The Virtue of Defects in Gauge Theories and 2d CFTs

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Introduction

- Solving exactly four dimensional gauge theories is out of reach
- The constraints imposed by supersymmetry have resulted in novel insights into the dynamics of gauge theories:
  - non-perturbative effects
  - strong coupling dynamics (Seiberg-Witten)
  - dualities (S-duality, Seiberg dualities, mirror symmetry...)
  - holography
  - mathematical connections
- Ubiquity of non-local operators in gauge theories
  - Wilson loops
  - 't Hooft loops
  - surface operators
  - domain walls

  Play a central role in dualities

- The slogan for this talk is:

$\boxed{\text{2d Toda CFTs compute exactly susy observables in 4d } \mathcal{N}=2 \text{ gauge theories}}$

- The slogan for this talk is:

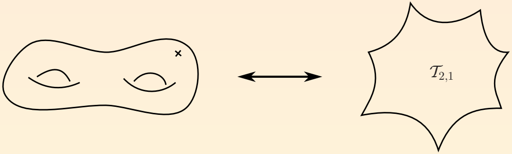
2d Toda CFTs compute exactly susy observables in 4d $\mathcal{N} = 2$ gauge theories

- The dictionary is:

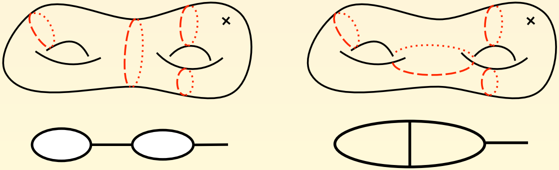
4d Gauge Theory	Toda CFT
partition function on S^4	CFT correlation function
Wilson-'t Hooft loops	topological defects (degenerate)
surface operators	insertion of degenerate operator
domain walls:	
1) Janus	generalized loop operators
2) symmetry breaking	topological defects (non-degenerate)
3) duality walls	insertion of Moore-Seiberg groupoid element

Background

- Associate a 4D $\mathcal{N} = 2$ theory $\mathcal{T}_{g,n}$ to a Riemann surface $C_{g,n}$



- "Lagrangian" description of $\mathcal{T}_{g,n} \longleftrightarrow$ pants decomposition σ of $C_{g,n}$
- Construct gauge theory from building blocks: trinions and tubes



- Partition function of 4D $\mathcal{N} = 2$ theory on S^4 given by Pestun

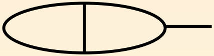
$$\mathcal{Z}^{(\sigma)}_{\mathcal{T}_{g,n}} = \int [da]\, \overline{Z}^{(\sigma)}_{\text{Nekrasov}} Z^{(\sigma)}_{\text{Nekrasov}}$$

Z_{Nekrasov}: deformation of the instanton partition function on R^4 Nekrasov

- 2D CFT correlators admit a holomorphically factorized form. In the pants decomposition $C_{g,n}$ with trivalent graph Γ_σ

$$\Big\langle \prod_{a=1}^{n} V_{m_a} \Big\rangle_{C_{g,n}} = \int d\nu(\alpha)\, \overline{\mathcal{F}}^{(\sigma)}_{\alpha,E}\, \mathcal{F}^{(\sigma)}_{\alpha,E}$$

$\nu(\alpha)$: 3pt functions of primaries, one for each vertex in Γ_σ



$\mathcal{F}^{(\sigma)}_{\alpha,E}$: conformal blocks, which capture contribution of descendants

- Partition function of $\mathcal{T}_{g,n}$ captured by 2d Toda correlation function

Alday, Gaiotto, Tachikawa

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- Modular invariance of 2d CFT $\Longrightarrow$ S-duality in 4d gauge theory

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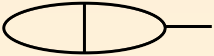

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- Modular invariance of 2d CFT$\Longrightarrow$ S-duality in 4d gauge theory

In the rest of talk I will sketch how to use 2d Toda CFT to compute loop operators and domain walls in 4d gauge theories

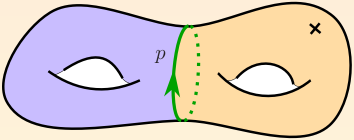
Drukker, Gaiotto, J.G.

Drukker, J.G., Okuda, Teschner

Alday, Gaiotto, Tachikawa, Verlinde

Topological Defect Operators in 2d CFTs

- 2D CFTs can be enriched by inserting defect loop operators



- A topological defect operator is transparent to the CFT stress tensor

Petkova,Zuber

$$\boxed{[T(z), \mathcal{O}_\mu(p)] = 0, \qquad [\overline{T}(\bar{z}), \mathcal{O}_\mu(p)] = 0}$$

p: (homotopy class) of curve

μ: a quantum number (representation)

- Semiclassically, $\mathcal{O}_\mu(p)$ can be thought of as a Wilson loop in the 2d CFT

$$\mathrm{Tr}_\mu \mathrm{P} \exp \oint_p \mathcal{L}$$

where $\mathcal{L}$ is a flat connection

- Since $\mathcal{O}_\mu(p)$ commutes with $T(z)$ and $\overline{T}(\bar{z}) \implies$

$$\mathcal{O}_\mu(p) = D_{\mu\alpha}\, 1_{R_\alpha \otimes \bar{R}_{\alpha^*}}$$

- topological defect $\Longleftrightarrow$ "permutation boundary state" upon folding

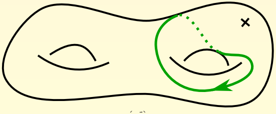
- Topological defect operators in RCFTs determined by modular invariance

$$\boxed{\mathcal{O}_\mu(p) = \frac{S_{\mu\alpha}}{S_{1\alpha}}\, 1_{R_\alpha \otimes \bar{R}_{\alpha^*}}}$$

where S are the modular matrices of the CFT

$$\chi_\mu(\tau) = \sum_\alpha S_{\mu\alpha}\chi_\alpha(-1/\tau)$$

- Topological loop operators *act* on the space of conformal blocks



$$\left[\mathcal{O}_\mu \circ \mathcal{F}^{(\sigma)}\right]_{\alpha,E} = \int d\nu(\alpha')\, \mathcal{O}_\mu(\alpha,\alpha')\, \mathcal{F}^{(\sigma)}_{\alpha',E}$$

- Alternative description of Verlinde loop operators in 2d CFTs
- Measure the monodromies induced by $V_\mu(z)$ on conformal blocks

- Representations labeled by Liouville momentum α of primary operator

$$V_\alpha = e^{2\alpha\phi} \qquad \text{with} \qquad \Delta = \alpha\,(Q-\alpha)$$

  - ZZ topological defects. Labeled by a *degenerate* representation:

$$2\alpha_{r,s} = \frac{1-r}{b} + (1-s)b \qquad \text{where} \qquad r,s \in \mathbb{Z}_+$$

The corresponding Virasoro character is given by

$$\chi_{r,s}(\tau) = \frac{q^{-(r/b+sb)^2/4}(1-q^{rs})}{\eta(\tau)}$$

$\Longrightarrow$

Sarkissian

$$\boxed{\mathcal{O}_{r,s}(p) = \frac{\sin(2\pi ra/b)\sin(2\pi sab)}{\sin(2\pi a/b)\sin(2\pi ab)}}$$

where $a = \alpha - Q/2$

- FZZT topological defects. Labeled by a *non-degenerate* representation:

$$\mu = \frac{Q}{2} + m \qquad \text{where} \qquad m \in i\mathbb{R}$$

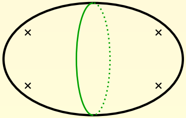
The corresponding Virasoro character is given by

$$\chi_\mu(\tau) = \frac{q^{-m^2}}{\eta(\tau)}$$

$\Longrightarrow$

$$\boxed{\mathcal{O}_\mu(\alpha) = -\frac{\cos(4\pi a m)}{2\sin(2\pi a/b)\sin(2\pi a b)}}$$

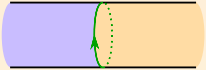
- These expressions apply whenever the topological defect wraps a curve encircling a tube in the pants decomposition of $C_{g,n}$



- We can use topological defects to calculate exactly gauge theory observables

ZZ Topological Defect Operators in 4d Gauge Theories

- Liouville correlators in the presence of $\mathcal{O}_{r,s}(p)$ around tubular curve



$$\Big\langle \mathcal{O}_{r,s}(p) \Big\rangle_{C_{g,n}} = \int d\nu(\alpha)\, \frac{\sin(2\pi r a/b)\sin(2\pi s a b)}{\sin(2\pi a/b)\sin(2\pi a b)} \overline{\mathcal{F}}^{(\sigma)}_{\alpha,E}\, \mathcal{F}^{(\sigma)}_{\alpha,E}$$

- At $b=1$, a basis of ZZ topological defects is given by $\mathcal{O}_{1,2j+1}(p)$

$$\frac{\sin(2\pi a(2j+1))}{\sin(2\pi a)} = \sum_{m=-j}^{j} e^{4\pi i m a} \equiv \mathrm{Tr}_j e^{2\pi i a}$$

Therefore

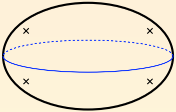
$$\boxed{\langle \mathcal{O}_{1,2j+1}(p) \rangle_{C_{g,n}} = \int [da]\, \mathrm{Tr}_j\, e^{2\pi i a}\, \overline{Z}_{\text{Nekrasov}}\, Z_{\text{Nekrasov}}}$$

$\Longrightarrow$ Expectation value of Wilson loop in 4d gauge theory on S^4

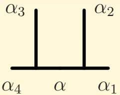
- What is the 2d CFT description of 't Hooft loops in the 4d gauge theory?

The electric and magnetic charges of loop operators encoded in the choice of curve in $C_{g,n}$ Drukker, Morrison, Okuda

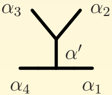
$\Longrightarrow$ Consider topological defect operator which traverses a tube in $C_{g,n}$



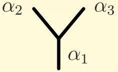
- Must determine the monodromy induced by $V_\mu(z)$ on the conformal blocks
- Monodromy generated by 3 basic moves, relating conformal blocks in different channels: Moore,Seiberg
  - Fusion move:



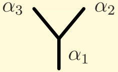
$$\equiv \int d\alpha' \, F_{\alpha\alpha'}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]$$



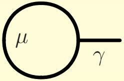
  - Braiding move:



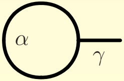
$$\equiv e^{i\pi(\Delta(\alpha_1)-\Delta(\alpha_2)-\Delta(\alpha_3))}$$



  - S-move:

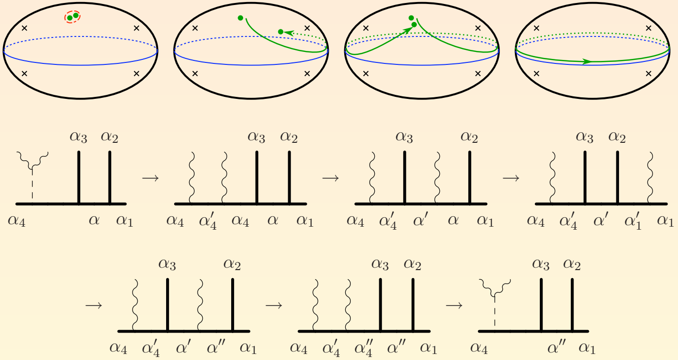


$$= \int d\alpha \, S_{\mu\alpha}(\gamma)$$

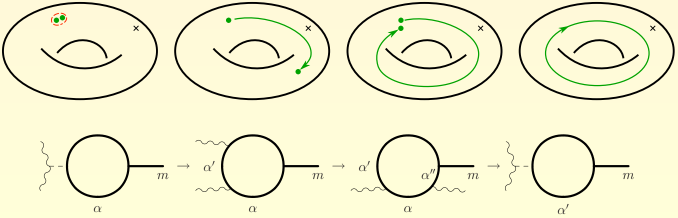


- Monodromy captured by the Fusion and Braiding matrices of the CFT

- 't Hooft loop in $\mathcal{N}=2$ SU(2) SYM w/ $N_F=4$



- 't Hooft loop in $\mathcal{N}=2^*$ SU(2) SYM on S^4



- Topological defect shifts the momentum labeling the tubes that it traverses

$$\Big[\mathcal{O}_\mu \circ \mathcal{F}^{(\sigma)}\Big]_{\alpha,E} = \int d\nu(\alpha')\,\mathcal{O}_\mu(\alpha,\alpha')\,\mathcal{F}^{(\sigma)}_{\alpha',E}$$

- Computed by concatenating the fusion and braiding matrices associated to tracing the curve in $C_{g,n}$

- Can obtain explicit formulae for vev of 't Hooft loops in 4d $\mathcal{N}=2$ gauge theories on S^4. For example, for $\mathcal{N}=2^*$:

$$\boxed{2\int d\alpha\,\sin(2\pi\alpha)\,\overline{\mathcal{G}}_{\alpha,m}\big[\sin(\pi(2\alpha-m))\mathcal{G}_{\alpha-1/2,m}+\sin(\pi(2\alpha+m))\mathcal{G}_{\alpha+1/2,m}\big]}$$

- Consistent with the action of S-duality!

$\Longrightarrow$ ZZ topological defects in Liouville CFT capture the exact vev of supersymmetric loop operators in 4d $\mathcal{N}=2$ gauge theories

$$\boxed{\text{ZZ Defect Operators} = \text{Gauge Theory Loop Operators}}$$

- What 4d gauge theory observables do FZZT topological defects compute?

Toda CFT Topological Defects in 4d Gauge Theories

- Toda CFT is based on a Lie algebra $\mathfrak{g}$ and admits a W-algebra symmetry

$$W^{(2)}, W^{(3)}, \ldots, W^{(r)} \qquad r = \mathrm{rank}(\mathfrak{g})$$

- Toda CFTs are only partially understood (e.g. no formula for 3 pt function)
- Representations labeled by Toda momentum μ of primary operator

$$e^{\langle \mu, \phi \rangle} \qquad \text{with} \qquad \Delta = \langle Q, \mu \rangle - \frac{1}{2}\langle \mu, \mu \rangle$$

- Constructed the topological defects in Toda CFTs. Their expectation values are (at $b = 1$):

    - Degenerate Topological defects. Labeled by representation R of $\mathfrak{g}$:

$$\boxed{\langle \mathcal{O}_R(p) \rangle_{C_{g,n}} = \int [da]\, \mathrm{Tr}_R\, e^{2\pi i a}\, \overline{Z}_{\text{Nekrasov}}\, Z_{\text{Nekrasov}}}$$

$\Longrightarrow$ Expectation value of Wilson loop in 4d gauge theory on S^4

- Semi-Degenerate Topological defects. Labeled by:

  m: purely imaginary Toda momentum

  r: representation of $\mathfrak{h} \subset \mathfrak{g}$

  Toda CFT correlator with insertion of topological defect yields:

$$\boxed{\langle \mathcal{O}_{m,r}(p) \rangle_{C_{g,n}} = \int d\nu(\alpha) \frac{\prod_{e>0}^{e\in\mathfrak{h}} \sin^2(\pi\langle a,e\rangle)}{\prod_{e>0}^{e\in\mathfrak{g}} \sin^2(\pi\langle a,e\rangle)} \overline{\mathcal{F}}_{\alpha,E}^{(\sigma)} \mathcal{F}_{\alpha,E}^{(\sigma)} e^{2\pi i\langle m,a\rangle} \mathrm{Tr}_r(e^{2\pi i a})}$$

Claim: 4d $\mathcal{N}=2$ gauge theory in the presence of a susy domain wall on S^3

Symmetry Breaking Domain wall:

- 3d $\mathcal{N}=2$ vector multiplet fields taking values in $\mathfrak{g}-\mathfrak{h}$ vanish
- $\mathrm{Tr}_r(e^{2\pi i a})$: Wilson loop supported on the wall
- m: FI parameters for the $U(1)$ factors localized on the wall
- $\prod_{e>0}^{e\in\mathfrak{g}} \sin^2(\pi\langle a,e\rangle)$: one-loop determinant from 3d $\mathcal{N}=2$ vector multiplet
- Ratio removes precisely the 3d $\mathcal{N}=2$ vector multiplet fields in $\mathfrak{g}-\mathfrak{h}$

- Computation of the 4d path integral reproduces the 2d CFT answer $\Longrightarrow$

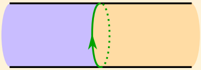
$$\boxed{\text{Semi-degenerate Defect Operators} = \text{Symmetry Breaking Domain Walls}}$$

2d CFT Description of other 4d Gauge Theory Domain Walls

- Janus Domain Wall: Susy wall across which the coupling constant jumps

$$\tau_N = \frac{\theta_N}{2\pi} + \frac{4\pi i}{g_N^2}\,, \qquad \tau_S = \frac{\theta_S}{2\pi} + \frac{4\pi i}{g_S^2}$$

Corresponds to inserting a "generalized defect operator" on $C_{g,n}$ in Toda CFT



$$f \circ T_1 = T_2 \qquad \bar{f} \circ \overline{T}_1 = \overline{T}_2$$

$\bar{f} \neq f^* \Rightarrow$ left and right movers on different Riemann surfaces (different moduli)

$$\langle \mathcal{O}_{1,\{f,\bar{f}\}}(p) \rangle_{C_{g,n}} = \int d\nu(\alpha)\, \overline{\mathcal{F}}^{(\sigma)}_{\alpha,E}(q)\, \mathcal{F}^{(\sigma)}_{\alpha,E}(q')$$

- Computation of the 4d path integral reproduces the 2d CFT answer $\Longrightarrow$

$$\boxed{\text{Generalized Defect Operators} = \text{Janus Domain Walls}}$$

Conclusion

- Loop operators enrich the correlation functions in 2d CFTs
- Non-rational 2D CFT's compute exact quantities in 4D gauge theories
- 2d CFT loop operators capture the dynamics of non-local gauge theory operators

Wilson-'t Hooft operators

domain wall operators

- Non-local operators bound to provide insights into gauge theory dualities
- Toda CFT is a window into non-lagrangian 4d gauge theories