theta_j + \int dq \prod_{j=1}^{N} S(p_j,q)\, e^{-mLE(q)}$$

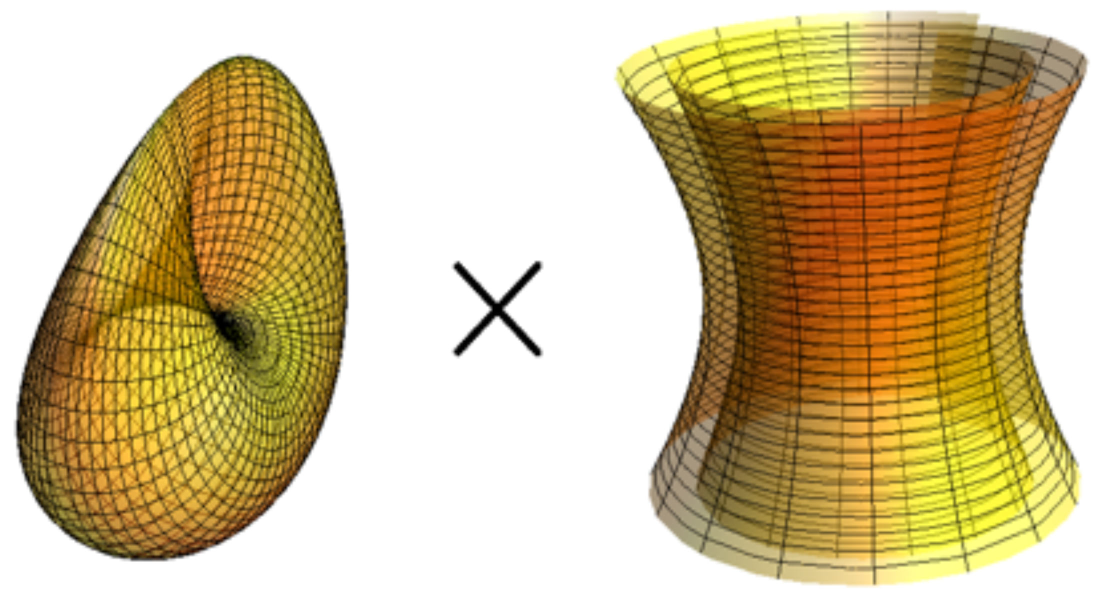
$$\frac{L}{2\pi}\Big(E_{\theta_{\underbrace{\{0,0,\dots,0\}}_{m\text{ times}}}} - E_0\Big) = \frac{1}{8\pi} e_0^2(L)\, m(m+2) \approx \frac{m(m+2)}{4\log(\omega/L)}$$

$$\frac{L}{2\pi}E_{\vec n_1\vec n_2\vec n_3\dots}(L) \simeq -\frac{1}{4} + \sum_{k=1}^{N}\sum_{\alpha=1}^{3} |n_k^{(\alpha)}|$$

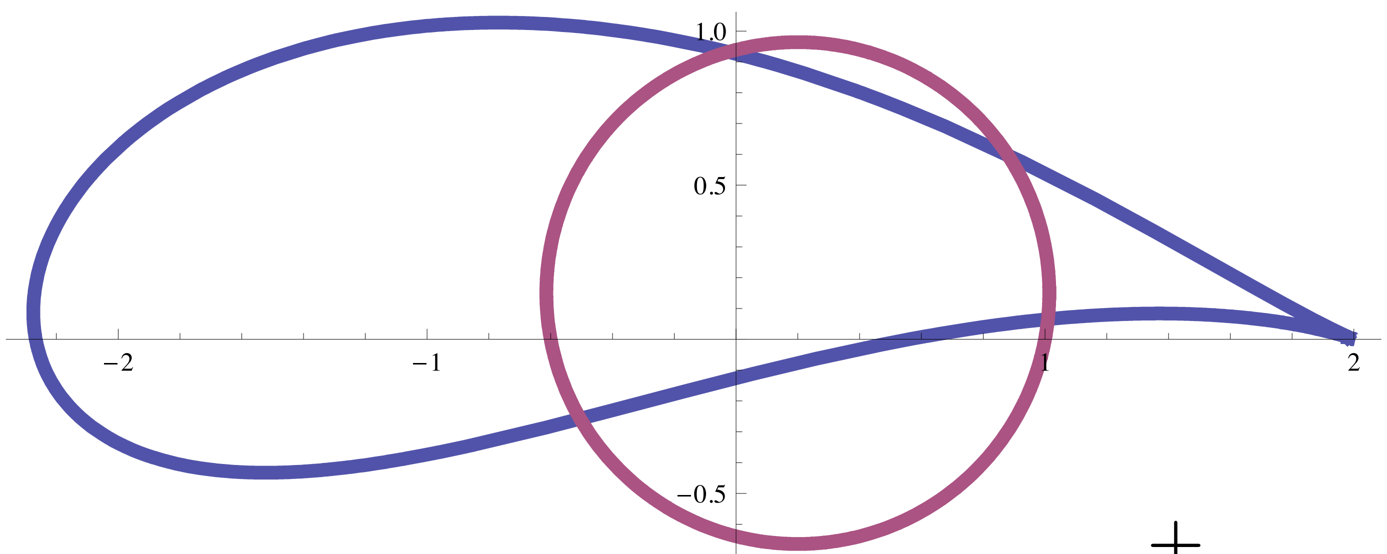
[Gromov, Kazakov, PV]

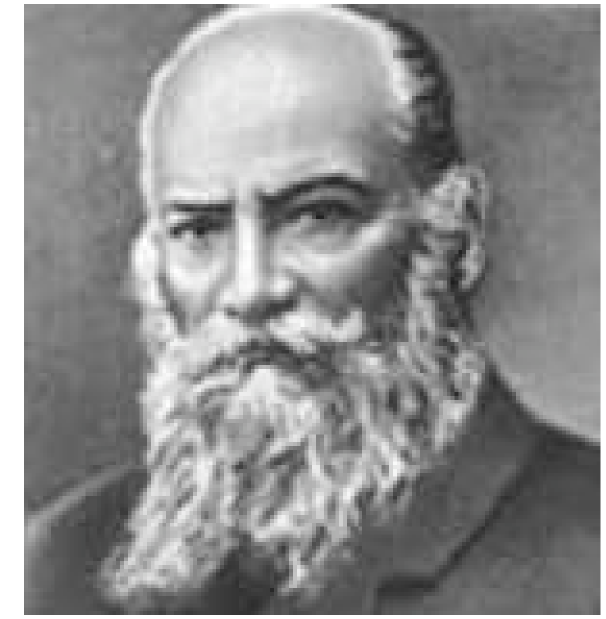
Full spectrum of planar ABJM





Explicitly



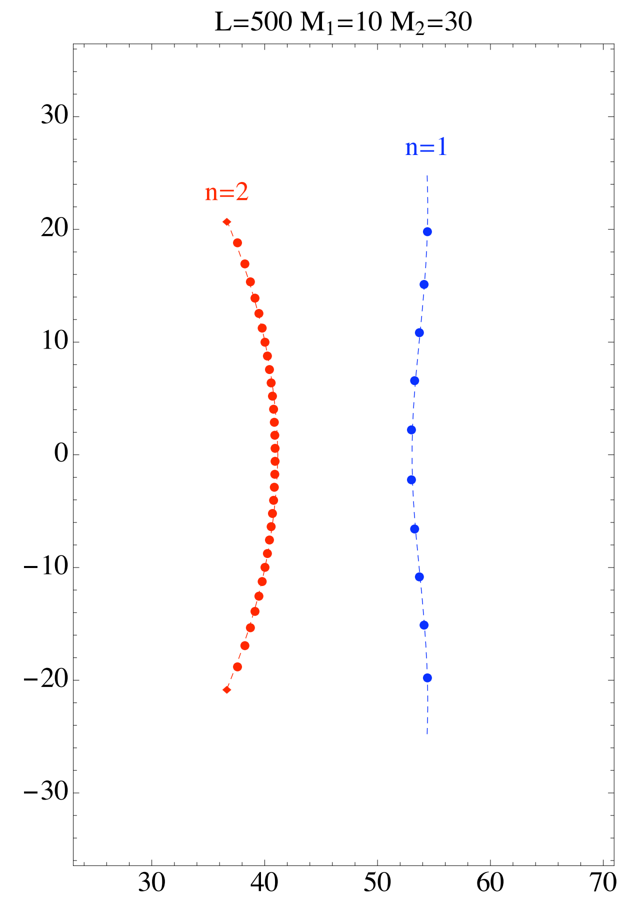


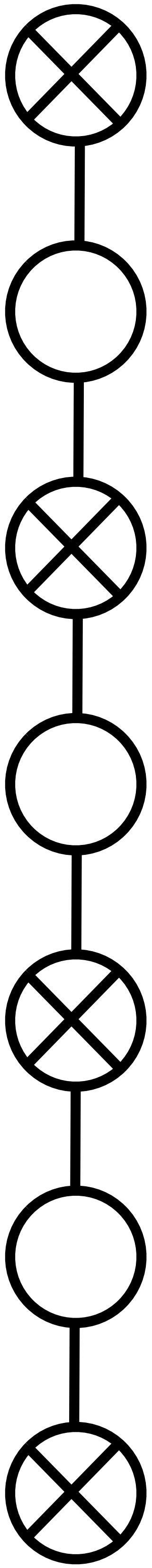
$$\frac{x^+}{x^-} = e^{ip}$$

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{g}$$

$$x^+ + \frac{1}{x^+} + x^- + \frac{1}{x^-} = \frac{2u}{g} = 2\left(x + \frac{1}{x}\right)$$

[Beisert, Staudacher; Beisert, Eden, Staudacher;
Beisert, Hernandez, Lopez, Aryutunov-Frolov]





$$1 = \frac{\mathbf{u_1} - u_2 + \frac{i}{2}}{\mathbf{u_1} - u_2 - \frac{i}{2}} \frac{\mathbf{x_1} - \frac{1}{x_4^+}}{\mathbf{x_1} - \frac{1}{x_4^-}},$$

$$1 = \frac{\mathbf{u_2} - u_2 - i}{\mathbf{u_2} - u_2 + i} \frac{\mathbf{u_2} - u_1 + \frac{i}{2}}{\mathbf{u_2} - u_1 - \frac{i}{2}} \frac{\mathbf{u_2} - u_3 + \frac{i}{2}}{\mathbf{u_2} - u_3 - \frac{i}{2}},$$

$$1 = \frac{\mathbf{u_3} - u_2 + \frac{i}{2}}{\mathbf{u_3} - u_2 - \frac{i}{2}} \frac{\mathbf{x_3} - x_4^+}{\mathbf{x_3} - x_4^-}$$

$$1 = \left(\frac{\mathbf{x_4^-}}{\mathbf{x_4^+}}\right)^L \frac{\mathbf{u_4} - u_4 + i}{\mathbf{u_4} - u_4 - i} \frac{x_1 - \frac{1}{\mathbf{x_4^-}}}{x_1 - \frac{1}{\mathbf{x_4^+}}} \frac{\mathbf{x_4^-} - x_3}{\mathbf{x_4^+} - x_3} \frac{x_7 - \frac{1}{\mathbf{x_4^-}}}{x_7 - \frac{1}{\mathbf{x_4^+}}} \frac{\mathbf{x_4^-} - x_5}{\mathbf{x_4^+} - x_5} \sigma_{\mathrm{BES}}(\mathbf{u_4}, u_4),$$

$$1 = \frac{\mathbf{u_5} - u_6 + \frac{i}{2}}{\mathbf{u_5} - u_6 - \frac{i}{2}} \frac{\mathbf{x_5} - x_4^+}{\mathbf{x_5} - x_4^-}$$

$$1 = \frac{\mathbf{u_6} - u_6 - i}{\mathbf{u_6} - u_6 + i} \frac{\mathbf{u_6} - u_7 + \frac{i}{2}}{\mathbf{u_6} - u_7 - \frac{i}{2}} \frac{\mathbf{u_6} - u_5 + \frac{i}{2}}{\mathbf{u_6} - u_5 - \frac{i}{2}},$$

$$1 = \frac{\mathbf{u_7} - u_6 + \frac{i}{2}}{\mathbf{u_7} - u_6 - \frac{i}{2}} \frac{\mathbf{x_7} - \frac{1}{x_4^+}}{\mathbf{x_7} - \frac{1}{x_4^-}},$$

+ some square roots
(for mirror theory)

$$\prod_{k=1}^{K_5} \frac{u_{6,j} - u_{5,k} + \frac{i}{2}}{u_{6,j} - u_{5,k} - \frac{i}{2}}$$

Dressing Phase

$$\sigma \equiv \exp\left[i\chi^{++} + i\chi^{--} - i\chi^{+-} - i\chi^{-+}\right] \,, \; \chi^{\pm\pm} = \chi(u \pm i/2, v \pm i/2)$$

In physical kinematics:

[Beisert, Eden, Staudacher] (Dressing phase proposal)
[Dorey,Hofman,Maldacena] (Integral representation)

$$\chi(u,v) = \frac{1}{i} \oint_{|z_1|=1} \frac{dz_1}{2\pi} \oint_{|z_2|=1} \frac{dz_2}{2\pi} \frac{1}{z_1 - x^{\rm ph}(u)} \frac{1}{z_2 - x^{\rm ph}(v)} \log \frac{\Gamma(iw_1 - iw_2 + 1)}{\Gamma(iu_2 - iu_1 + 1)}$$

In the mirror kinematics:

[Aryutunov, Frolov: Volin; Gromov,Kazakov,PV]

$$\begin{aligned}\hat{\chi}(u,v) \quad \equiv \quad & \frac{1}{i} \int_{|z_1|>1} \frac{dz_1}{2\pi} \int_{|z_2|>1} \frac{dz_2}{2\pi} \left[\frac{1}{(z_1 - x^{\rm mir}(u))} - \frac{1}{(z_1 - \overline{x^{\rm mir}}(u))} \right] \times \\ & \times \left[\frac{1}{(z_2 - x^{\rm mir}(v))} - \frac{1}{(z_2 - \overline{x^{\rm mir}}(v))} \right] \log \frac{\Gamma(iu_1 - iu_2 + 1)}{\Gamma(iu_2 - iu_1 + 1)}\end{aligned}$$