Towards a classification of four dimensional N=2 gauge theories

What are N=2 gauge theories?

Renormalizable Lagrangians are constrained

- Pick a gauge group G, a matter representation R
- Pick gauge couplings, mass parameters
- A Lagrangian N=2 theory is labeled by G and R

Is this a classification?

- A few exotic theories were long know to exist.
- No Lagrangian, exceptional flavor symmetries E_6 , E_7 , E_8
- Can be used as ``matter'' by gauging flavors
- Hidden S-dualities relate different G and R

Computable quantities

Many protected quantities are exactly known

- IR Lagrangians: Seiberg-Witten theory
- Donaldson-Witten theory
- S-dualities.
- Nekrasov's instanton partition function.
- S^4 correlation functions from Pestun's localization
- IR Lagrangian on $R^3 \times S^1$: hyperkahler sigma models
- Superconformal indices

A dictionary ?

Can we organize N=2 theories in a way that

- Allows a uniform treatment of all theories
- Provides an algorithm to compute all protected quantities
- Does not require a UV Lagrangian

Can we label theories so that

- Distinct theories have distinct labels
- A ``well constructed'' label predicts a theory's existence

A peculiar construction

From 6d to 4d

- Take the maximally supersymmetric 6d field theory
  - labeled by ADE group G,
  - no Lagrangian, M-theory description

- Twisted compactification on C x $R^{3,1}$
  - Riemann surface C, any genus g
  - N=2 SUSY in 4d
- Add n defects on p_i x $R^{3,1}$, p_i points on C
  - defects labeled by SU(2) embedding in G
- Add defect lines

A powerful tool

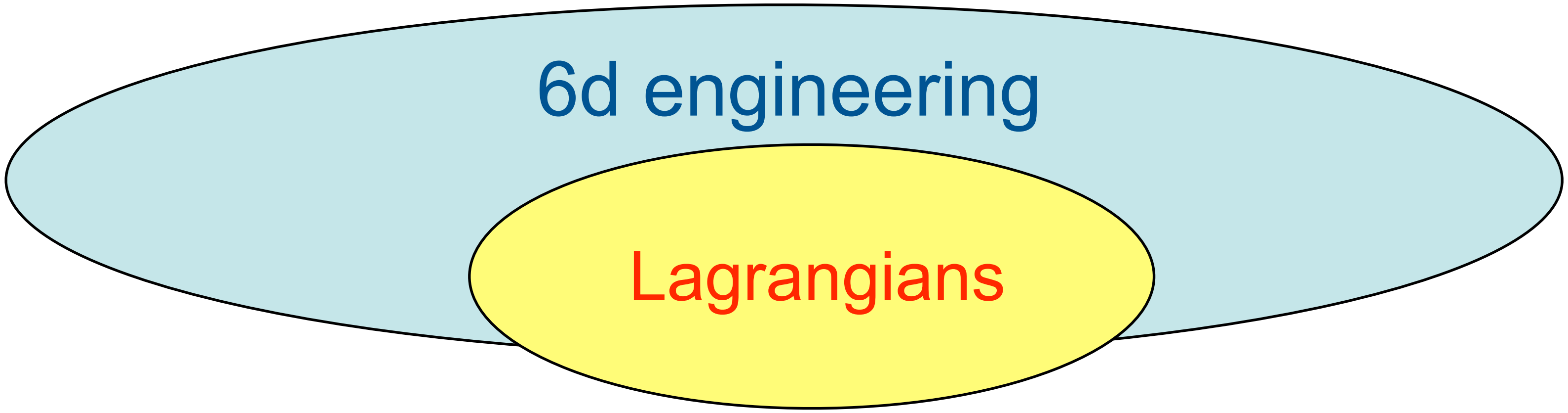
Flow to IR to get a 4d N=2 theory

- 4d theory labeled by ADE group and 2d data:
  - genus, number and type of punctures and defects.

Much can be computed from 2d data:

- SW curve as a cover of $C_{g,n}$
- S-dualities as Moore-Seiberg groupoid of $C_{g,n}$
- Z_{inst} as conformal block on $C_{g,n}$
- S^4 correlation functions from Toda 2d CFT on $C_{g,n}$
- $R^3 \times S^1$ calculations from Hitchin system on $C_{g,n}$
- M-theory holographic dual
- Superconformal index from TFT on $C_{g,n}$

A general construction?



Some 4d Lagrangians are missing a 6d parent

- What else are we missing?

Different 6d constructions may give same 4d theory

- How much are we overcounting?

Surface operators

BPS surface operators in N=2 theories

- Dimension 2 defects: non-dynamical ``cosmic strings''
- Preserve (2,2) SUSY in 2 dimensions!
- (2,2) theories in 2 dimensions are well studied
  - Calabi Yau sigma models

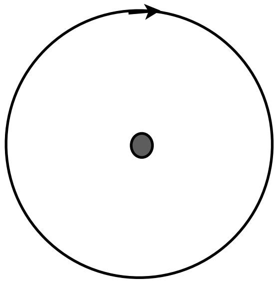
Borrow results for surface operators

- Quantum corrections to twisted chiral rings
- tt* geometry

Abelian surface operator

Surface operator: codimension 2 defect

- Gauge fields have a well-defined monodromy around it
  - Gukov-Witten
- $A = \alpha\, d\theta$



- Can add 2d ``theta angle'' $\eta \int F$
- (α, η) are circled valued
- exchanged by abelian e.m. duality
- $t_{2d} = \eta + \tau_{4d}\, \alpha$ lives on a torus of parameter τ_{4d}

Example: SU(2) surface operator

Embed a U(1) surface operator in SU(2)

- Make a choice of U(1) in SU(2)
- Use U(1) surface operator, parameter t_{UV}

Defect breaks SU(2) to U(1)

- Coupling to SU(2)/U(1)=CP^1 auxiliary non-dynamical dof
- Quantum-mechanically, it may become dynamical!
- CP^1 (2,2) 2d sigma model coupled to 4d gauge fields
- t_{UV} is Kahler parameter

IR Lagrangians

Coulomb branch

- Branch of vacua with massless abelian gauge fields only
- Complex adjoint scalars in gauge multiplets have vevs
  - $\langle \mathrm{Tr}\, \Phi_i^n \rangle = u_{i,n}$
  - Higgs gauge group to Cartan torus
  - Gauge fields + scalars a(u)

IR Lagrangians

Effective IR Lagrangian

- All determined by effective prepotential F(a)
- gauge couplings $\tau_{IR} = d^2F/da^2$

BPS particles

- BPS bound $M \geq |Z|$
- Central charge $Z = Q_m a_D + Q_e a$
  - $a_D = dF/da$ $\tau_{IR} = da_D/da$

Periods and Seiberg-Witten

Seiberg-Witten curve:

- Construct a family of curves $\Sigma(u)$ such that
- $\tau_{IR}(u)$ is period matrix of $\Sigma(u)$
  - enforces positivity of gauge couplings

- Pick a meromorphic differential $\lambda(u)$ on $\Sigma(u)$ such that
- $[a(u), a_D(u)]$ are periods of $\lambda(u)$ on cycles of $\Sigma(u)$
- Periods of $d\lambda/da = n + \tau_{IR}\, m$

IR Lagrangians

Coulomb branch and surface operators

- Basic assumption: no massless dof on surface operator
  - 4d scalar vevs give masses to 2d fields
  - massive 2d theory with choice of vacua i.

Effective IR Lagrangian

- IR couplings of abelian gauge fields to surface operator
- Controlled by effective twisted superpotential $W_i(a)$
- $t_{IR} = dW/da$

On IR twisted superpotentials

IR twisted superpotential

- Contains 2d couplings of 4d fields.
  - $t_{IR} = \eta_{IR} + \tau_{IR}\,\alpha_{IR}$
  - Periodicity captures 4d gauge couplings τ_{IR} !

Surface operator parameter space

- t_{UV} is an example of twisted coupling z
- To change in UV, add δz X to twisted superpotential
- x is a twisted chiral operator.
  - IR superpotential W(a, z). <X> = x = dW/dz

On IR twisted superpotentials

Surface operator parameter space

- z lives in some space C
- Choice of vacuum: (x,z) lives on a space $\Sigma(u)$, covers C
- $\lambda(u) = x\, dz$: one-form on $\Sigma(u)$
  - $dt_{IR}/dz = dx/da$

Paths in parameter space

- Consider the period of $dx/da\, dz$ on closed path in $\Sigma(u)$
- Integral must be a period of t_{IR}: $\Delta t_{IR} = n + \tau_{IR}\, m$

On IR twisted superpotentials

2d BPS particles

- BPS bound $M \geq |Z_{ij}|$
- $Z_{ij} = W_i - W_j =$ integral of $\lambda(u)$ from (x_i,z) to (x_j,z)

- Different paths: particles of different charges
- Difference of paths: periods of $\lambda(u)$ equal $Q_e\, a + Q_m\, a_D$

Seiberg-Witten geometry from surface operator!

Seiberg-Witten curve from surface operators

A simple dictionary

- C is Kahler moduli space of any surface operator
- “SW curve” is a cover of C, fiber is space of vacua.
- “SW differential” is x dz if couplings are δz x
- Different surface ops give very different C, x dz!

Is it a general statement?

- Given at least a gauge field...
- There is always a one-parameter family of surface ops

Surface operators and 6d

Simple surface operators in 6d

- z labels a point in $C_{g,n}$ i.e. $C = C_{g,n}$
- $\Sigma(u)$ is indeed SW curve

6d engineering produces a pair:

- A 4d theory together with a choice of surface operator
- 6d constructions distinguished by surface operator

Does it clarify the dictionary?

Inverting the map?

Pick a 4d theory T

And a one-parameter surface operator

- Compute C and behavior of Σ(u) and λ(u)
- Wrap appropriate 6d theory on C
- Engineer a 4d theory T’

Is T the same as T’?

- T’ may have a bigger Coulomb branch.
- T Coulomb branch is embedded in T’
- Conjecture: T is contained in T’ (Higgsing?)

tt* and Hitchin system

Analogy with 2d (2,2) theories on R x S^1

- Theory on a circle has vacua
- Bundle of vacua over parameter space
  - Natural hermitean connection D_i
  - Chiral ring acts on Ramond vacua
    - $x_i \, v_a = (c_i)_a{}^b \, v_b$

- D_i, c_i satisfy Hitchin-like equations
  - $[D^*,c]=0$ $[D^*,D] = [c^*,c]$ $[D,c^*]=0$
  - $[D,D]=0$ $[c,c]=0$ $[D^*,D^*]=0$ $[c^*,c^*]=0$

tt* and Hitchin system

4d theory on $R^3 \times S^1$

- Coulomb branch is bigger: Wilson lines on S^1
- It is a hyperkahler manifold M.
- For T' it is moduli space of solutions to Hitchin equations

M(T) is a hyperkahler submanifold of M(T')

- They are rare!
- Fixed points of symmetries, loci of ADE singularities...
- Strong constraint on T vs T' relation

Conclusions, future directions

To classify and study general N=2 theories...

- Look at surface operators
- Use tools from 6d engineering

Can we learn about N=1 theories?

- Many new N=1 theories from deformation of N=2
- Surface operators are BPS in N=1 as well