A Novel Abelian Gauge Theory

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Based on work to appear with

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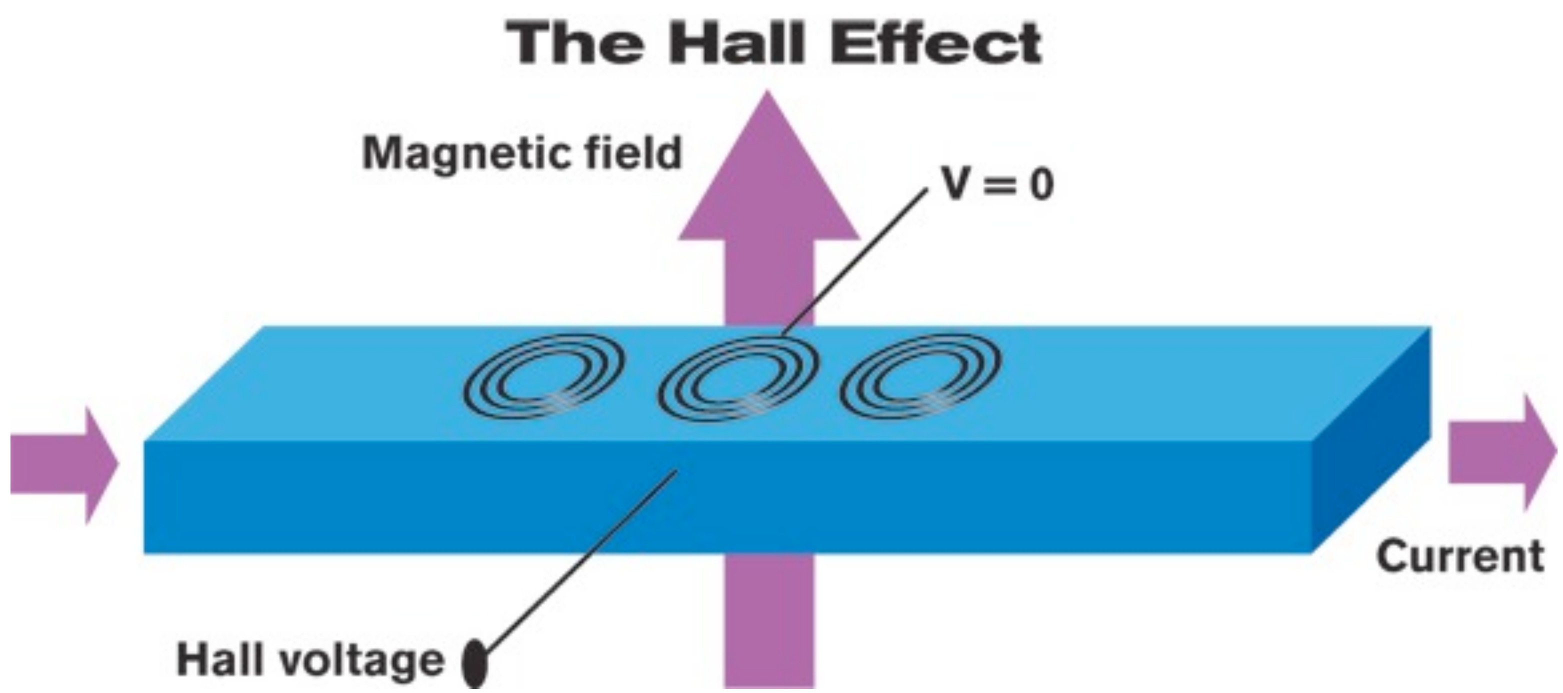
This talk has a few different motivations. Here are two:

I.



The physics of 2D electron gas in a (large) background magnetic field has been a rich, fruitful playground for both experimentalists and theorists. Famously, it is the home of the quantum Hall (and fractional quantum Hall) effect.

While if you are clever you can guess the ground state wavefunction for the system and derive much interesting physics in that way  , there is also a systematic effective field theory approach to understanding such 2D electron gases.

Zhang, Hansson, Kivelson; Frohlich, Zee

1) The essential physics is 2+1 dimensional.

2) The electromagnetic current is conserved:

$$\partial_\mu J^\mu = 0$$

3) We are interested in the physics at long distances and large time (= small wave number, low frequency).

4) P and T are broken by the external magnetic field.

In 2+1 dimensions, we can write the conserved current as the curl of a vector potential:

$$J^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda$$

This vector enjoys a gauge symmetry under which the current remains invariant.

Now, we apply standard logic: we expect the low-energy action to be governed by the lowest dimension operator(s) we can write down, consistent with the symmetries of the problem. The result is:

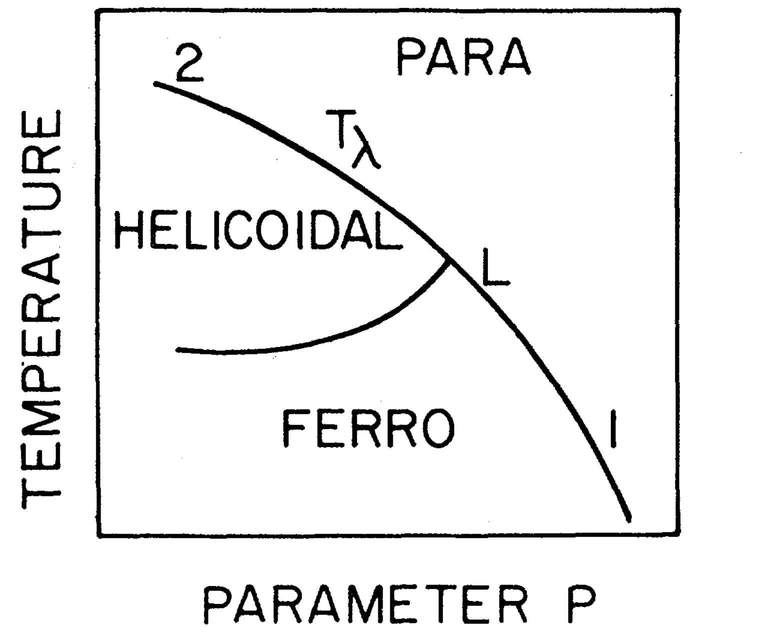
$$\mathcal{L} = \frac{k}{8\pi}\epsilon_{\mu\nu\rho}A_\mu\partial_\nu A_\rho + \frac{1}{e^2\Lambda}F_{\mu\nu}F^{\mu\nu} + \cdots$$

The theory is a Chern-Simons theory at low energies.

Using elementary reasoning starting from the abelian Chern-Simons theory, the phenomenology of the simplest odd-denominator filling fractions in the FQHE can be well explained.

Natural question: The irrelevant Maxwell perturbation here does not change the physics in the deep IR. Are there other extensions of the Chern-Simons theory that we can imagine, that would change observable properties in the IR? Might they arise in real systems?

2.



IR fixed points with non-trivial dynamical scaling

$$x \to \lambda x, \quad t \to \lambda^z t$$

are quite common in condensed matter systems (where Lorentz invariance is not a particularly natural symmetry).

An especially simple theory, which arises at critical points in many toy models of spin systems, is:

$$\mathcal{L} = \frac{1}{2} \int \ dt \ d^2x \ \left((\partial_t \phi)^2 - \kappa^2 (\nabla^2 \phi)^2 \right) \ .$$

This is a free scalar field theory with a line of fixed points, all with dynamical exponent z=2. It can be dualized to an abelian gauge theory:

$$\mathcal{L} = \int \ dt \ d^2x \, (E_i \partial_t A_i + A_t \partial_i E_i - H(E,A))$$

$$H(E,A) = \frac{1}{2g_2^2} (\partial_i E_j)^2 + \frac{1}{2g_3^2} B(A)^2$$

$$B(A) = \epsilon_{ij} \partial_i A_j \ .$$

With the gauge field scaling inversely to the coordinates, this also enjoys z=2 scaling, of course.

This theory has three potentially interesting perturbations. Two have been studied in the literature:

* One can perturb by the more standard quadratic in the electric field (which is clearly relevant):

$$\Delta L \sim E_i^2 \ .$$

* One can perturb by a quartic in the electric field, which is naively marginal:

Vishwanath, Balents, Senthil

$$\Delta L \sim (E_i^2)^2 \ .$$

With the sign of the perturbation that leaves the potential bounded below, this operator is marginally irrelevant.

* Finally, one can consider the theory with action:

$$L = \int dt d^2x \Big(\frac{1}{g_1^2}(E_i \partial_t A_i + A_0 \partial_i E_i) - \frac{1}{2g_2^2}(\partial_i E_j)^2 - \frac{1}{2g_3^2} B^2 + \frac{k}{2} \epsilon_{\mu\nu\rho} A_\mu \partial_\nu A_\rho \Big).$$

Or, formally solving for E and reinserting (valid at non-zero momenta):

$$L = \frac{1}{2} \int dt d^2x \Big(\frac{1}{g_2^2}(\partial_i E_j)^2 - \frac{1}{g_3^2} B^2 + k \epsilon_{\mu\nu\rho} A_\mu \partial_\nu A_\rho \Big),$$

$$E_i = \Big(\frac{g_2^2}{g_1^2} \Big) \frac{\partial_i A_0 - \partial_t A_i}{\nabla^2}, \quad B = \partial_1 A_2 - \partial_2 A_1.$$

This might be called the abelian "Lifshitz-Chern-Simons" theory. The Chern-Simons term looks like a marginal "perturbation" of the z=2 critical theory, and vice-versa.

This theory might be expected to arise if one combines the circumstances that normally give rise to z=2 scaling, with P and T violating background fields. It cannot be mapped (by a local transformation) into a scalar theory.

Henceforth, I will take the attitude that since this theory is a simple extension of two different physical theories that have played an important role in characterizing interesting phases of matter, and since it is eminently tractable, we should investigate its physics.

Ultimately, the usefulness of this theory will be determined by whether its phase structure and transport properties match those of observed systems. But we begin with a lightning discussion of more theoretical issues.

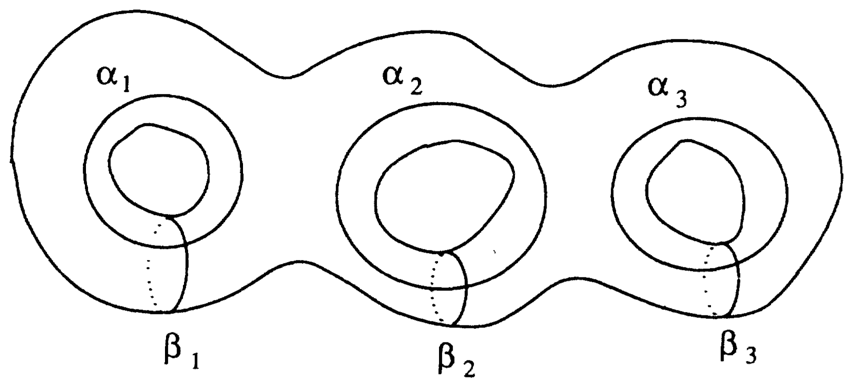
Interesting "topological" properties of the (L)CS theory

The Chern-Simons theory is (famously) a topological field theory, whose physical observables are correlated with interesting mathematical features of the space on which it lives (e.g. the Jones polynomials). Witten

Studies of "topologically ordered" states in condensed matter (gapped models with degenerate ground states which are not distinguished by a simple Landau order parameter) have taken on a life of their own.

But Chern-Simons theory is still the canonical example, and its two most interesting properties in this regard are:

* The existence of (almost) degenerate ground states, when the theory is formulated on a Riemann surface of genus g.



Wen;
Wen, Niu

In searching for ground states, one can restrict to zero-momentum modes of the gauge field (assuming the vacuum doesn't spontaneously break translation invariance).

In Maxwell-Chern-Simons theory, this yields a quantum mechanics problem with:

$$L = \int dt \Big[\frac{1}{2e^2 \Lambda} (\partial_t A_i)^2 + k(A_1 \partial_t A_2 - A_2 \partial_t A_1) \Big]$$

In the case of e.g. the torus, we can reduce this to a problem involving the two Wilson-line zero modes.

These Wilson lines themselves live on a (dual) torus, and are governed by the Lagrangian:

$$L = \int \; dt \;\, k\pi(\dot{x}y - \dot{y}x) + \frac{m}{2}(\dot{x}^2 + \dot{y}^2)$$

(where the mass term is generated by the Maxwell term).

This Lagrangian describes a charged particle moving on the dual torus, in the presence of a magnetic field of order k. The quantum mechanics of such a particle (i.e. the physics of the Landau levels) was studied by Haldane and Rezayi.

With

$$H = \frac{1}{2m}\left(-(\partial_x - iA_x)^2 - (\partial_y - iA_y)^2\right)$$

$$A_x = 0, A_y = Bx = 2\pi kx$$

they find that the ground state is k-fold degenerate. The wave functions of the ground states are given by:

$$\Psi_l(x,y) = \left(\sum_n e^{2\pi(x+iy)(nk+l) - \frac{(nk+l)^2}{k}\pi}\right)\ e^{-2\pi^2 Bx^2}, \quad l = 0, 1, ..., k-1$$

↑

(This thing is a theta function)

The result can be generalized to higher genus surfaces (with theta functions again playing a starring role). One finds, at level k and genus g:

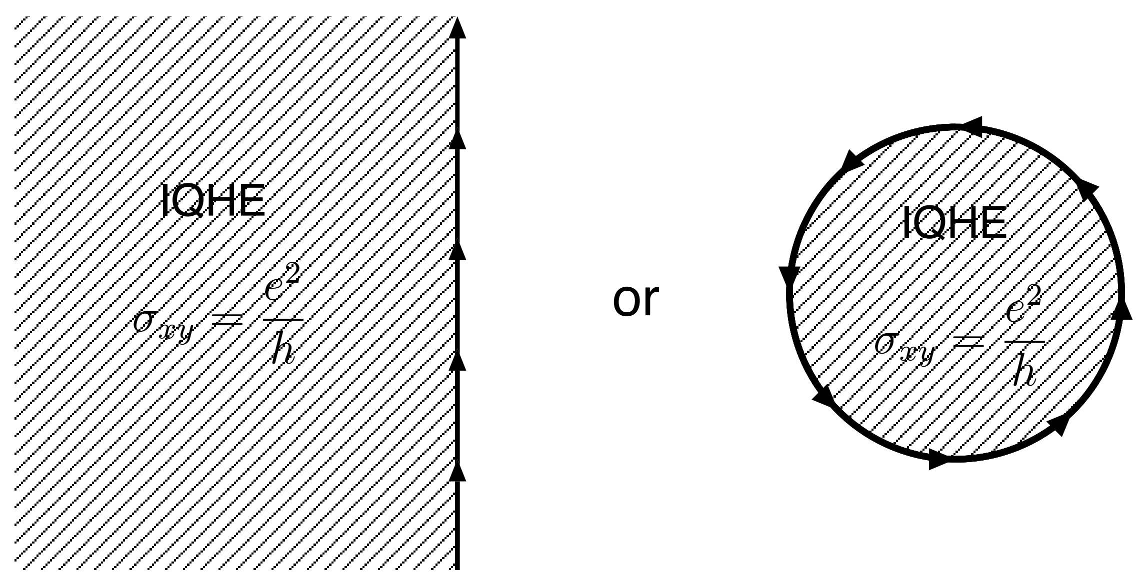
$$\text{Result}: k^g \text{ degenerate ground states}$$

More precisely, on a system with finite size L, the splitting of these ground states goes like

$$\Delta E \sim \text{Exp}(-Lm)$$

where m is related to the gap and the mass of the lightest quasiparticle excitations.

* The existence of chiral edge modes, when the theory is placed on a space with boundary:



This is more or less self-explanatory. The derivation of the existence of the edge modes most elegantly proceeds from anomaly considerations:

When formulated on a space with boundary, the gauge-variation of the Chern-Simons action yields:

$$\delta A = \partial f \rightarrow$$

$$\delta S \sim \int_{\partial \Sigma_3} f \wedge F \ .$$

For the level k theory, this lack of gauge invariance can be fixed by adding a boundary chiral fermion:

$$S_{\mathrm{boundary}} = \frac{k}{4\pi} \int \ dt \ dx \ \lambda^* (i\partial_t - \partial_x)\lambda \ .$$

Such "edge modes" are characteristic of topological phases, and in fact play a central role in some recent efforts to classify topological insulators.

Ryu, Schnyder, Furusaki, Ludwig; Kitaev

Both the chiral edge modes and the degenerate ground states of these topologically ordered systems, have been implicitly assumed to be closely related to the gapped nature of the bulk theory.

One of the interesting aspects of the abelian Lifshitz-Chern-Simons theory is that it exhibits both the chiral edge modes and the ground-state degeneracy, while manifesting a gapless critical theory in the bulk. (This is in contrast to Maxwell-Chern-Simons, which is massive).

Deser, Jackiw, Templeton

The persistence of the edge modes is clear.

* Their presence is still required to cancel a non-vanishing gauge variation when the LCS theory is placed on a space with boundary.

* Perturbation theory in the Lifshitz gauge couplings could in principle perturb the Lagrangian for the chiral edge modes, but this purely chiral theory has no relevant perturbations. It is thus robust under the addition of the Lifshitz terms, with small gauge couplings.

The ground state degeneracy persists for the following intuitive reason. Let's consider the theory with a regulating (marginally irrelevant) perturbation:

$$\Delta L \sim -\lambda (E_i^2)^2$$

* The degeneracy within each Landau level of the "particle in a magnetic field" (that arises by quantizing the moduli space of flat connections) remains.

* An extra degeneracy that would arise even in the pure Lifshitz gauge theory (from the undetermined expectation value of the E-field) is lifted by the regulator.

Linear Response and Phase Structure of the LCS theory

Now, we study the phase structure and transport properties as a function of the single relevant coupling and the Chern-Simons level.

To do this, we couple the external electromagnetic field to the current of the statistical gauge field

$$J_\mu = \epsilon_{\mu\nu\rho}\partial_\nu A_\rho$$

$$\delta L = \int dt d^2x \; J^\mu V_\mu .$$

We then integrate it out (quadratic action!), to obtain:

$$L = \frac{1}{2}\int d\omega d^2p \; V_\mu K_{\mu\nu} V_\nu$$

We can then obtain e.g. the conductivity in each phase, by using the simple relation:

$$\sigma_{jk}(\omega, p) = \frac{K_{jk}(\omega, p)}{i\omega}$$

which follows from Ohm's law

$$\frac{\delta L}{\delta V_i} = J_i = K_{ij} V_j = \sigma_{ij} E_j .$$

Phase 1: The Critical Theory

The Lagrangian of the (free) scale-invariant theory is:

$$L = \frac{1}{2} \int d\tau d^2 x \Big[\frac{(\partial_t A_i)^2 + (\partial_i A_0)^2}{-g_E^2 \nabla^2} - \frac{(\partial_i A_j)^2}{g_3^2} + k \epsilon_{\mu\nu\rho} A_\mu \partial_\nu A_\rho + 2 \epsilon_{\mu\nu\rho} V_\mu \partial_\nu A_\rho \Big].$$

The field equations allow one to solve for A in terms of V:

$$\begin{pmatrix} A_0 \\ A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{g_E^2} & -ikp_2 & ikp_1 \\ ikp_2 & \frac{\omega^2}{g_E^2 p^2} - \frac{p^2}{g_3^2} & ik\omega \\ -ikp_1 & -ik\omega & \frac{\omega^2}{g_E^2 p^2} - \frac{p^2}{g_3^2} \end{pmatrix}^{-1} \begin{pmatrix} 0 & ip_2 & -ip_1 \\ -ip_2 & 0 & -i\omega \\ ip_1 & i\omega & 0 \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \\ V_2 \end{pmatrix}$$

$$=: \mathbf{G}^{-1}\mathbf{H} \begin{pmatrix} V_0 \\ V_1 \\ V_2 \end{pmatrix}$$

Then, we can re-write the action in terms of V as:

$$L_{\text{eff}} = \frac{1}{2} \int d\omega d^2p \; \mathbf{V}^t \Big[(\mathbf{G}^{-1}\mathbf{H})^t \mathbf{H} - 2\mathbf{H}\mathbf{G}^{-1}\mathbf{H} \Big] \mathbf{V}$$

and solve for the response function:

$$\mathbf{K} = ((\mathbf{G}^{-1}\mathbf{H})^t - 2\mathbf{H}\mathbf{G}^{-1})\mathbf{H}.$$

We find a finite compressibility

$$c = \lim_{p \to 0} K_{00} = \frac{3g_3^2}{1 + g_E^2 g_3^2 k^2}$$

and an AC conductivity

$$K_{11} = K_{22} = \frac{3g_3^2 g_E^2 p^2 \omega^2 (-g_E^2 p^4 + g_3^2 \omega^2)}{2g_3^2 g_E^2 p^4 \omega^2 - g_3^4 \omega^4 + g_E^4 (-p^8 + g_3^2 k^2 p^4 \omega^2)}$$

which vanishes quadratically at low momentum

$$K_{11} = -3g_E^2 p^2 + \mathcal{O}(p^6)$$

In other words, the critical theory is an insulator.

Phase II: The Massive Phase

In this phase, we add back the electric field couplings present in the Maxwell term:

$$L=\int dtd^2x\Big((E_i\partial_t A_i+A_0\partial_i E_i)-\frac{e^2}{2}E_i^2-\frac{\ell^2}{2}(\partial_i E_j)^2-\frac{1}{2g_3^2}B^2+\frac{k}{2}\epsilon_{\mu\nu\rho}A_\mu\partial_\nu A_\rho+V_\mu J^\mu(A)\Big).$$

Only the "G-matrix" is modified as compared with the calculation in the previous phase:

$$\mathbf{G}=\begin{pmatrix}\frac{p^2}{e^2+\ell^2p^2} & -ikp_2 & ikp_1\\ ikp_2 & \frac{\omega^2}{(e^2+\ell^2p^2)}-\frac{p^2}{g_3^2} & ik\omega\\ -ikp_1 & -ik\omega & \frac{\omega^2}{(e^2+\ell^2p^2)}-\frac{p^2}{g_3^2}\end{pmatrix}.$$

Now we find a compressibility:

$$c = \lim_{p \to 0} \frac{3 g_3^2 p^2}{e^2 g_3^2 k^2 + (1 + g_3^2 \ell^2 k^2) p^2}.$$

This vanishes at low momentum, as one would expect; the IR physics should be that of the pure Chern-Simons theory.

The AC conductivity takes the form:

$$K_{11} = K_{22} = -\frac{3\omega^2 \left(\frac{-p^2}{g_3^2} + \frac{\omega^2}{e^2 + \ell^2 p^2}\right)}{\frac{p^4}{g_3^4} + \omega^2 \left(-k^2 - \frac{2p^2}{e^2 g_3^2 + g_3^2 \ell^2 p^2} + \frac{\omega^2}{(e^2 + \ell^2 p^2)^2}\right)}.$$

The leading correction in the Maxwell coupling, expanding about the Lifshitz-Chern-Simons critical point, is:

$$K_{11} = \frac{3e^2\omega^2}{e^4k^2 - \omega^2} + \mathcal{O}(p^2).$$

* At level k=0, the resulting conductivity shows a pole in the imaginary piece, corresponding (via Kramers-Kronig) to a delta function in the real conductivity. This is the superconducting behavior of the pure Maxwell theory.

* At non-zero k, the longitudinal conductivity vanishes at zero momentum, indicating insulating behavior.

* In addition, a non-zero k the Hall conductivity is controlled by:

$$K_{12} = -i\omega/k$$

I.e., we have a quantum Hall system for non-zero k, as expected.

Phase III: The Anisotropic Phase

Now, we study the theory with a "wrong sign" quadratic term in the electric field, stabilized by a quartic perturbation. The Hamiltonian is:

$$H[E,B] = -e^2 E_i^2 + \frac{\ell^2}{2}(\partial_i E_j)^2 + \frac{\lambda}{4}(E_i^2)^2 + \frac{1}{2g_3^2}B^2.$$

As a result, the electric field condenses (this is the analogue of the helical phase neighboring the Lifshitz point in ferromagnets):

$$E_i^2 = \tfrac{2e^2}{\lambda}, A_i = 0.$$

Choosing (WLOG) the condensate to be along the x-axis, we find:

$$H[E,B] = \frac{1}{2}e^2E_x^2 + \frac{\ell^2}{2}(\partial_i E_y)^2 + \frac{1}{2g_3^2}B^2 + \mathcal{O}(E^3)$$

Formally integrating out the electric field, we find the Lagrangian:

$$L = \frac{1}{2}\int dt d^2x \Big[\frac{(\partial_t A_x)^2}{e^2} + \frac{(\partial_t A_y)^2}{-\ell^2\nabla^2} - \frac{(\partial_i A_j)^2}{g_3^2} + k\epsilon_{\mu\nu\rho}A_\mu\partial_\nu A_\rho + 2V_\mu J^\mu(A)\Big].$$

Integrating out the statistical gauge field, we find:

$$\mathbf{G} = \begin{pmatrix} \frac{p_1^2}{e^2} + \frac{p_2^2}{\ell^2 p^2} & -ikp_2 & ikp_1 \\ ikp_2 & -\frac{p^2}{g_3^2} & 0 \\ -ikp_1 & 0 & -\frac{p^2}{g_3^2} \end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix} 0 & ip_2 & -ip_1 \\ -ip_2 & 0 & 0 \\ ip_1 & 0 & 0 \end{pmatrix}.$$

Now, the static response functions are anisotropic. For instance, if we turn on an external electric field in the x-direction, we find:

$$c^1 = \lim_{p_1 \to 0} \lim_{p_2 \to 0} K_{00} = \lim_{p_1 \to 0} \frac{3g_3^2 p_1^2}{e^2 g_3^2 k^2 + p_1^2} = 0,$$

$$\sigma_H^1 = \lim_{p_1 \to 0} \lim_{p_2 \to 0} K_{02} = \frac{1}{k}.$$

Alternatively, if we turn on an external electric field in the y-direction, we find:

$$c^2 = \lim_{p_2 \to 0} \lim_{p_1 \to 0} K_{00} = \frac{3g_3^2}{1 + g_3^2 \ell^2 k^2},$$

$$\sigma_H^2 = \lim_{p_2 \to 0} \lim_{p_1 \to 0} \frac{K_{01}}{ip_2} = \frac{g_3^2 \ell^2 k}{1 + g_3^2 \ell^2 k^2}.$$

These behaviors are the same as those which we found in the massive phase and at the LCS critical point, respectively.

The dynamic response is similarly anisotropic, with longitudinal conductivity determined by:

$$K_{11} = -\,3\ell^2 p^2$$
$$K_{22} = -\,3e^2 + \mathcal{O}(p^2).$$

This system therefore shows quantum Hall-like behavior in the "1" direction, and superconducting behavior in the "2" direction! (Assuming the delta function peak is not broadened by disorder, which one expects not to happen for systems of this general type).

If the conserved current in this theory is interpreted as the electromagnetic current, then this theory describes a transition between a quantum Hall state (the massive phase), and a state which supports a current in the ground state (the anisotropic phase). Presumably the system then breaks up into domains (similar to a ferromagnet) so there is no net macroscopic current. Various phases of the 2D electron gas which have vaguely similar anisotropic behavior, or which have non-zero currents (at least when the sample is bombarded by external microwaves), have already been observed. It would be very interesting (but perhaps too much to expect) if this theory could provide a framework for some of these experiments.

Mani et al; Zudov et al;
Andreev, Aleiner, Millis;
Durst, Sachdev, Read, Girvin

To conclude, I would like to mention that gravitational systems with a Lifshitz-like metric (which enjoys a symmetry under scaling with dynamical exponent z)

$$ds^2 = L^2 \left(-r^{2z} dt^2 + r^2 d\mathbf{x}^2 + \frac{dr^2}{r^2} \right),$$

S.K., Liu, Mulligan

have been a subject of some study in recent years. In addition to their intrinsic interest as potential duals for systems with dynamical scaling, they arise in certain attempts to explain strange metallic behavior; are the backreacted geometries of probe fermions in some gravity realizations of non-Fermi liquids; and capture the near-horizon limits of Maxwell-Dilaton black holes in AdS.

Hartnoll, Polchinski, Silverstein, Tong;
Goldstein, S.K., Prakash, Trivedi

It would be nice to have candidate large N field theories that could match to some of these gravity backgrounds. A nonabelian extension of the abelian theories I discussed here, without a Chern-Simons coupling (for now), is given by the action:

$$S[A_0, A_i, E_i] = \int d^2x d\tau \mathrm{Tr}\Big(\frac{1}{g_1^2}(E_i \partial_\tau A_i + A_0 D_i E_i) - \frac{1}{2g_2^2}(D_j E_i)^2 + \frac{1}{2g_3^2}B^2 + i\frac{\kappa}{2}\epsilon_{ij}[E_i, E_j]B + \lambda[E_i, E_j]^2\Big),$$

$$D_i V = \partial_i V - i[A_i, V] = (\partial_i V^a + f^{abc} A_i^b V^c)t^a,$$

$$B(A) = \partial_1 A_2 - \partial_2 A_1 - i[A_1, A_2] = (\partial_1 A_2^a - \partial_2 A_1^a + f^{abc} A_1^b A_2^c)t^a$$

We are currently finishing a study of the most basic aspects of the RG structure of this theory, with SU(N) gauge group. However, that would be the subject for another talk.