Holographic Superconductors with Paper and Pencil

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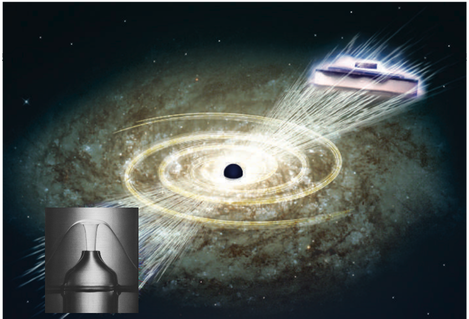

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AdS/CFT and Superconductivity

$$S = \frac{1}{2\kappa^2}\int d^{d+1}x\sqrt{-g}(R-2\Lambda) - \frac{1}{4g^2}\int d^{d+1}x\sqrt{-g}F_{\mu\nu}F^{\mu\nu}$$

$$-\int d^{d+1}x\sqrt{-g}\left(|(\partial - iqA)\Psi|^2 + V(|\Psi|)\right) \ .$$

- With $\Lambda < 0$ and $d = 3$ or 4, dual interpretation as strongly interacting superfluid or superconductor.
- Extensive studies since the work of Gubser (2008) and Hartnoll, H, and Horowitz (2008).
- Most results are numerical.
- I want to talk about some recent analytic progress.

The Duality

	high T	low T
gravity	black hole with $\mathbf{E}$ (and $\mathbf{B}$)	black hole with $\mathbf{E}$, scalar hair (and $\mathbf{B}$)
field theory	normal phase with $J^t \neq 0$ (and $\mathbf{B}$)	superfluid/conducting phase

A Few Questions

What can we calculate in such a model?

- equation of state, correlation functions, transport coefficients, phase diagram

What can we learn from such a model?

- universal properties?
- some specific example that is a real world system?

A Nice Early Paper

Horowitz and Roberts (2008)

- mini-survey of different holographic superconductors
- $d = 3$ and 4 with $V(|\Psi|) = m^2|\Psi|^2$ for a handful of m.
- worked in a $\kappa \to 0$ limit (weak gravity, probe limit) — fixes background and decouples Abelian-Higgs sector

One example: $d = 4$ and $m^2 L^2 = -4$ (with L defined via $\Lambda = -6/L^2$)

- saturates Breitenlohner-Freedman bound
- corresponds to a field theory operator with $\Delta = 2$ that will serve as an order parameter

Some Numeric results for this example

$$
\begin{aligned}
T_c &= 0.253\,(J^t)^{1/3}\ , \\
\langle O_2 \rangle &\approx (6.7\,T_c)^2 \left(1 - \frac{T}{T_c}\right)^{1/2}\ , \\
\lim_{\omega\to 0} G^{xx}(\omega, 0) &\approx 97.4\,T_c^2 \left(1 - \frac{T}{T_c}\right)\ .
\end{aligned}
$$

Here $G^{xx}(\omega, k)$ is the Fourier transformed $J^x J^x$ correlation function.

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$$

I found

$$
0.253 = \frac{1}{2^{1/3}\pi} \ , \quad 6.7 = \left(\frac{144}{7}\right)^{1/4} \pi \ , \quad 97.4 \approx \frac{72}{7}\pi^2 \ .
$$

How I know

- I am glad that Horowitz and Roberts chose to work in the canonical ensemble at fixed J^t. Otherwise, they might have noticed
$$T_c = \mu/2\pi \ .$$
- We will work at fixed temperature $\pi T = 1$ and allow μ to vary ($\mu_c = 2$). In the gravity dual, that means setting
$$ds^2 = \frac{1}{u^2}\left(-f\,dt^2 + d\mathbf{x}^2 + \frac{du^2}{f}\right)$$
where $f = 1 - u^4$. A black D3-brane in the Poincare patch.
- In the probe limit and in the absence of scalar hair, the gauge field must satisfy the equation
$$\partial_u \frac{1}{u}(\partial_u A_t) = 0$$
which is solved by $A_t = (1 - u^2)\mu$.

- μ acts like a negative mass term for the scalar:

$$u^3 \left[\frac{f \psi'}{u^3} \right]' = -\frac{A_t^2}{f} \psi + \frac{m^2 L^2}{u^2} \psi$$

  where we made the gauge choice $\Psi^* = \Psi = \psi / \sqrt{2}$.
- $\psi < \infty$ at the black hole horizon $u = 1$.
- We set the source term $j = 0$ at the boundary $u = 0$:

$$\Psi = j\, u^2 \ln u - \langle O_2 \rangle u^2 + \ldots$$

!!! ψ has a zero mode solution when $\mu = 2$:

$$\psi = \frac{u^2}{1 + u^2} \; .$$

Solution near the Phase Transition

We have an expansion of the solution near the critical point of the form:

$$\begin{aligned} A_t &= \mu_c(1 - u^{d-2}) + \epsilon^2 A_{t(2)}(u) + \dots \\ \psi &= \epsilon\psi_1(u) + \epsilon^3\psi_3(u) + \dots \end{aligned}$$

where for the model we just looked at $\langle O_2 \rangle \sim \epsilon$ is the size of the order parameter, $\mu_c = 2$, $\psi_1 = u^2/(1 + u^2)$, $d = 4$.

We have an expansion in even powers of ϵ for A_t and an expansion in odd powers of ϵ for ψ.

Form of this expansion closely related to critical exponents.

Critical Exponents

Recall the critical exponents of a thermal phase transition:

$$\begin{aligned} C = T\frac{\partial S}{\partial T} &\sim (T_c - T)^{-\alpha} \\ \langle O \rangle &\sim (T_c - T)^{\beta} \\ \frac{\partial \langle O \rangle}{\partial j} &\sim (T_c - T)^{-\gamma} \\ j &\sim \langle O \rangle^{\delta} \end{aligned}$$

where the last holds at $T = T_c$.

There exist scaling relations $\alpha = 2 - \beta(\delta + 1)$ and $\gamma = \beta(\delta - 1)$.

Computing β

We can compute $\beta = 1/2$ from the boundary expansion of A_t:

$$A_t(0) = \mu_c + \epsilon^2 A_{t(2)}(0)$$

implies

$$\mu = \mu_c + \epsilon^2 \frac{1}{48}$$

implies

$$\epsilon = \sqrt{48(\mu - \mu_c)} \; .$$

This is basically the second result from Horowitz and Roberts.

I won't bother to derive the third result for you, but let me compute δ instead.

Computing δ

We can compute $\delta = 3$ from the boundary expansion of ψ:

$$\psi(u) = \epsilon\,\psi_1(u) + \epsilon^3\,\psi_3(u)$$

as $u \to 0$.

Remembering to freeze the chemical potential to its critical value, $\mu = 2$, we find that we have to introduce a source term to ψ_3.

We find $\langle O_2 \rangle \sim \epsilon$ while $j \sim \epsilon^3$.

We could now verify $\alpha = 0$ and $\gamma = 1$ directly, but let me skip that.

See also Maeda, Natsuume, Okamura (2009); Iqbal, Liu, Mezei, Si (2010)

Other Results

- One can verify the transition is second order through computing the change in the free energy.
- One can calculate the speed of second sound near the phase transition
$$c_2^2 = \frac{6}{7}(\mu - 2)$$
using thermodynamic identities.
- One can calculate the two point correlation functions near the phase transition. Pole structure:
$$960\omega^3 + 56i(12k^2 + \epsilon^2)\omega^2 - 12(16k^2 + 3\epsilon^2)k^2\omega$$
$$-i(48k^4 + 16k^2\epsilon^2 + \epsilon^4)k^2 = 0 \ .$$

Universality?

If we change the mass, we lose our analytic solution, but we can play around with higher order terms:

$$V(|\Psi|) = m^2|\Psi|^2 \quad \to \quad m^2|\Psi|^2 + B|\Psi|^4$$
$$|\partial\Psi - iA\Psi|^2 \quad \to \quad (1 + A|\Psi|^2)|\partial\Psi - iA\Psi|^2$$

These terms affect the physics:

$$\text{Order Parameter:} \quad \langle O_2 \rangle \sim \left(\frac{\mu - 2}{1 + 4A + 4B} \right)^{1/2}$$

$$\text{Free Energy Difference:} \quad \Omega_{\mathrm{sf}} - \Omega_{\mathrm{vac}} \sim -(1 + 4A + 4B)\epsilon^4$$

If A and B get too large, we lose the second order phase transition! (There is also a change to c_2^2, but I'll skip it.)

More General Actions

We may also want to consider terms that are not so well behaved at $\Psi = 0$:

$$\begin{aligned} V(|\Psi|) = m^2|\Psi|^2 &\rightarrow m^2|\Psi|^2 + B|\Psi|^a \\ |\partial\Psi - iA\Psi|^2 &\rightarrow (1 + A|\Psi|^{a-2})|\partial\Psi - iA\Psi|^2 \end{aligned}$$

where $2 < a \leq 4$. This change will affect the critical exponents of the phase transition.

Franco, Garcia-Garcia, Rodriguez-Gomez (2009)

Solution near the Phase Transition

We have an expansion of the solution near the critical point of the form:

$$\begin{aligned} A_t &= \mu_c(1-u^{d-2}) + \epsilon^{a-2} A_{t(a-2)}(u) + \ldots \\ \psi &= \epsilon\psi_1(u) + \epsilon^{a-1}\psi_{a-1}(u) + \ldots \end{aligned}$$

The case we studied originally has $a = 4$.

This a is related to the critical exponents $\beta = 1/(a-2)$ and $\delta = a-1$ by a trivial generalization of the earlier arguments.

It follows from scaling that $\alpha = (a-4)/(a-2)$ and $\gamma = 1$.

Beyond Critical Exponents

The calculation of the critical exponents is independent of the existence of an analytic solution.

The analytic solution does two things:

- Gives the constant of proportionality in the definition of the critical exponents.
- Allows one to see when such a series solution in ϵ exists: $1 + 4A + 4B > 0$ condition (for $a = 4$).

Not the first analytic superconductor

p-wave in 3+1 dimensions in probe limit

- replace abelian higgs with a nonabelian SU(2) gauge field

$$-\frac{1}{4g^2}\int d^5x\,\sqrt{-g}F^a_{\mu\nu}F^{a\mu\nu} \ .$$

- choose an ansatz: $A = \phi\tau^3\,dt + w\tau^1\,dx$,
- w develops a nontrivial profile when $\phi = (1-u^2)\mu$ becomes too large
- boundary value of A^a_μ now a vector order parameter
- phase transition at $\mu_c = 4$ with

$$w = \frac{u^2}{(1+u^2)^2} \ .$$

Basu, He, Mukherjee, and Shieh; Herzog and Pufu

What about arbitrary spin?

$$\frac{1}{\sqrt{-g}}\partial_u(\sqrt{-g}g^{i_1 i_1}\cdots g^{i_s i_s}g^{uu}\partial_u\Psi) = g^{i_1 i_1}\cdots g^{i_s i_s}(g^{tt}A_t^2\Psi + m^2\Psi)$$

where $\Psi(u) = h_{i_1 i_2 \cdots i_s}$ becomes

$$\Psi'' + \left(\frac{f'}{f} + \frac{2s+1-d}{u}\right)\Psi' + \left(\frac{A_t^2}{f^2} - \frac{m^2 L^2}{u^2 f}\right)\Psi = 0 \ .$$

Take $m^2 L^2 = 4(s-1)$, $\mu = 2(s+1)$.

$$\Psi(u) = \frac{u^2}{(1+u^2)^{s+1}}$$

is a zero mode signaling onset of a phase transition.

Some remarks

- The differential equation for Ψ is closely related to a Heun equation, i.e. a second order order differential equation with 4 regular singular points. Our zero mode is a Heun polynomial. Is there some deeper symmetry principle at work here?
- There are well known issues in coupling $s \geq 2$ fields to a U(1) gauge field even in flat backgrounds — causality, ghosts, tachyons. Clearly, these need to be addressed at some level if we go beyond $s = 1$. The $s = 2$ case relevant for d-wave superconductors.
- Low temperature limit (at nonzero density) is sensitive to irrelevant terms in the action. Is the physics at $T > 0$ more universal? That seemed to be a lesson from AdS/QCD.