Note Title 3/17/2010

Scattering Amplitudes + The Grassmannian

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⋮

The Goal

Find a "weak-weak" dual theory to compute scatt. amplitudes, without usual evolution through spacetime.

Emergent Spacetime (Emergent QM?)

Start with $\mathcal{N}=4$ SYM – "Harmonic Oscillator of 21'st century".

Outline

(I) Cast of characters

★ $\underbrace{\text{Conformal} + \text{Dual}}_{\text{Yangian}}$ Conformal

Twistor Space

"Momentum" Twistor Space

★ Leading Singularities of Amplitudes

(II) Dual Theory for L.S. + the Grassmannian

$$\mathcal{L}_{n,k}[\mathcal{W}] = \underbrace{\frac{1}{\mathrm{vol}\,GL(k)} \int \frac{d^{k\times n} C_{\alpha a}}{(12..k)..(n1..k-1)}}_{\text{integral over } G(k,n),\ k\text{-planes in } n \text{ dim, with natural measure}} \times \prod_{\alpha=1}^{k} \delta^{4|4}(C_{\alpha a}\mathcal{W}_a)$$

Multidim. contour integral $\rightarrow$ residues + residue theorems.

Conjectures (huge amount of evidence + moral understanding why they must be true..)

- $\mathcal{L}_{n,k}$ generates all Yangian Invariants + relations between them!
- $\mathcal{L}_{n,k}$ contains all L.S.'s

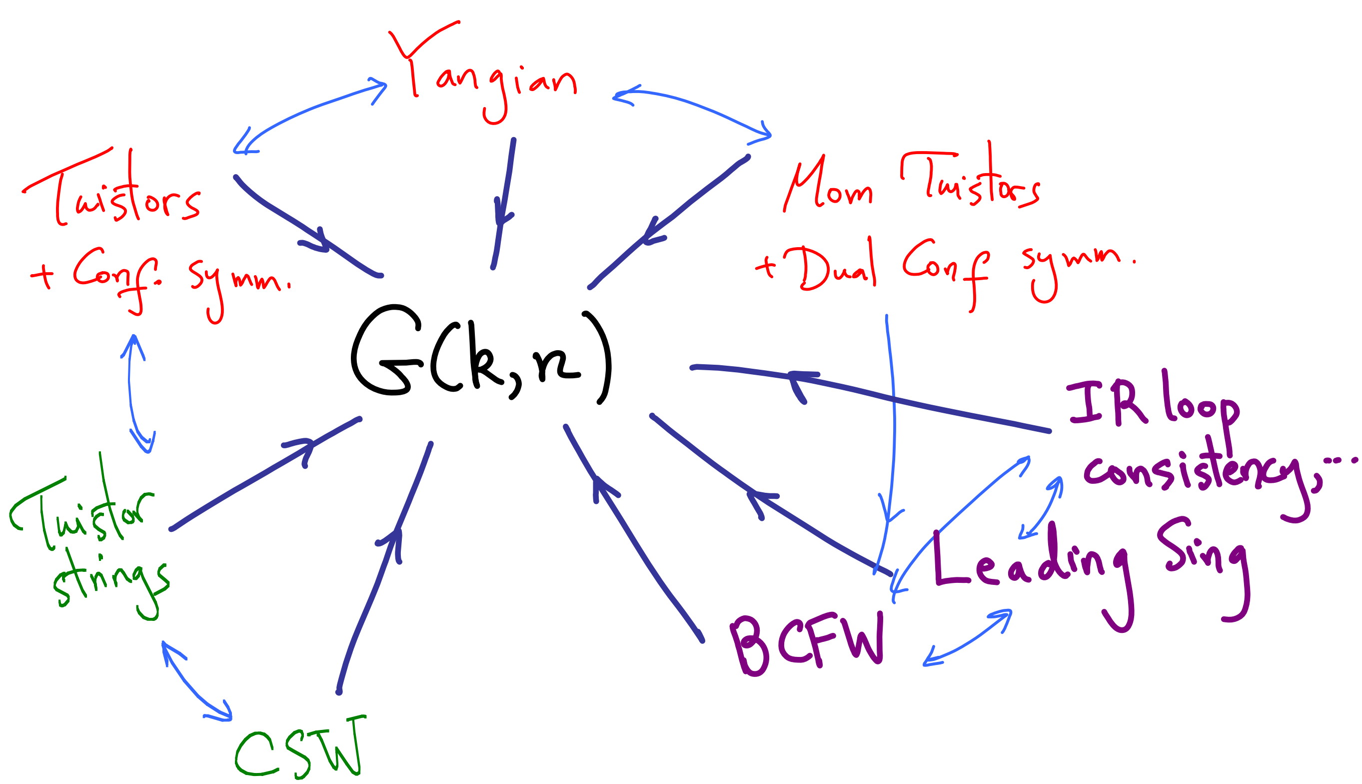
Local Spacetime

Consistent full loop ampl.

(Also very likely: all residues are L.S.'s)

III) Emergent Spacetime from $G(k,n)$:

- Natural Contours from "Particle Interpretation" in $G(k,n) \xrightarrow[\text{def}]{\text{contour}}$ CSW + light-cone gauge spacetime Lagrangian.
- Connected to twistor string theory by natural deformation with parameter "t"; t independence from same contour deformation as above!



(IV) G(k,n) knows about full loop amp.

- The rich set of relationships between leading sing. encoded in G(k,n) has a purpose in life – they naturally motivate + make possible the direct construction of a new set of perfectly well-defined, IR finite objects.

- These turn out to co-incide with the so called "ratio function" – stripping the M_{MHV} off the full amplitude!

$$M_{n,k}^{\text{full}} = M_{n,MHV}^{\text{full}} \, \mathcal{R}_{n,k}$$

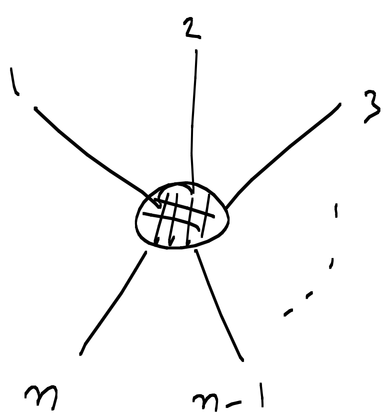
(Drummond, Henn, Korchemsky, Sokatchev)

Contain IR div.

Juan's talk tomorrow

Finite + Dual Conf. Inv.

Knowing all LS's + the relations they satisfy is a powerful guide to computing $R_{n,k}$ in a new way, directly - form of new way of computing $R_{n,k}$ strongly suggests an extended Grassmannian dual theory for this object.



Color-stripped, Planar.

$p = \lambda \cdot \tilde{\lambda}$. $|\tilde{\eta}\rangle = e^{Q \cdot \tilde{\eta}} |+1\rangle$.

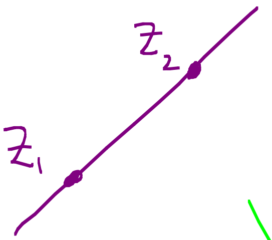
$M_n[\lambda_a, \tilde{\lambda}_a, \tilde{\eta}_a]$ $\quad a = 1, .., n$

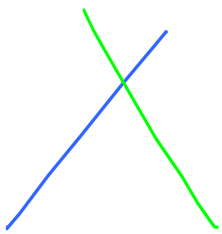
$$M_n = \sum_{k=0}^{n} M_{n,k}$$

net $\tilde{\eta}$ # 4k.
k - helicity gluons.

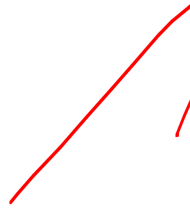
Twistor Space

• $Z = \begin{pmatrix}\mu \\ \lambda\end{pmatrix}$





Spacetime



$\mu = X\alpha$

null ray

• $X = \dfrac{\mu_1\lambda_2 - \mu_2\lambda_1}{\langle 12\rangle}$

• Y

• X $\longleftrightarrow$ null $\quad (X-Y)^2 = 0$

Connection with usual mom. space amps easy in (2,2) signature

$$\mathcal{M}[\mathcal{W}_a] = \int d^2\lambda_a\, e^{i\tilde{\mu}_a\lambda_a}\mathcal{M}(\lambda_a, \tilde{\lambda}_a, \tilde{\eta}_a)$$

$$\uparrow$$

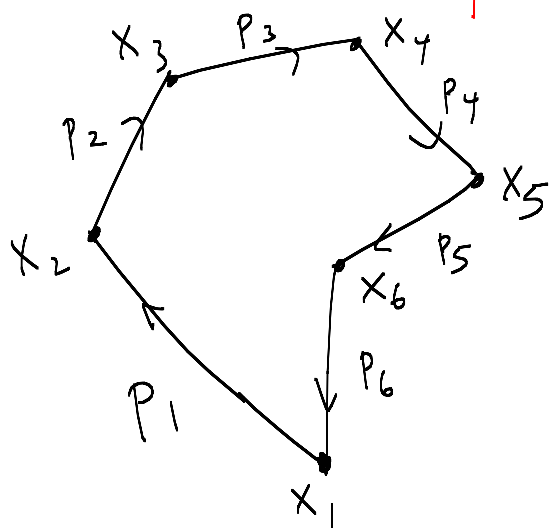
$$\begin{pmatrix}\tilde{\mu} \\ \tilde{\lambda} \\ \hdashline \tilde{\eta}\end{pmatrix}$$

Superconformal just $PSL(4|4)$

$\mathcal{M}(t_a\mathcal{W}_a) = \mathcal{M}(\mathcal{W}_a)$;

$\mathcal{W}_a$ live in $\mathbb{CP}^{3|4}$

Dual (Super) Conformal Symmetry

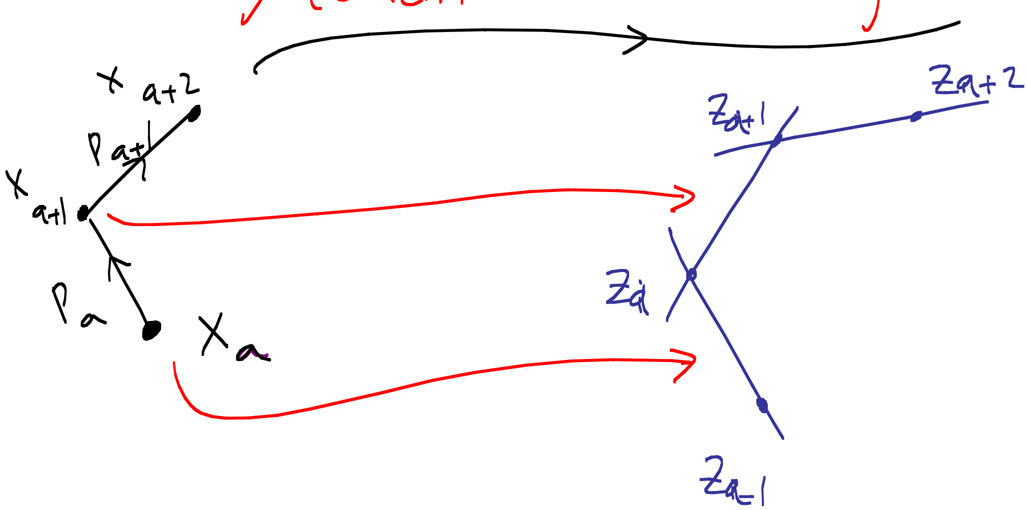


$$p_a = x_{a+1} - x_a$$

Conf. transf. on this x space.

(Alday, Maldacena ; Drummond, Henn, Korchemsky, Sokatchev, ..)

"Momentum" Twistor Space (Hodges)

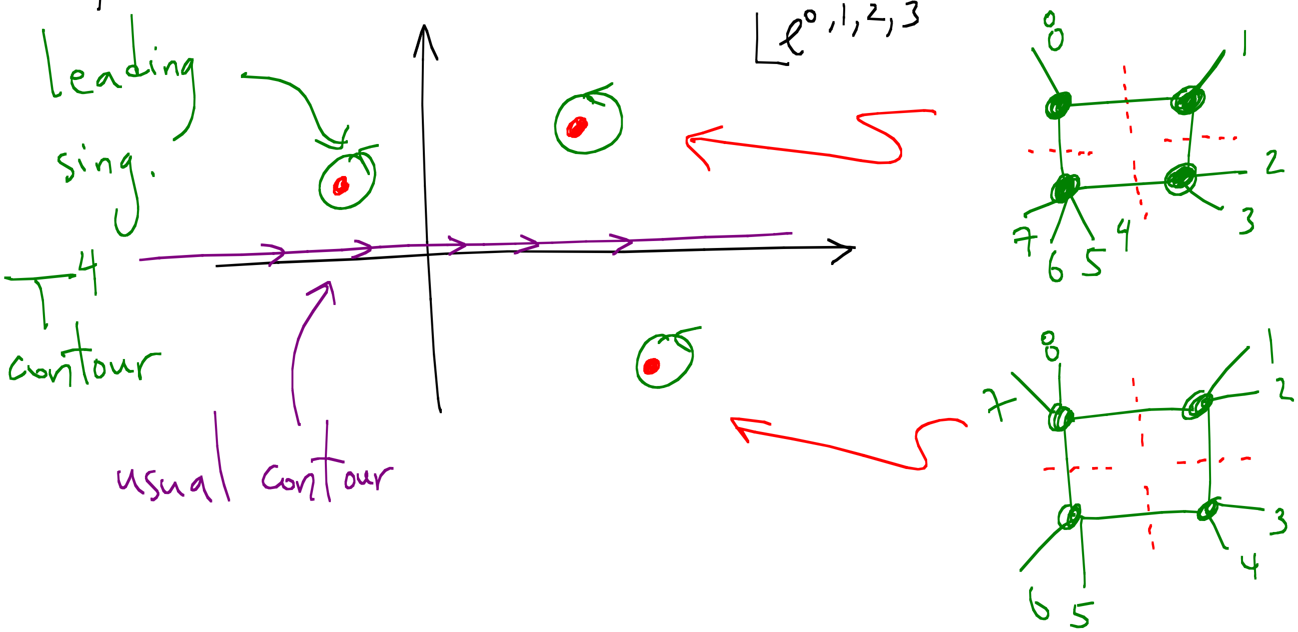


$$Z_a = \begin{pmatrix} \mu_a \\ \lambda_a \\ \eta_a \end{pmatrix}, \qquad \tilde\lambda_a = \frac{\langle a-1\; a\rangle \mu_{a+1} + \text{cyclic}}{\langle a-1\; a\rangle \langle a\; a+1\rangle}$$

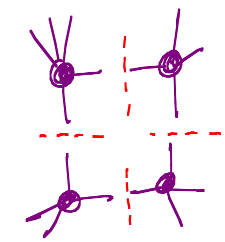
$$\tilde\eta_a = \frac{\langle a-1\; a\rangle\, \eta_{a+1} + \text{cyc.}}{\langle a-1\; a\rangle \langle a\; a+1\rangle}$$

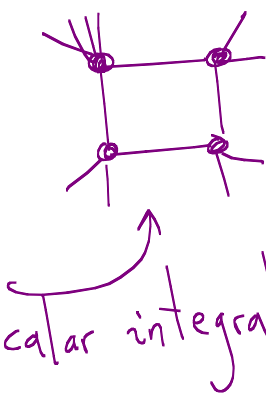
Leading Singularities

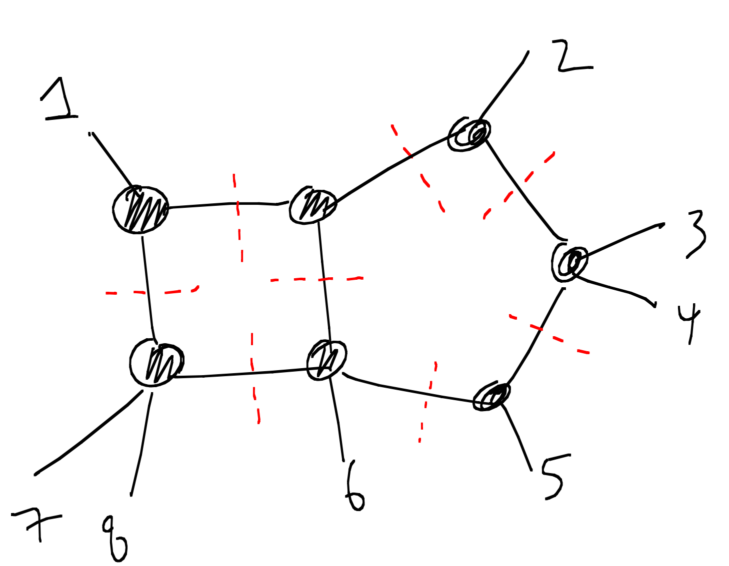
Amp. IR divergent @ 1-loop.

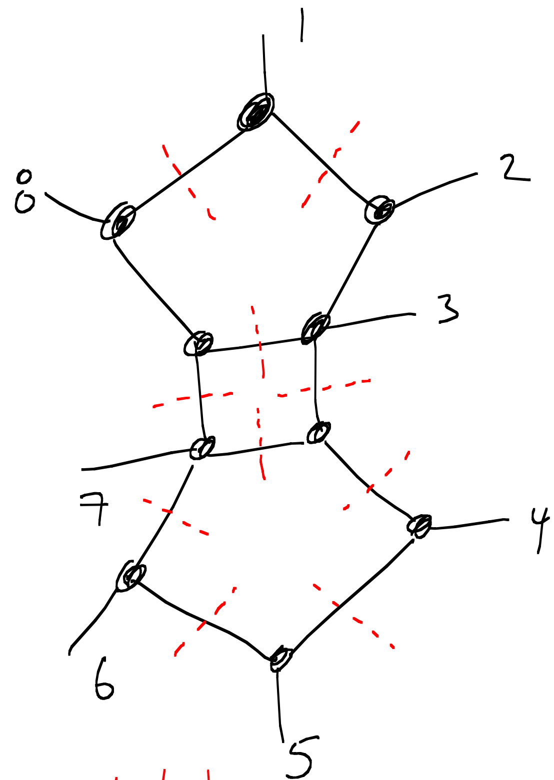


At 1-loop, the leading sing. completely determine the amplitude!

$A^{1\text{-loop}} = \sum$  BCF

$\times$ 



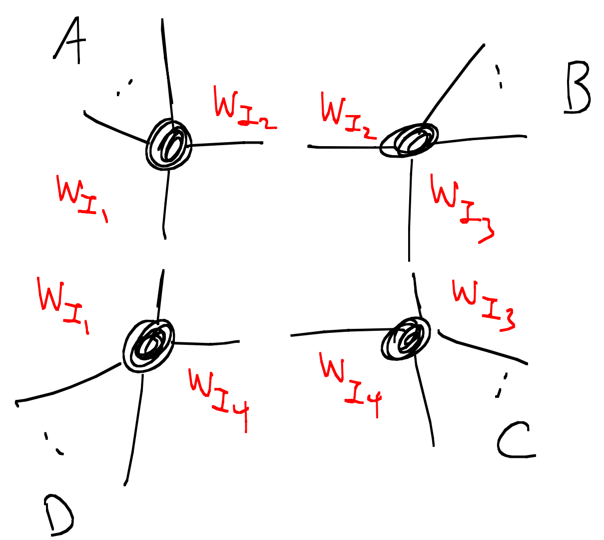


T^{4L} contours @ L loops. Completely IR finite.

Conjecture: determine full amplitude just like @ 1-loop. (Cachazo)

Leading Sing. in Twistor Space

Kaplan, Mason + Skinner



$$\int D^{3|4} W_{I_1} \cdots D^{3|4} W_{I_4} \; M_A(W_{I_1}, W_2, \ldots) \cdots M_D(W_{I_4}, W_{I_1}, \ldots)$$

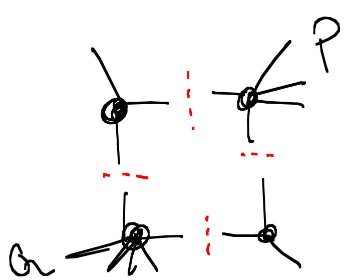
So L.S. are manifestly superconformal invariant.

Morally – given that the L.S. are the Loop amplitudes evaluated on natural compact contours, if the theory has Yangian symmetry only broken by IR div, the L.S. should be Yangian invariant. Dual superconf. invariance is not completely obvious though – but has been shown w/ a bit of work in many cases (+ may be proven in general soon).

- Leading Singularities are algebraic functions of the kinematical variables. They are recognizeable as coefficients of "Polylogs" in the full amplitude

e.g.

$$\mathcal{A}^{1\text{-loop}} \supset (LS) \times Li_2\left[1 - \frac{st}{P^2 Q^2}\right] + \cdots$$



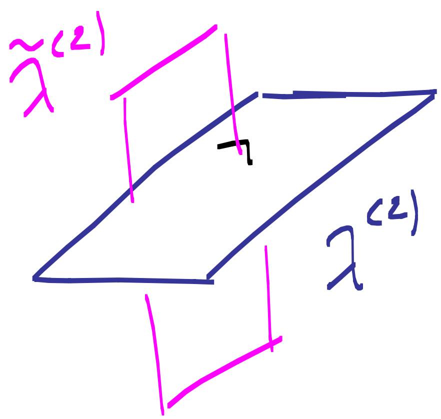
has a double discontinuity in s, t.

The Proposed Duality

- Start by thinking about momentum conservation afresh!

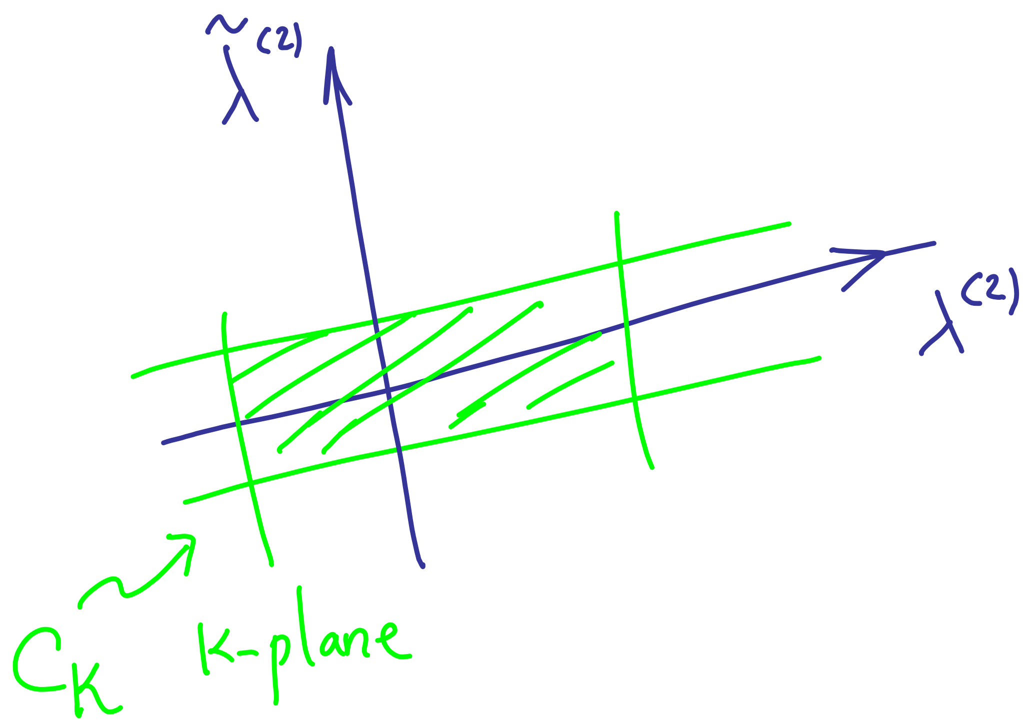
$$\lambda_\alpha^a \,, \quad \tilde{\lambda}_{\dot\alpha}^a$$

L^n



mom. conservation:

$$\lambda \cdot \tilde\lambda = 0.$$



Note: parity invariant since

$\lambda \longleftrightarrow \tilde{\lambda}$

k plane $\longleftrightarrow$ n-k plane

Note: impossible for $k = 0, 1, n-1, n$.

Good!

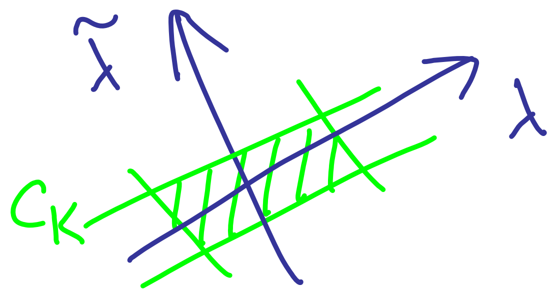
Eqns:

$$C = \begin{bmatrix} \vec{c}_1 \\ \vdots \\ \vec{c}_k \end{bmatrix} = C_{\alpha a}$$

Invariance under $GL(k)$ $\quad C_{\alpha a} \to L_\alpha^{\ \beta} C_{\beta a}$.

Space of k-planes in n-dim : Grassmannian $G(k,n)$

$$\dim G(k,n) = kn - k^2 = k(n-k)$$



$$\int d^{2\times k}\rho_\alpha \; \underbrace{\delta^2\left[C_{\alpha a}\rho_\alpha - \lambda_a\right]}_{C \text{ contains } \lambda} \; \underbrace{\delta^2\left[C_{\alpha a}\tilde{\lambda}_a\right]}_{C \text{ orthogonal to } \tilde{\lambda}} \; \underbrace{\delta^4\left[C_{\alpha a}\tilde{\eta}_a\right]}_{\text{SUSY partner}}$$

Motivation : preserve $GL(k)$

This object looks even simpler in twistor space:

$$\int d^2\lambda \; e^{i\tilde{\mu}\lambda} \int d\rho^{2\times k} \, \delta^2[\rho_\alpha C_{\alpha a} - \lambda_a] = \delta^2[C_{\alpha a}\tilde{\mu}_a]$$

so we have

$$\delta^{4|4}[C_{\alpha a} \mathcal{W}_a] \,, \quad \mathcal{W}_a = \begin{pmatrix} \tilde{\mu}_a \\ \tilde{\lambda}_a \\ \tilde{\eta}_a \end{pmatrix}$$

MANIFESTLY SUPERCONFORMAL INVARIANT

$k = 0, 1, n-1, n$: no possible planes.

$k = 2$ unique : $C = 2$ plane.

General k : integrate over all k-planes!

$$\int \frac{d^{k\times n} C_{\alpha a}}{(12\ldots k)(23\ldots k{+}1)\ldots(n1\ldots k{-}1)}$$

simplest + most natural GL(k) invariant measure!

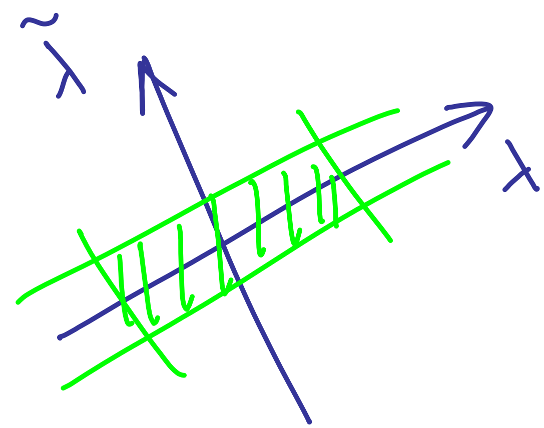
$(m_1 \ldots m_k)$: $k\times k$ minor of C made of columns $m_1, \ldots, m_k$.

So

$$\mathcal{Z}_{n,k} = \underbrace{\int \frac{d^{k\times n} C_{\alpha a}}{(12..k)\cdots(n1..k-1)}}_{\text{Simplest Natural integral over Grassmannian}} \underbrace{\prod_{\alpha=1}^{k} \delta^{4|4}\left[C_{\alpha a} \mathcal{W}_a\right]}_{\text{Simplest dependence on external particle data}}$$

Stripping off mom. conserving δ-function, we have

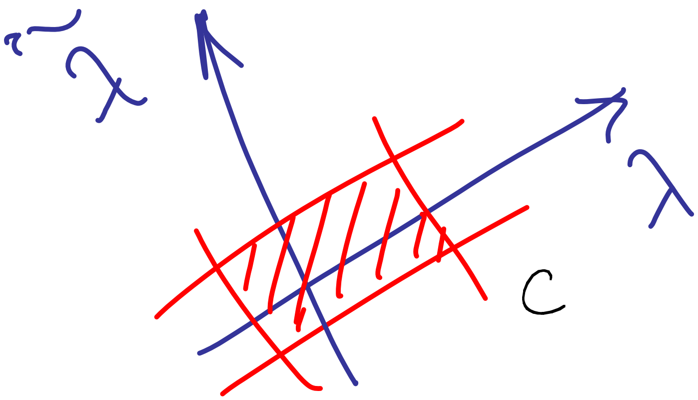
$$L_{n,k} = \int \frac{d^{(k-2)(n-k-2)}\tau}{(12\ldots k)(\tau)\ \cdots\ (n1\ldots k-1)(\tau)}$$



τ's parametrize freedom of $(k-2)$ plane in $(n-4)$ dim.

We think of this as multidimensional contour integral.

Manifest Dual Superconformal Invariance



C contains λ plane:

so really an integral over

$(k-2)$ planes in n dimensions!

Natural linear transformation mapping $k \times k$ minors to $(k-2)\times(k-2)$ minors ...

$$\mathcal{D}_{\alpha a} = Q_{ab} C_{\alpha b} = \frac{\langle a-1\, a\rangle C_{\alpha\, a+1} + \langle a+1\, a-1\rangle C_{\alpha a} + \langle a\, a+1\rangle C_{\alpha\, a-1}}{\langle a-1\, a\rangle\langle a\, a+1\rangle}$$

$$\tilde{\lambda}_a = Q_{ab}\, \mu_b$$

$$\tilde{\eta}_a = Q_{ab}\, \eta_b$$

$$\mathcal{Z}_a = \begin{pmatrix} \lambda_a \\ \mu_a \\ \hline \eta_a \end{pmatrix}$$

We discovered momentum twistors!

$$\mathcal{L}_{n,k} = \frac{\delta^4(\Sigma p)\,\delta^8(\Sigma \lambda\tilde{\eta})}{\langle 12\rangle \cdots \langle n1\rangle}$$

$$\times \int \frac{d^{(k-2)\times n}\mathcal{D}_{\alpha a}}{(1 \cdots k-2) \cdots (n\,1 \cdots k-3)}\; \delta^{4|4}\left[\mathcal{D}_{\alpha a}\mathcal{Z}_a\right]$$

Dual Conformal Invariance

(Mason + Skinner)

Note in this argument, the nature of the measure $(12..k)..(n1..k-1)$, with consecutive minors, was crucial.

It is possible to prove (Drummond + Ferro, Korchemsky + Sokatchev) that is we consider

$$\int d^{k\times n}C\, f(C)\, \delta^{4|4}(C.\mathcal{W}),$$

under action of Yangian generators, transforms into total derivative only if

$$f(C) = \frac{1}{(12..k)..(n1..k-1)}.$$

Then, residues are Yangian Invariant.

Let's go back to ordinary superconformal invariance.
We were led to general objects of the form

$$\int d^{k\times n} C \;\; f(C) \prod_{\alpha=1}^{k} \delta^{4|4}\left(C_{\alpha a} \mathcal{W}_a \right)$$

from our picture for momentum conservation. But actually, it is hard to come up with any other way of writing superconformal invariants! [In fact Korchemsky, Sokatchev claim every superconformal invariant is of this form].

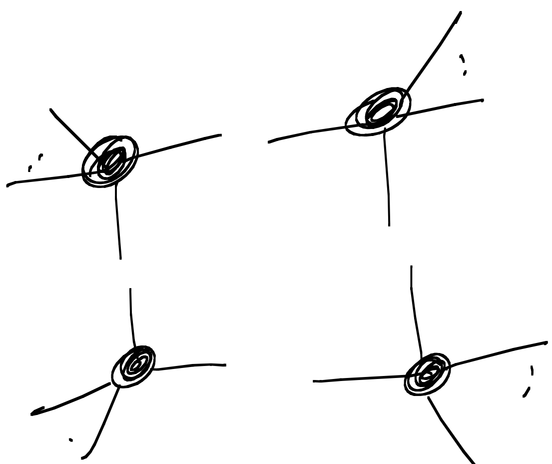
If this is true —

$$\int \frac{d^{k\times n} C}{(12\cdots k)\cdots(n1\cdots k-1)} \prod_{\alpha=1}^{k} \delta^{4|4}(C_{\alpha a}\mathcal{W}_a)$$

is the unique way of writing Yangian invariants.

Grassmannian Kinematics

Kaplan,
Mason + Skinner

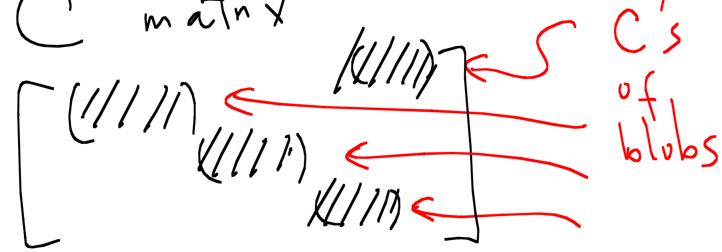


If every blob is of the form

$$\int f(C)\ \delta^{4|4}(C\cdot\mathcal{W}),$$

the integral over $\mathcal{W}_{\text{Internal}}$ is trivially done, and the big object has the same form with a bigger C matrix

$C_{\text{big}} \sim$



So we can identify the point in the big $G(K,N)$ going with this L.S.

Grassmannian Dynamics

Highly non-trivial that with the "good" measure, C_{big} is a pole with correct residue to match leading sing!

Direct proof seems very hard. Indirect proof could be:

- ☆ Solidly prove $\mathcal{I}_{n,k}$ generates all Yangian Invariants
- ☆ Solidly prove all L.S. are Yangian Invariant

} Very likely true

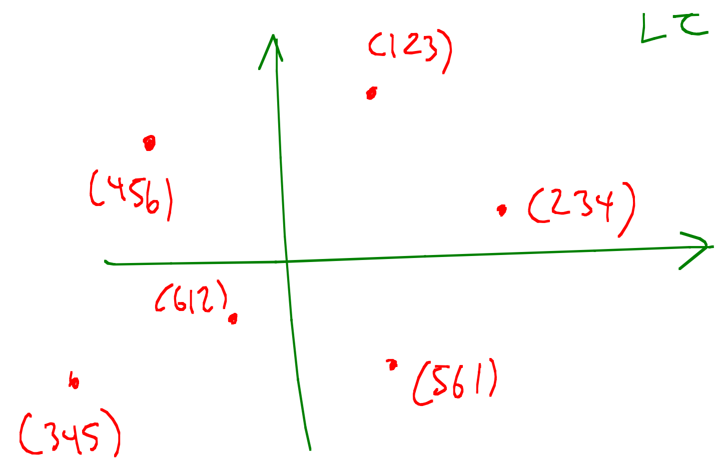
⟶ then we know it works + also how to identify each L.S. as a residue.

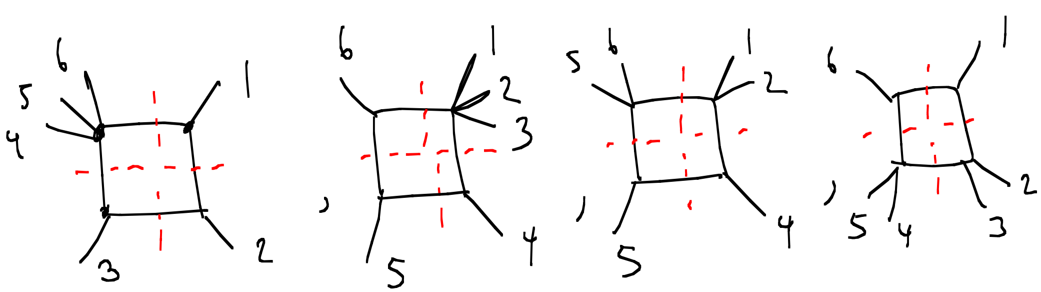
Quick Examples

First non-trivial $k=3,\ n=6$, NMHV, $(k-2)(n-k-2) = 1$ variable!

$$\mathcal{Z}_{6,3} = \int \frac{d\tau}{(123)(\tau) \ldots (612)(\tau)}$$

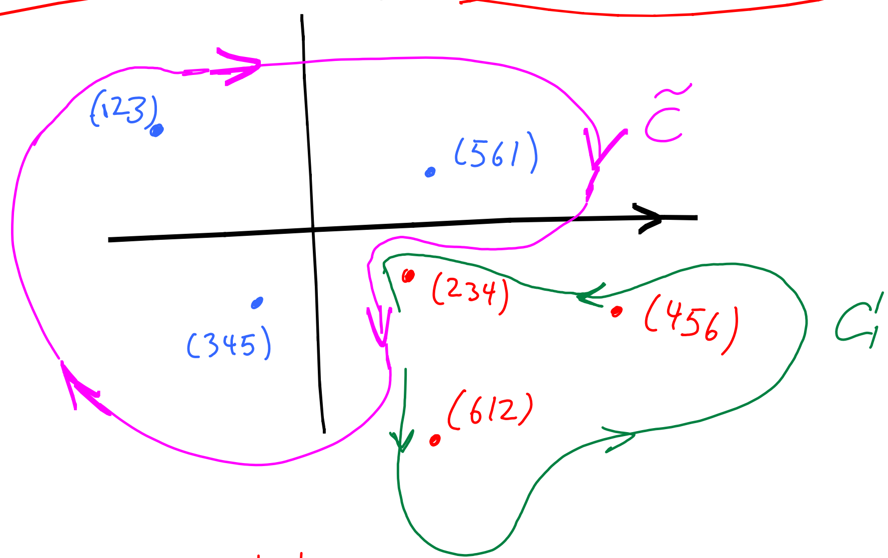
each minor linear in τ



e.g.  : $\{(123) - (456)\}$

[LS exhausted @ 1 loop!]

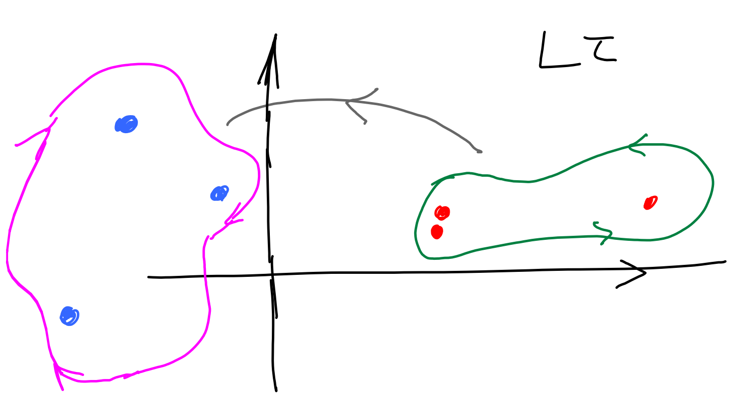
Tree Amplitude



[Unique choices respecting cyclic symmetry]

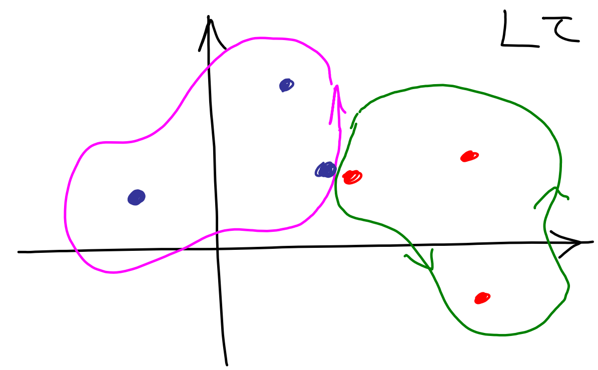
- residues : BCFW terms
- residues : P[BCFW] terms
- Cauchy : BCFW = P[BCFW] = Remarkable 6-term identity!

Spurious Poles



Contour can be deformed away from singularity

Physical Poles



Can't deform contour to avoid singularity

General Comments

- Residues of $\mathcal{L}_{n,k}$ can be very complex algebraic functions coming from solving multi-polynomial eqns.
- $\mathcal{L}_{n,k}$ not only gives the Yangian invariants $\leftrightarrow$ L.S; but also (amazingly non-trivial!) relations between them, following from more powerful higher-dimensional res. theorems. Hopeless to discover these relations any other way!

Emergent Spacetime

For 6 pts, the choice of residues yielding the tree amplitude was fixed by cyclicity. Not so for 7 pts + above. What Grassmannian principle can fix it?

"Particle Interpretation"

Find a single variety $(f_1, \ldots, f_m) = 0$ giving the contour, such that "adding a particle" is accomplished with the new variety $(f_1, \ldots, f_m, f_{m+1})$.

Can determine this from $5 \to 6$

no contour

known contours.

So e.g. for NMHV we write

$$\int d^{(n-5)}\tau \; \frac{h(\tau)}{\underbrace{f_1(\tau) \quad f_{n-5}(\tau)}_{\text{defines the variety}}} \quad \text{where} \quad \frac{h}{f_1 \cdots f_{n-5}} = \frac{1}{(123) \cdots (n12)}$$

Can extend to all tree amps.

Answer for NMHV

6pt: $(1) + (3) + (5)$

7pt: $(1)(2) + (1)(4) + (1)(6) + (3)(4) + (3)(6) + (5)(6)$

BCFW!

$= -\left[(2) + (4) + (6)\right]$

$(2)(3) + (2)(5) + (2)(7) + (4)(5) + (4)(7) + (6)(7)$

P(BCFW)!

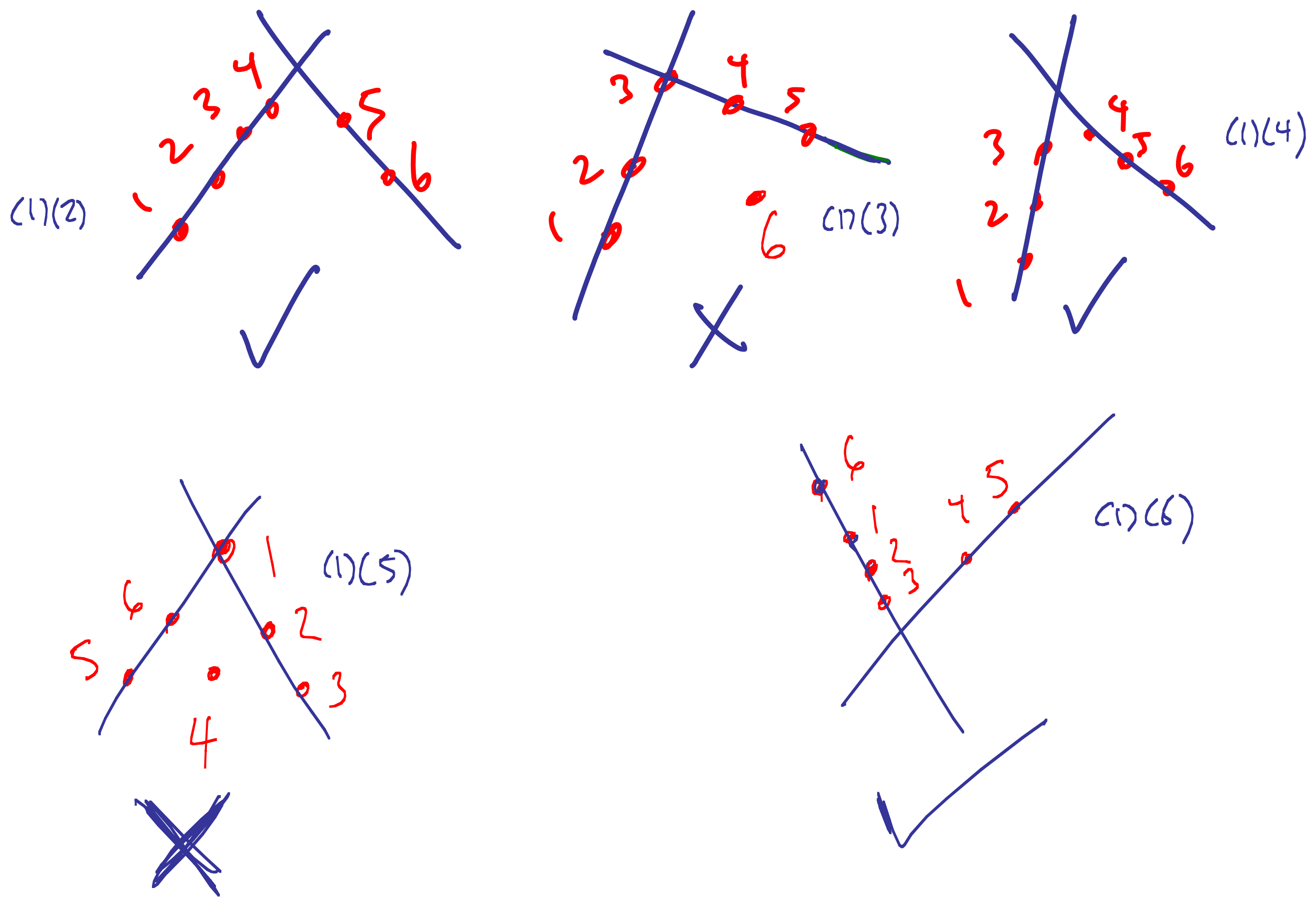
$=$ residue theorem

In general

$$\sum \underbrace{(e_1)(o_2)\cdots}_{n-5\text{ factors}} \quad \text{or} \quad (\text{even} \longleftrightarrow \text{odd})$$

Strikingly Combinatorial. Manifests Yangian symmetry. Absence of spurious poles $\Longleftrightarrow$ residue theorem guaranteeing equality of both terms. But can we see spacetime more directly?

$$(1) \longrightarrow (1)(2) + (1)(3) + (1)(4) + (1)(5) + (1)(6)$$



Check ✓ guys: CSW terms! (Essentially) light-cone gauge $\mathcal{L}$ Feynman rules!

In (1) + (3) + (5) : bad guys (with spurious poles!) cancel in pairs by antisymmetry of residues ; left with all $3 \times 3 = 9$ CSW diagrams! Very easy to prove general result for NMHV:

$$\text{Grassmannian} \xrightarrow[\text{contour deformation}]{\delta \text{ relaxing}} \begin{array}{c}\text{CSW + Spacetime } \mathcal{L} \\ \text{in Light-Cone gauge}\end{array}$$

More Generally

$$G(k,n) \xrightarrow[\text{contour deformations}]{\text{recursive } \delta\text{-relaxing}} \text{Spacetime Physics in Light-cone gauge}$$

[Passing through "Risager recursion" $\longleftrightarrow$ CSW]

Relation to Twistor Strings

One could have started @ the very beginning, with the Grassmannian picture

$$\int \frac{d^{k\times n} C}{\mathrm{vol}\, GL(k)} \, f(C) \prod_\alpha \delta^{4|4}(C_{\alpha a} \mathcal{W}_a)$$

impose $(k-2)(n-k-2)$ constraints

$$= \int_{\Sigma'} d^{2n-4}\sigma \prod_\alpha \delta^{4|4}(C_{\alpha a}(\sigma) \mathcal{W}_a)$$

"Particle interpretation": $\Sigma = (\text{something})^n$!

only choice something $= \mathbb{C}^2$, $(\mathbb{C}^2)^n / GL(2)$

has dim $2n-4$.

$$\mathbb{C}^2 \longrightarrow \mathbb{C}^k$$

$$\sigma = \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \sigma_1^k \\ \sigma_1^{k-1}\sigma_2 \\ \vdots \\ \sigma_2^k \end{pmatrix} \equiv \sigma^V$$

"Veronese Map"

So $\overline{V}$

$$C_{\alpha a}[\sigma] = \begin{pmatrix} \sigma_{(1)}^{\overline{V}} & \sigma_{(2)}^{\overline{V}} & \cdots & \sigma_{(n)}^{\overline{V}} \end{pmatrix}$$

has a natural "particle interpretation" in the Grassmannian.

$$\mathcal{T}_{n,k}[\mathcal{W}] = \frac{1}{\mathrm{vol}(GL(2))} \int \frac{d^2\sigma_1 \cdots d^2\sigma_n}{(12)\cdots(n1)} \, \delta^{4|4}\left[C_{\alpha a}[\sigma] \mathcal{W}_a \right]$$

is the most natural way to integrate over all k-planes with the Veronese "particle interpretation".

This is [RSV connected prescription for] Witten's twistor string theory.

(Dolan, Goddard)
(Spradlin, Volovich, Wen)
(Mason + Skinner), ..

For instance 6 pt NMHV, translating into the $G(k,n)$ picture

$$\int d\tau \, \frac{(135)}{(123)(345)(561)} \times \underbrace{\frac{1}{(123)(345)(561)(246) - (234)(456)(612)(351)}}_{\text{enforces 6 pts in } \mathbb{CP}^2 \text{ are on a conic!}}$$

In this very simple case, contour deformation easily shows equivalence to original $G(k,n)$ picture.

Can show equivalence directly Global Residue Theorem for 7, 8 pts, but involves increasing amounts of cleverness + gymnastics ....

... But there is a more indirect argument.

Taking our original "particle interpretation" contour

$$\int d\tau \, \frac{h(\tau)}{f(\tau)}$$

it is possible to smoothly deform $f(\tau)$ into one that matches the form from the connected prescription!

$$\int d\tau \, \frac{(135)}{(123)(345)(561)} \times \frac{1}{t\,(123)(345)(561)(246) - ((234)(456)(612)(351))}$$

$t = 0$: our form

$t = 1$: connected prescription

Trivial in this case – but generalizes

can take e.g.

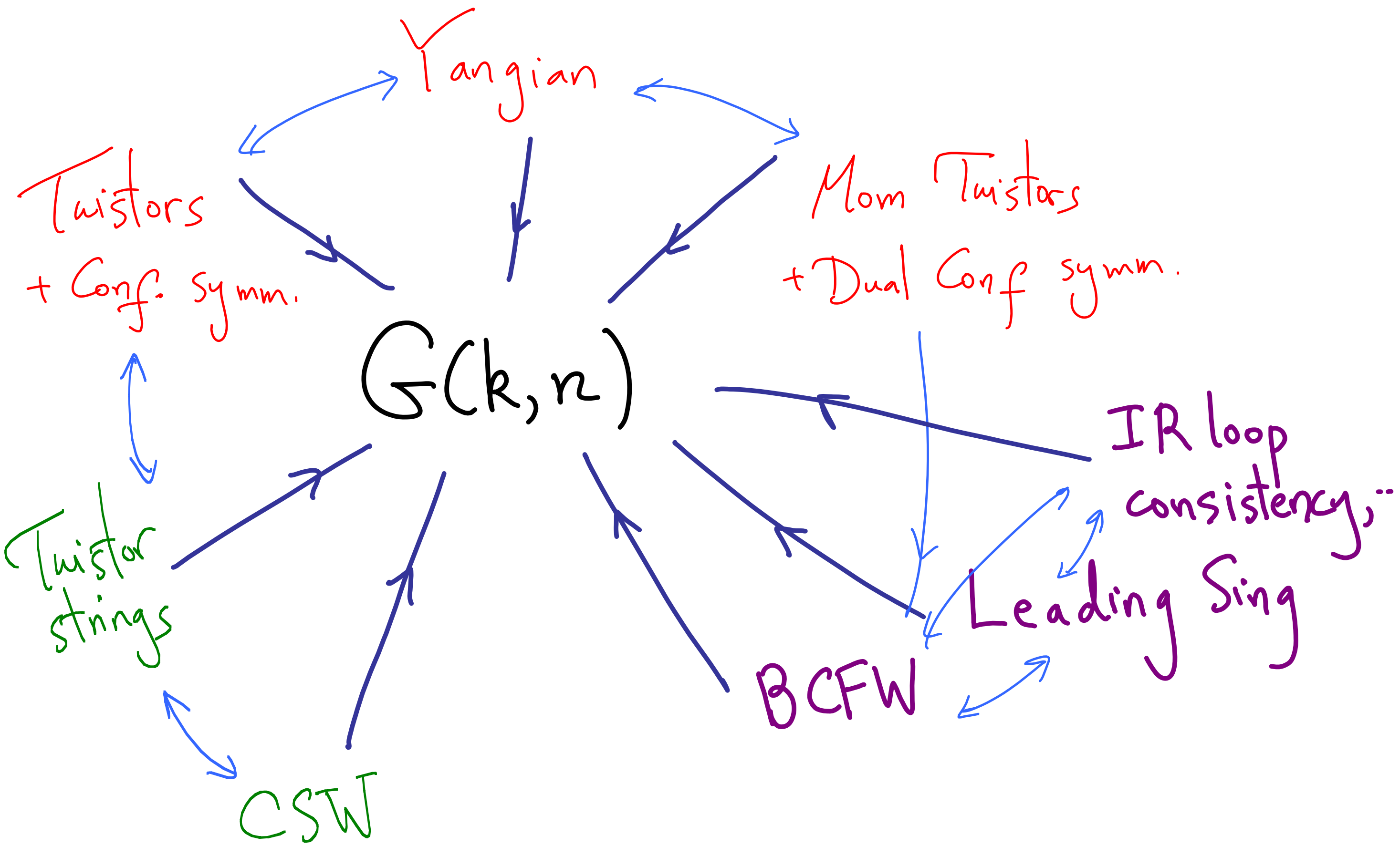
$$f_6^{(t_1)} = (456)(612)(234)(315) - t_1 (123)(345)(561)(246)$$

$$f_7^{(t_2)} = (567)(712)(235)(136) - t_2 (671)(123)(351)(725)$$

$(t_1 = 0, t_2 = 0)$: $\mathcal{Z}$ $\longrightarrow$ "Quarks"

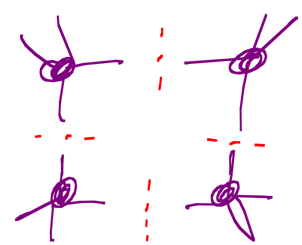
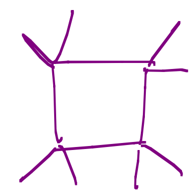
$(t_1 = 1, t_2 = 1)$: connected prescription $\longrightarrow$ "Hadrons"

+ t independ. follows in general from same δ-relaxing contour deformation argument as exposed CSW.


Yangian
Twistors
+ Conf. symm.
Mom Twistors
+ Dual Conf symm.
G(k, n)
IR loop
consistency, ...
Twistor
strings
Leading Sing
BCFW
CSW

Beyond Leading Singularities

All the remarkable relations between L.S. suggest they want to be further unified – but how?

1-loop natural idea $\sum$  $\times$  {Happens to be right!}

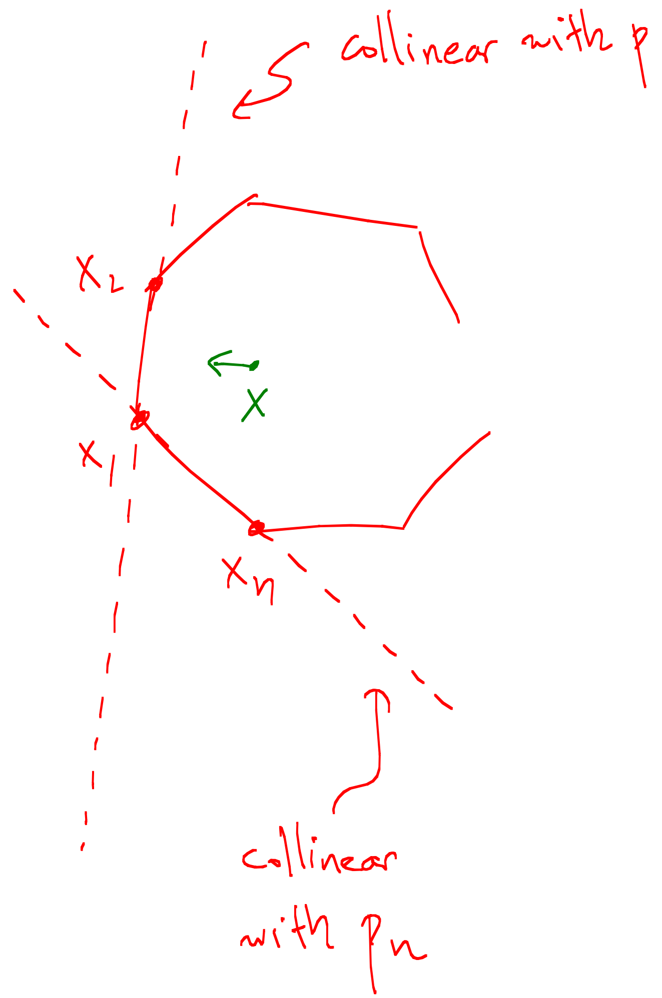
__But__, unpleasant because IR divergent. And, matches L.S. to integrals 1 – 1, doesn't take advantage of relations!

General 1-loop integral

$$\int \frac{d^4x \; \mathcal{N}}{(x-x_1)^2 \cdots (x-x_n)^2}$$

IR finite: $\mathcal{N}$ should vanish on these lines!

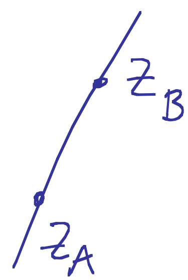
[IR finite $\Longrightarrow$ Dual Conformal Invariant]



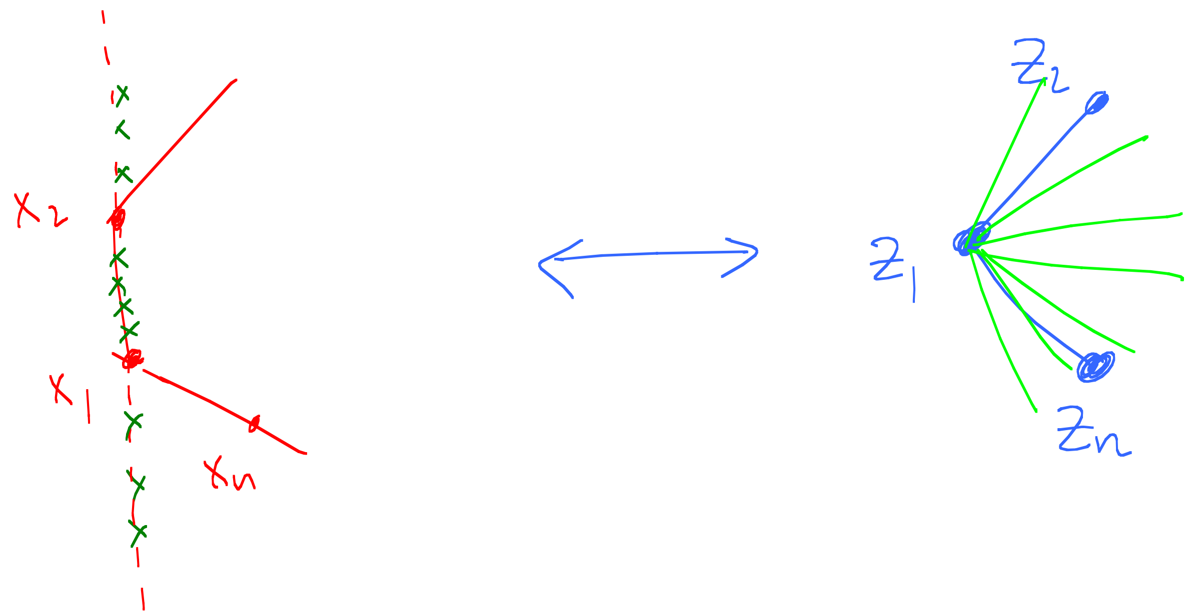
Rephrase in Momentum Twistor Language:

$$\int d^4x \quad \longleftrightarrow \quad \int_{G(2,4)} \frac{dZ_A \, dZ_B}{\text{vol}\, GL(2)}$$

$$x \quad \longleftrightarrow$$



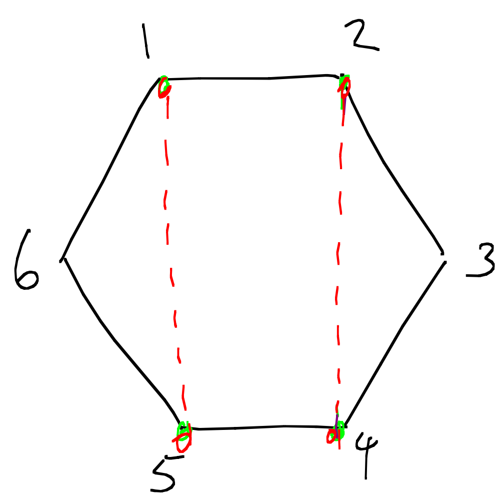
$$\frac{1}{(x-x_i)^2} \quad \longleftrightarrow \quad \frac{1}{\langle AB\, Z_{i-1} Z_i \rangle}$$



Lines in $(n\ 1\ 2)$ intersecting 1.

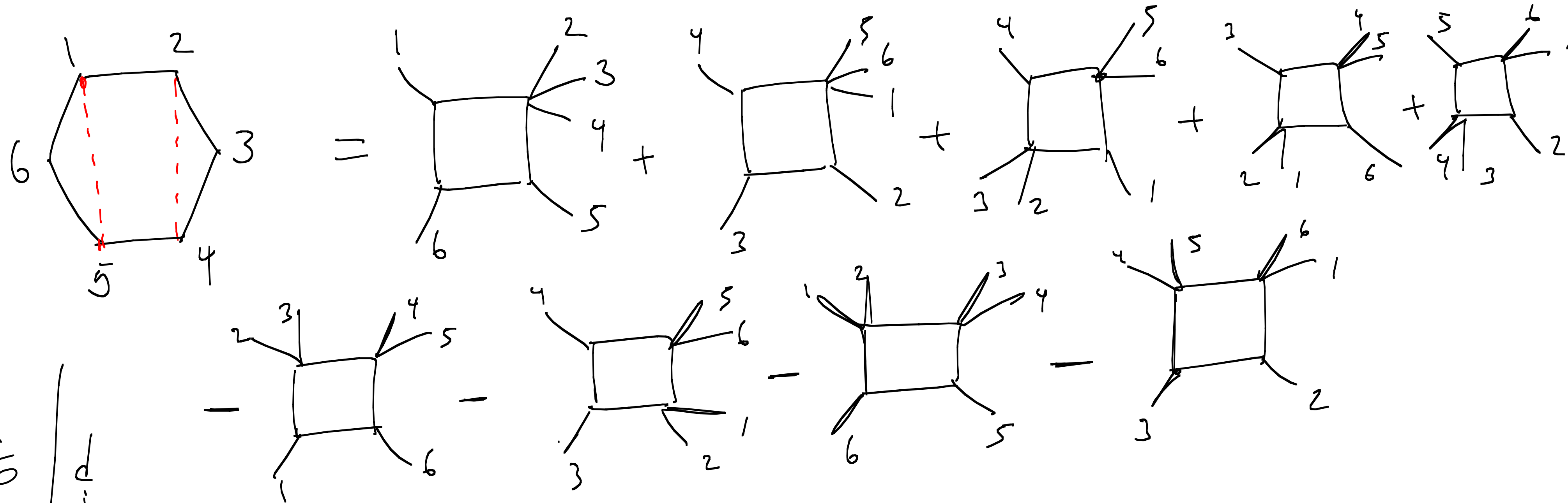
So $\mathcal{N}$ must vanish for these conf.

Easy! Starts w/ 6 points



$$I_6 = \frac{1}{\mathrm{vol}\,GL(2)} \int d^4 Z_A\, d^4 Z_B \, \frac{\langle AB15\rangle \langle AB24\rangle \langle 6312\rangle \langle 6345\rangle}{\langle AB12\rangle \cdots \langle AB61\rangle}$$

Of the $\binom{6}{4} = 15$ L.S, 6 vanish. Very remarkably, a residue thm shows that the rest are all ± 1!



compute directly

$$= \log u_1 \log u_2 + \mathrm{Li}_2(1-u_1) + \mathrm{Li}_2(1-u_2) + \mathrm{Li}_2(1-u_3)\ !$$

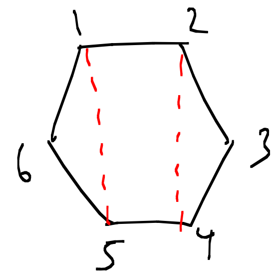
BOXES ARE BAD

- We have only 3 such objects but 15 L.S. Can we possibly match all LS with them, due to relations? Need 12 miracles!

Almost! Matching 3, we find

$$\Delta LS = M_6^{\text{tree}}, \ 12 \text{ times!}$$

So, we've found a beautiful finite object

$$\mathcal{R} = \left[(4)-(5)+(6)\right]$$  $+\ 2$ cyclic

such that

$$\mathcal{R} \quad + \quad M^{\text{tree}} \times \Big(\underbrace{\text{combination of boxes}}_{\text{IR div.}} \Big)$$

matches all the L.S. of the amplitude.

This object had previously been found by Drummond, Henn, Korchemsky + Sokatchev "in the opposite direction":

$$\mathcal{M}_{n,k} = \mathcal{M}_{n,MHV} \times \mathcal{R}_{n,k}$$

$$\left(= \mathcal{M}_{tree,MHV} \times W \times \mathcal{R}_{n,k} \right)$$

Finite + Dual-Conf. Inv.

but $\mathcal{R}$ was produced in the "9 box" form.

But our point of view suggests that we start from $G(k,n)$, the leading singularities + all their known relations, + find a dual theory to compute it directly.

At 1-loop: simple counting. There are a basis of $\frac{1}{2} n (n-5)^2$ such manifestly IR finite integrals — (sums of between 3 → 9 boxes). $n(n-4)$ residue thm. miracles must happen for Δ L.S. $= M^{\text{tree}}$.

They happen of course (general argument @ 1-loop due to Brandhuber et. al.)

Then extend to higher loops — <u>many</u> relations unexploited!

Of course what we really want to do is put the LS + the integrals together: striking that they are written already as

$$\int_{G(2,4)} \times \int_{G(k,n)}$$

- Striking too, that the modern understanding of the wonderful properties of dilogs + Polylogs — so ubiquitous in loop amplitudes — associates these functions with volumes of Polytopes in the Grassmannian [Gelfand + Macpherson '82, '88, Goncharov, ..]

- So it appears that a well-defined an attainable goal is a theory for

$$\frac{\mathcal{M}_{n,k}}{\langle \text{Wilson loop} \rangle} = \mathcal{R}_{n,k}.$$

Spectacular recent progress here @ strong coupling → Juan tomorrow.

- But I suspect this separation between "$\langle W \rangle$" and "$\mathcal{R}$" is artificial — we will have to reformulate + understand the IR div. in the end!