"Strings 2010"

Mitchell Institute, College Station, March 19

Y-system for the Spectrum of Planar AdS/CFT: News and Checks

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(ENS, Paris)

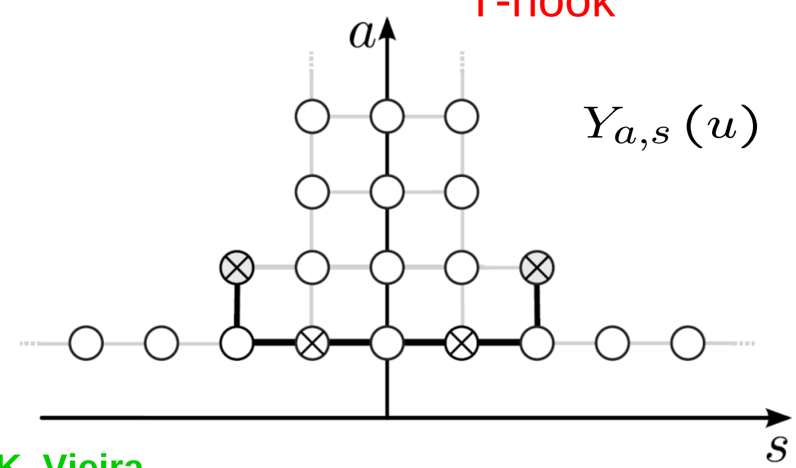
Collaborations with N.Gromov, A.Kozak, S.Leurent, Z.Tsuboi and P.Vieira

Outline

- Integrability for QFT in dimensions D>2? Possible for some SYM's at D=3,4…
- Based on AdS/CFT duality to 2D superstring σ-models on AdS-type backgrounds
- String σ-models in light-cone gauge look like "massive non-relativistic" 2D QFT's
- All popular 2D integrability tools applicable: S-matrix, asymptotic Bethe ansatz (ABA), TBA for finite size spectrum, ...
- .... Y-system (for planar AdS_5/CFT_4 , AdS_4/CFT_3 ,...)

$$\frac{Y_{a,s}(u+\frac{i}{2})\,Y_{a,s}(u-\frac{i}{2})}{Y_{a+1,s}\,Y_{a-1,s}}=\frac{\left[1+Y_{a,s+1}\right]\;\left[1+Y_{a,s-1}\right]}{\left[1+Y_{a+1,s}\right]\;\left[1+Y_{a+1,s}\right]}$$

T-hook



Conjecture: it calculates anomalous dimensions of all local operators in planar N=4 SYM theory at any coupling

Gromov,V.K.,Vieira

N=4 SYM and a string on $AdS_5 \times S^5$

$$\mathcal{S}_{SYM} = \frac{1}{\lambda}\int d^4x \,\mathrm{Tr}\left(F^2 + (\mathcal{D}\Phi)^2 + \bar{\Psi}\mathcal{D}\Psi + \bar{\Psi}\Phi\Psi + [\Phi,\Phi]^2\right)$$

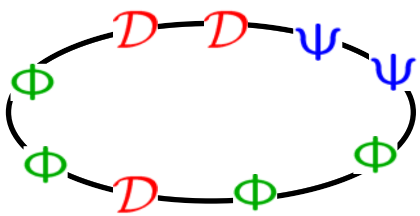
$\leftrightarrow$

$$\mathcal{S}_{sigma} = \sqrt{\lambda}\int d\tau \int_0^L d\sigma \left[\left(\partial\vec{X}\right)^2 + \Lambda(\vec{X}^2 - I) + \text{fermions}\right]$$

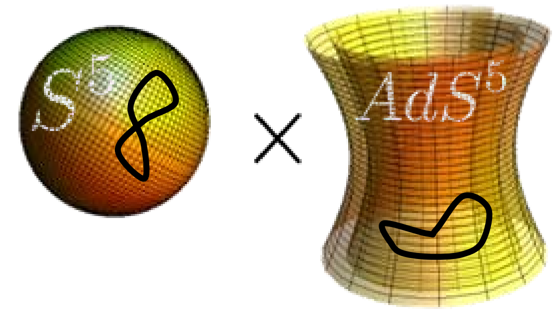
Metsaev-Tseytlin sigma model of GS superstring

CFT/AdS duality

$$\mathcal{O}(x) = \mathrm{Tr}\,[\mathcal{D}\mathcal{D}\Psi\Psi\Phi\Phi\mathcal{D}\Psi\ldots](x) \quad \leftrightarrow$$



$$\lambda = g_{YM}^2 N = \frac{R^4}{\alpha'^2} = g_s N$$



$$\mathcal{O}_A(\Lambda x) \to \Lambda^{\Delta_A(\lambda)}\mathcal{O}_A(x)$$

$$\hat{D}\,\mathcal{O} = \Delta\,\mathcal{O}$$

Dilatation operator
as a spin chain hamiltonian

Dimension of a local operator = Energy of a string state

Global superconformal symmetry → psu(2,2|4) ← isometry of background

*

Integrability in planar AdS/CFT

- Origins of YM integrability: Lipatov's reggeon Hamiltonian

Lipatov
Faddeev,Korchemsky

- Perturbative integrability in1,2,3,4,..∞ –loops in N=4 SYM

Minahan,Zarembo,
Beisert,Kristijanssen,Staudacher

- Classical integrability of string σ-model on $AdS_5\times S^5$
  Finite gap solution and quasilcassics

Bena,Roiban,Polchinski
V.K.,Marshakov, Minahan, Zarembo,
Beisert,V.K.,Sakai,Zarembo
Gromov,Vieira

- S-matrix, asymptotic Bethe ansatz (ABA). Cusp dimension

Arutyunov,Frolov,Staudacher
Staudacher, Beisert
Janik
Hernandez,Lopez
Beisert,Eden,Staudacher

- Finite size corrections and mirror model

Ambjorn,Janik,Kristijanssen
Arutyunov,Frolov
Bajnok,Janik

- Y-system for all operators at all couplings, Konishi dimension numerics

Gromov , V.K, Vieira

- TBA for AdS/CFT, confirming and clarifying the Y-system

Bombardelli,Fioravanti, Tateo
Gromov , V.K, Kozak,Vieira
Arutyunov, Frolov

- New integrable AdS/CFT's: AdS4/CFT3, AdS3/CFT2,.....

Minahan,Zarembo,
Gromov , V.K, Vieira
Babichenko,Stefanski,Zarembo
Zarembo

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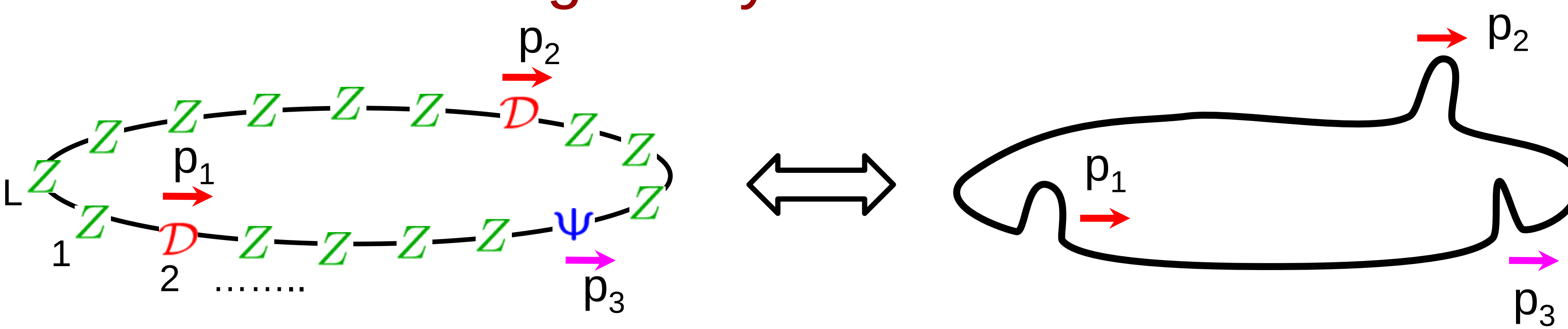
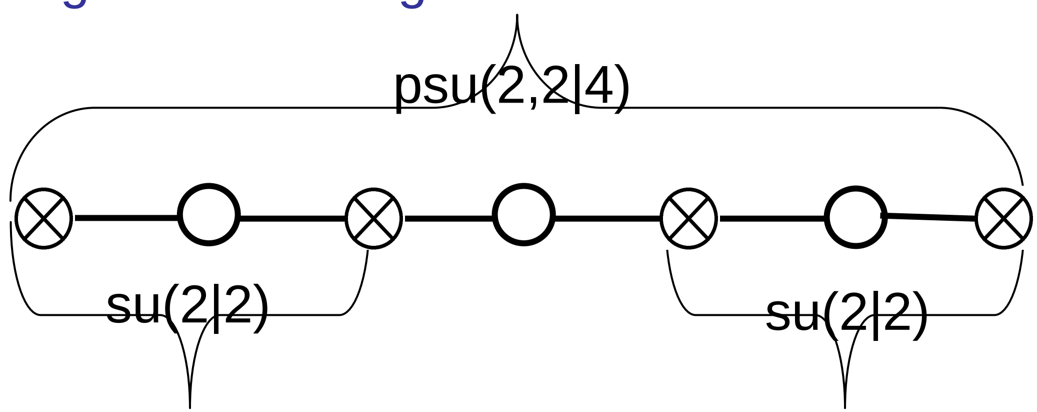
Integrability and S-matrix



• Light cone gauge breaks the global and world-sheet Lorentz symmetries :



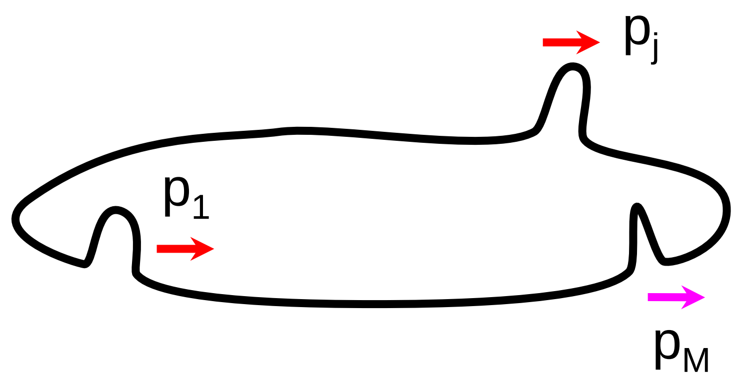
• S-matrix of AdS/CFT via bootstrap of A.&Al.Zamolodchikov (Hubbard R-matrix!)

$$\hat{S}_{PSU(2,2|4)}(p_1,p_2) = S_0^2(p_1,p_2) \times \hat{S}_{SU(2|2)}(p_1,p_2) \times \hat{S}_{SU(2|2)}(p_1,p_2)$$

• Asymptotic integrability : factorized scattering, asymptotic Bethe ansatz…

Asymptotic Bethe Ansatz (ABA)

$$0 = \left(e^{iLp_j} \prod_{k\neq j}^{M} \hat{S}(p_j, p_k) - 1 \right) \Bigg| \quad \Bigg\rangle$$



- This periodicity condition is diagonalized by nested Bethe ansatz
- Asymptotic dispersion relation for one « particle »

Santambogio,Zanon
Beisert,Dippel,Staudacher

$$\epsilon(p) = \sqrt{1 + \lambda \sin^2 \frac{p}{2}}$$

$$—Z—Z—\mathcal{D}—Z—Z—Z—Z—$$

- Energy of state

$$E = \Delta - L = \sum_{j=1}^{M} \epsilon(p_j) \quad + \quad O\left(e^{-c\,L}\right)$$

finite size corrections,
important for short operators!

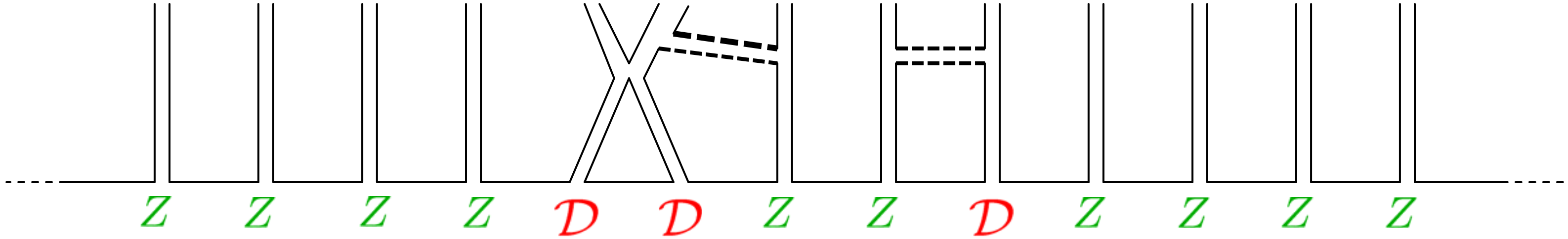
- Dispersion in Zhukovski parametrization:

$$p(u) = i \log \frac{x(u+i/2)}{x(u-i/2)}, \qquad \epsilon(u) = \cdots$$

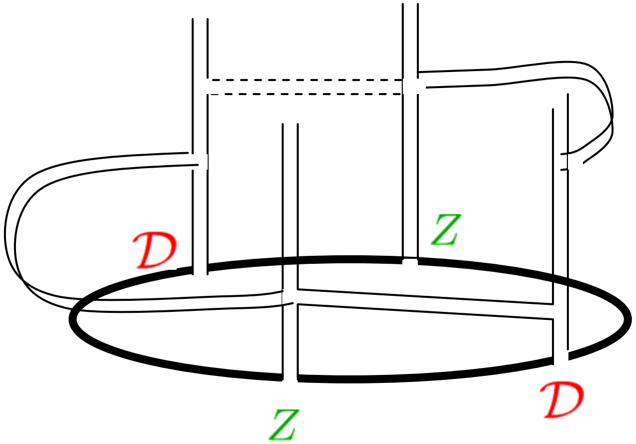
$$x + \frac{1}{x} = \frac{u}{\sqrt{\lambda}}$$

Perturbation theory and wrapping effects

- Unwrapped Feynman graphs - described by ABA



- Wrapped Feynman graphs - beyond the ABA - Y-system

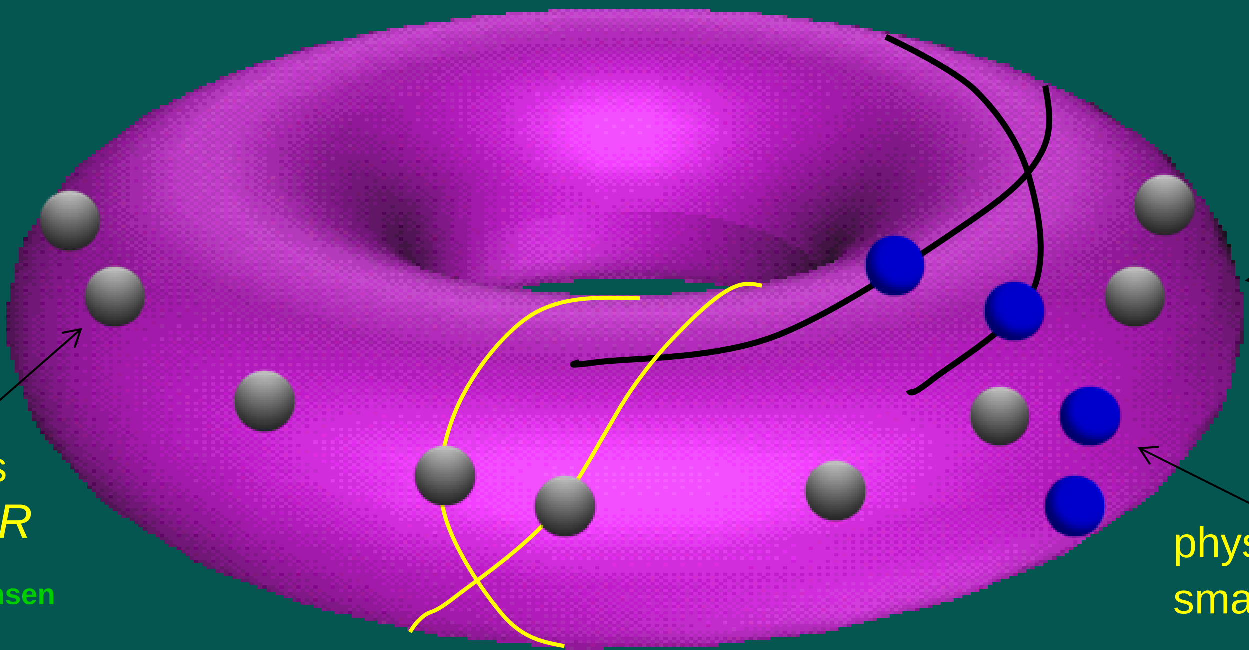


$$\mathcal{O}_{Konishi} = \mathrm{Tr}\ [\mathcal{D}, Z]^2$$

- We need to take into account the finite size effects

Yang-Yang TBA for finite size (Al.Zamolodchikov trick)

- 2d vacuum consists of a dense gas of virtual (mirror) “particles”
- Excited states: a few physical particles added
- Scattering: mir-mir, mir-phys, phys-phys



Ambjorn,Janik,Kristjansen
Arutyunov,Frolov

$$\mathcal{Z} = \sum_n e^{-E_n^{mir}(R)L} = \sum_n e^{-E_n^{ph}(L)R} \xrightarrow[R\to\infty]{} e^{-E_0^{ph}(L)R}$$

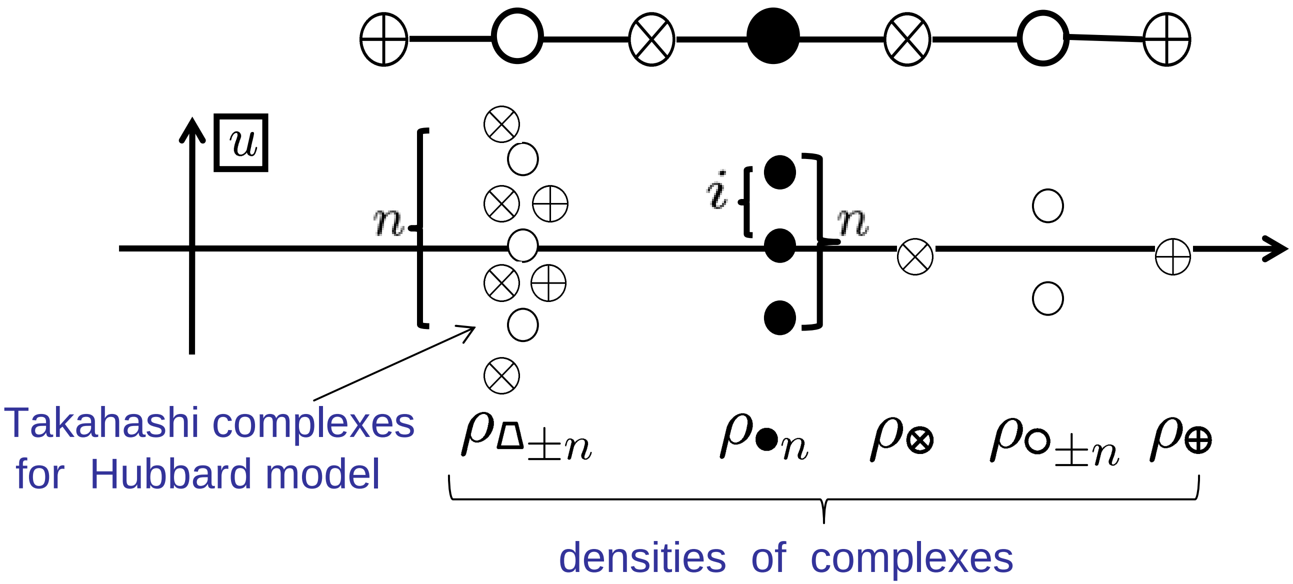
- Large R : mirrror rapidities localize on poles of S-matrix → bound states
- Fusion is simple on a *u*-plane (rapidity)

$$S_{ab}^{cd}(p(u),p(v)) \sim \frac{1}{u-v+i}$$

Bound states and TBA in AdS/CFT

Gromov,V.K.,Vieira
Bombardelli,Fioravanti,Tateo
Gromov,V.K.,Kozak,Vieira
Arutyunov,Frolov

N.Dorey



- Thermodynamic Bethe equations:

$$\rho_A + \bar{\rho}_A = \delta_{A,\bullet_n} \cdot p^{mir}_{\bullet_n} + \sum_{A'} K_{A',A} * \rho_{A'}$$

index

$$A = \{\circ_{\pm n}, \oplus_{\pm}, \otimes_{\pm}, \triangle_{\pm n}, \bullet_n\}$$

- Kernels (scattering phases of complexes)

$$K_{A',A} = \frac{\partial}{\partial u'} \log \mathcal{S}_{A',A}(u',u)$$

*

Y-system for the spectrum of AdS/CFT

- TBA: Minimizing free energy at finite temperature T=1/L

$$f = \underbrace{L \sum_{A\in\bullet_n} \int \rho_A \ \epsilon_A^{mir}}_{\text{energy}} - \underbrace{\sum_A \int \left[\rho_A \log\left(1+\frac{\bar\rho_A}{\rho_A}\right) - \bar\rho_A \log\left(1+\frac{\rho_A}{\bar\rho_A}\right)\right]}_{\text{entropy}}$$

- Extremum condition – TBA equation

$$\log Y_A = \delta_{A,\bullet_n} L\,\partial_u \epsilon_{\bullet_n}^{mir} + \sum_{A'} K_{A,A'} * \log\left(1+Y_{A'}(u)\right)$$

only ratios enter!

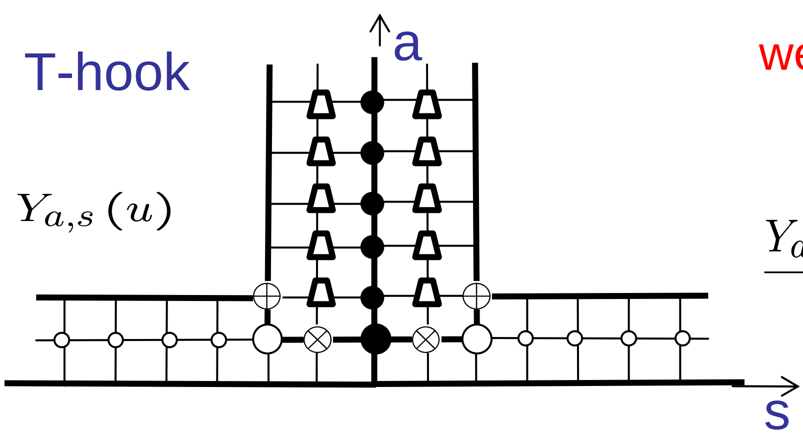
$$Y_A(u) = \frac{\bar\rho_A(u)}{\rho_A(u)}$$

- Acting by discrete Laplace operator on both sides of this eq.

$$\Delta f_a(u) \equiv f_a(u+i/2) + f_a(u-i/2) - f_{a+1}(u) - f_{a-1}(u)$$

T-hook

$Y_{a,s}(u)$



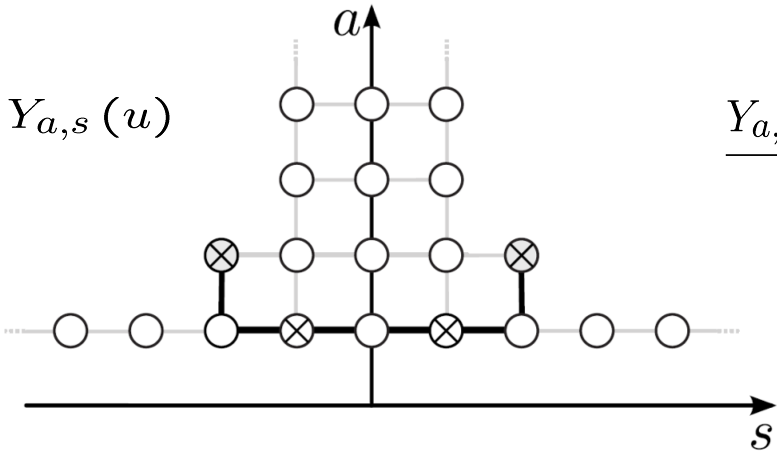
we get the universal Y-system!

$$\frac{Y_{a,s}(u+i/2)\,Y_{a,s}(u-i/2)}{Y_{a+1,s}\,Y_{a-1,s}} = \frac{\left[1+Y_{a,s+1}\right]\,\left[1+Y_{a,s-1}\right]}{\left[1+Y_{a+1,s}\right]\,\left[1+Y_{a+1,s}\right]}$$

*

Y-system for excited states of AdS/CFT

$Y_{a,s}\,(u)$



$$\frac{Y_{a,s}(u+\frac{i}{2})\,Y_{a,s}(u-\frac{i}{2})}{Y_{a+1,s}\,Y_{a-1,s}}=\frac{\left[1+Y_{a,s+1}\right]\,\left[1+Y_{a,s-1}\right]}{\left[1+Y_{a+1,s}\right]\,\left[1+Y_{a+1,s}\right]}$$

Gromov,V.K.,Vieira

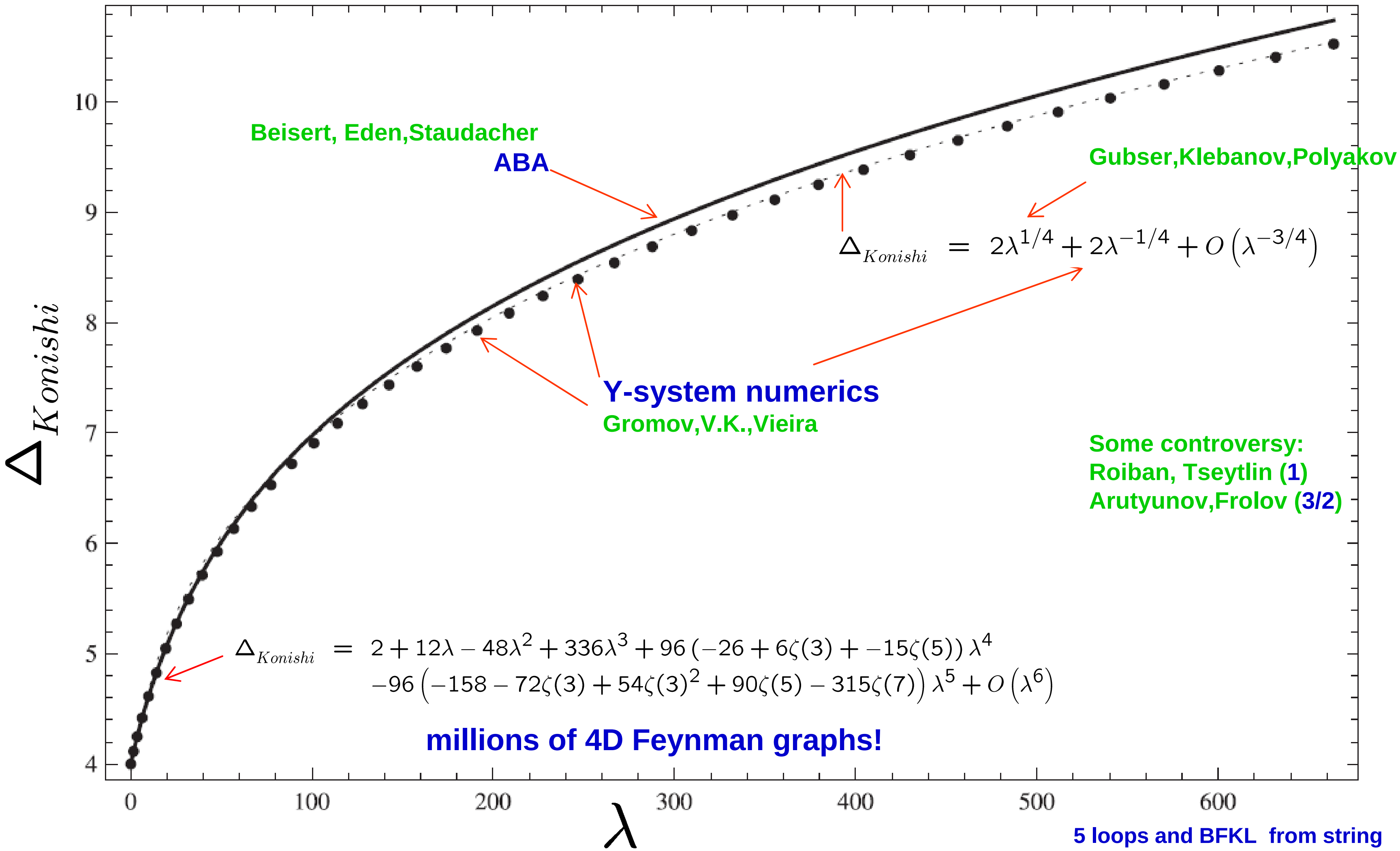
- Anomalous dimensions: $\Delta = f_{min} = \int \frac{du}{2\pi i}\,\partial_u p_a^{mir}\,\log\left(1+Y_{a,0}\right)$

- u_j obey the exact Bethe eq.: $Y_{1,0}(u_j)+1=0$

Bazhanov, Lukyanov, Zamolodchikov
Dorey,Tateo
Fioravanti,Quattrini,Ravanini

For AdS/CFT Gromov,V.K.,Kozak,Vieira

Konishi dimension numerics and various checks for **Y-system**



4 loops from SYM
Fiamberti,Santambrogio,Sieg,Zanon
Velizhanin
4 loops from string
Bajnok,Janik
Gromov,V.K.,Vieira

- Y-system passes all known tests

5 loops and BFKL from string
Bajnok,Janik,Lukowski
Lukowski,Rej,Velizhanin,Orlova
Arutyunov, Frolov
Balog, Hegedus

Fiamberti,Santambrogio,Sieg
Minahan,Ohlsson,Sax,Sieg

Y-system is integrable by itself !

- Full quantum AdS/CFT Y-system obeys a discrete integrable classical Hirota dynamics

- This should help to reduce the spectral problem to a ***finite*** system of nonlinear equations

Y-system and Hirota eq.: discrete integrable dynamics

- Relation of Y-system to T-system (Hirota equation)
  . (the Master Equation of Integrability!)

$$Y_{a,s} = \frac{T_{a,s+1}\,T_{a,s-1}}{T_{a+1,s}\,T_{a-1,s}}$$

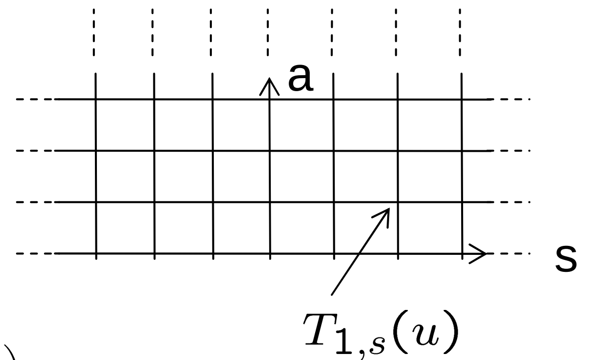
$$T_{a,s}(u+\tfrac{i}{2})T_{a,s}(u-\tfrac{i}{2}) = T_{a,s-1}(u)\,T_{a,s+1}(u)+T_{a+1,s}(u)\,T_{a-1,s}(u)$$

- Discrete classical integrable dynamics!

- General solution in half-plane lattice:

$$T_{a,s}(u) = \mathrm{Det}_{\,1\le k,j\le a}\; T^{[k+j]}_{1,s+k-j}$$

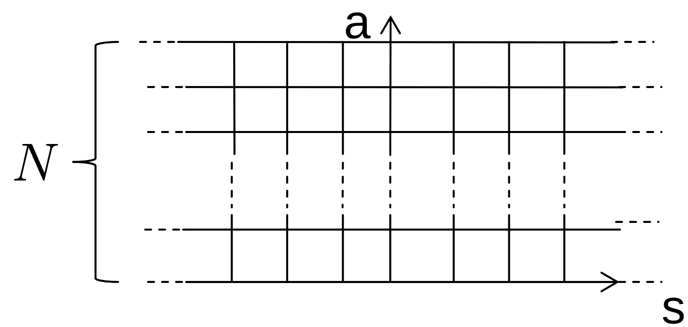
$$f^{[k]} \equiv f\left(u+\frac{i\,k}{2}\right)$$



For spin chains :
Bazhanov,Reshetikhin
V.K.,Vieira

More interesting solution of Hirota equation ....

- General solution in a strip:



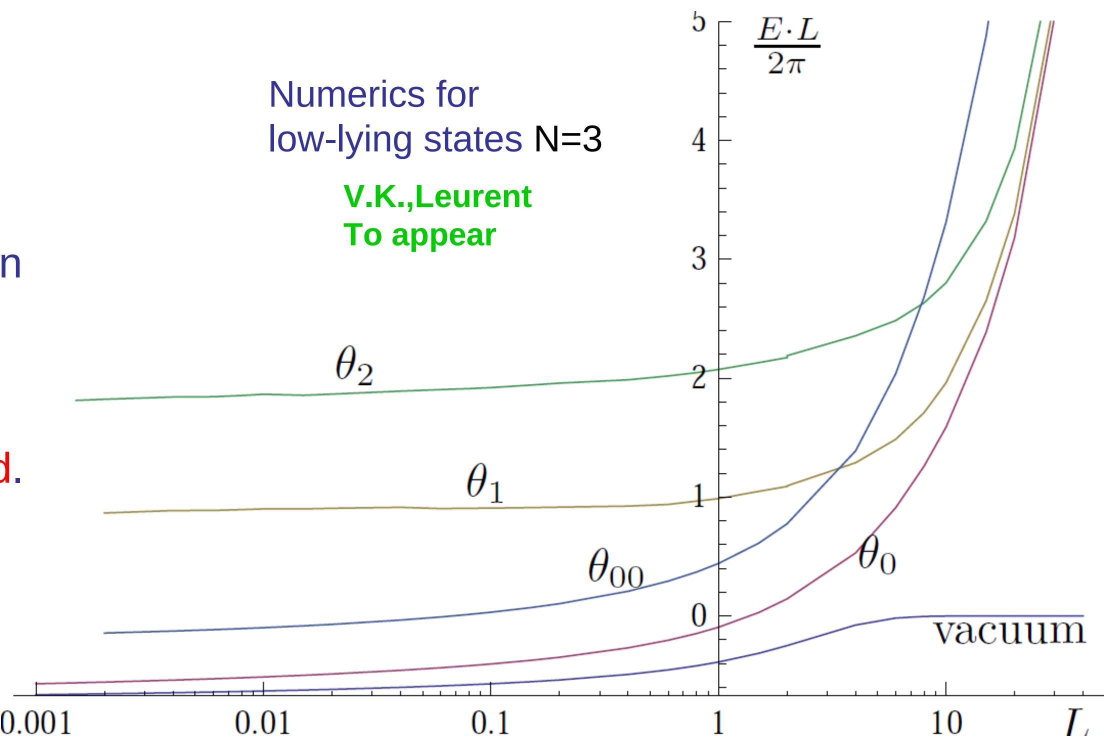
$$T_{a,s}(u) = \left| \begin{array}{ccc} \bar{q}_1^{[-s-1]} & \cdots & \bar{q}_N^{[-s-1]} \\ \cdots & \cdots & \cdots \\ \bar{q}_1^{[-s-a]} & \cdots & \bar{q}_N^{[-s-a]} \\ q_1^{[s-a-1]} & \cdots & q_N^{[s-a-1]} \\ \cdots & \cdots & \cdots \\ q_1^{[s-N]} & \cdots & q_N^{[s-N]} \end{array} \right| \begin{array}{l} \left.\vphantom{\begin{array}{c} a \\ a \\ a \end{array}}\right\} a \\ \left.\vphantom{\begin{array}{c} a \\ a \\ a \end{array}}\right\} N-a \end{array}$$

Krichever,Lipan, Wiegmann,Zabrodin

$$q^{[k]} \equiv q(u + i\,k/2)$$

Numerics for low-lying states N=3

V.K.,Leurent To appear



- This can be efficiently used for solution of various sigma models at finite size. Example: exact finite system of non-linear integral equations for the spectrum of SU(N) principal chiral field.

Gromov,V.K.,Vieira
V.K.,Leurent

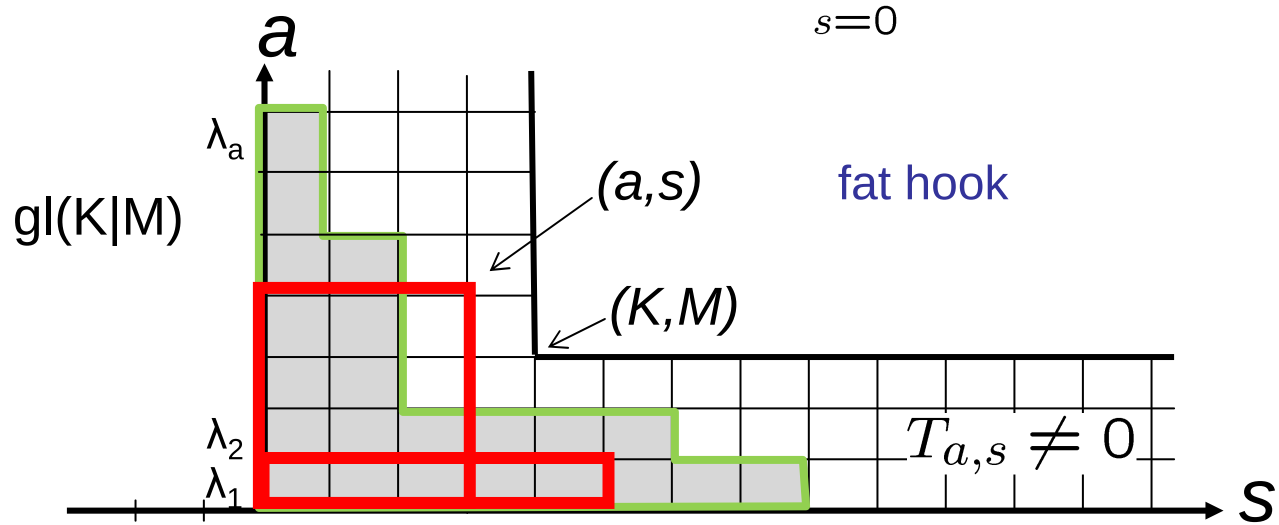
Hirota equation and characters of supergroups

- Jacobi-Trudi formula for GL(K|M) characters in general irrep $\lambda=\{\lambda_1,\lambda_2,\ldots,\lambda_a\}$

$$T_{\{\lambda\}}[g] = \det_{1\le i,j\le a} T_{1,\lambda_i-i+j}[g]\,, \qquad g\in GL(K|M).$$

super-characters
in symmetric irreps

$$w(z) \equiv \mathrm{sdet}\ (1-zg)^{-1} = \sum_{s=0}^{\infty} T_{1,s}[g]\, z^s$$



- For rectangular Young tableaux (a,s) : Hirota eq. with fat hook b.c.:

$$T_{a,s}^2 = T_{a+1,s}T_{a-1,s} + T_{a,s+1}T_{a,s-1}$$

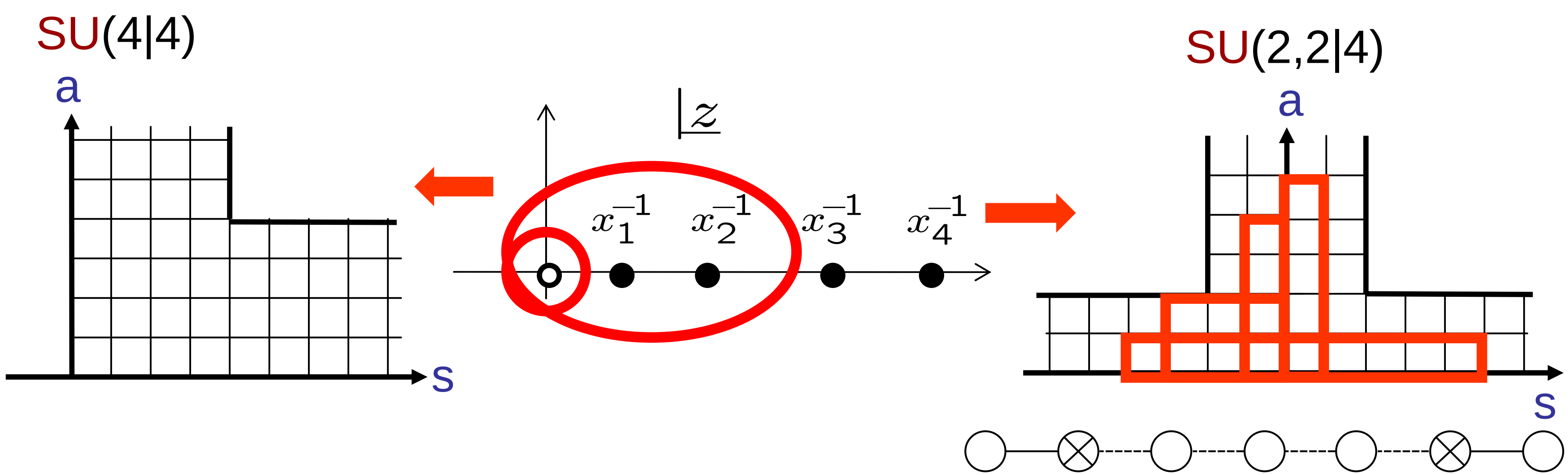
For super-spin chains:
Tsuboi
V.K.,Zabrodin,Sorin

Super-characters: Fat Hook of U(4|4) and T-hook of SU(2,2|4)

$$w(z;g) = \frac{(1-zy_1)(1-zy_2)}{(1-zx_1)(1-zx_2)}\frac{(1-zy_3)(1-zy_4)}{(1-zx_3)(1-zx_4)} = \sum_s T_{1,s}\, z^s$$

$$g = \mathrm{diag}\{x_1, \cdots, x_K | y_1, \cdots, y_M\}$$

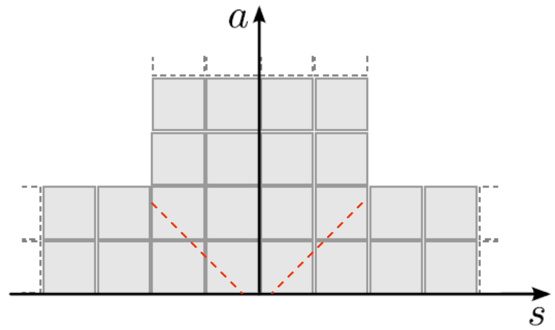
$$T_{1,s} = \oint \frac{dz\, z^{-s-1}}{2\pi i} w(z;g)$$



Unitary highest weight representations of u(2,2|4) !

Constructed by:
Kwon
Cheng,Lam,Zhang

Character solution of T-hook for u(2,2|4)



Gromov,V.K.,Tsuboi

- Solution in finite 2×2 and 4×4 determinants (analogue of the 1-st Weyl formula)

$$T_{a,s} = \begin{cases} (-1)^{(a+1)s}\left(\dfrac{x_3 x_4}{y_1 y_2 y_3 y_4}\right)^{s-a} \dfrac{\det\left(S_i^{\theta_{j,s+2}}\, y_i^{\,j-4-(a+2)\theta_{j,s+2}}\right)_{1\le i,j\le 4}}{\det\left(S_i^{\theta_{j,0+2}}\, y_i^{\,j-4-(0+2)\theta_{j,0+2}}\right)_{1\le i,j\le 4}} \,, & a \ge |s| \\[2ex] \dfrac{\det\left(Z_i^{(1-\theta_{j,a})}\, x_i^{\,2-j+(s-2)(1-\theta_{j,a})}\right)_{1\le i,j\le 2}}{\det\left(Z_i^{(1-\theta_{j,0})}\, x_i^{\,2-j+(0-2)(1-\theta_{j,0})}\right)_{1\le i,j\le 2}} \,, & s \ge +a \end{cases}$$

$$\theta_{j,s} = \begin{cases} 1\,, & j > s \\ 0\,, & j \le s \end{cases}$$

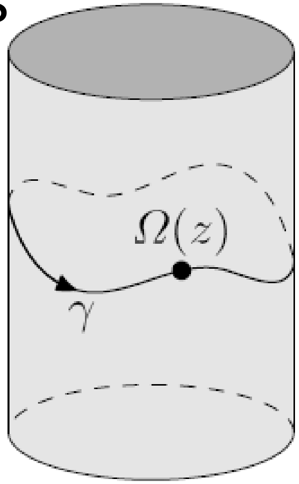
$$S_i = \frac{(y_i - x_3)(y_i - x_4)}{(y_i - x_1)(y_i - x_2)}$$

$$Z_i = \frac{(x_i - y_1)(x_i - y_2)(x_i - y_3)(x_i - y_4)}{(x_i - x_3)(x_i - x_4)}$$

- Generalization to full quantum case (u-dependence) possible!

Gromov,V.K.,Leurent,Tsuboi
(in progress)
See also: Hegedus

Classics and quasiclassics from u(2,2|4) characters



Gromov
Gromov,V.K.,Tsuboi

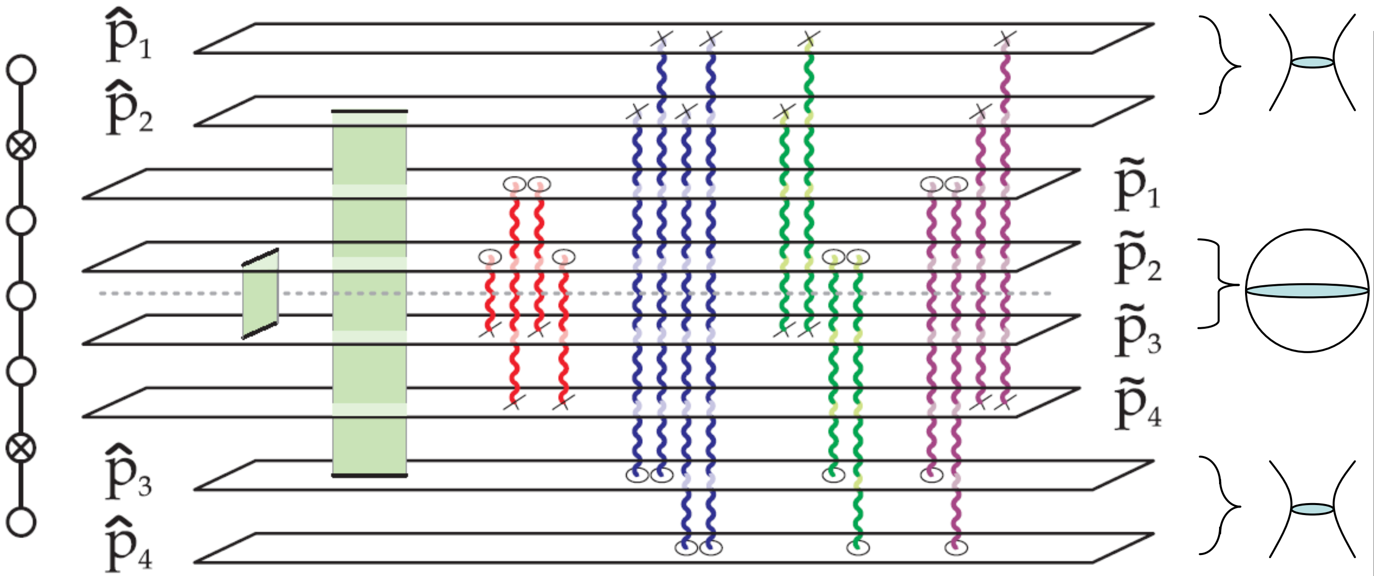
- Monodromy matrix and algebraic curve for quasimomenta

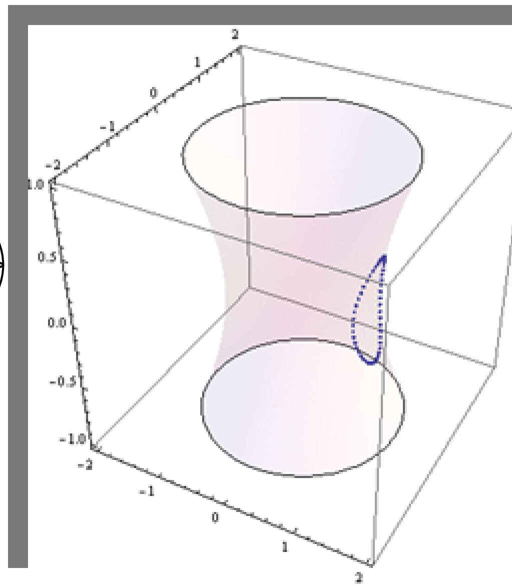
Bena,Roiban,Polchinski'02

$$(d+\mathcal{A}(z))\wedge(d+\mathcal{A}(z))=0, \qquad \Omega(z)=P\exp\oint\mathcal{A}(z)d\sigma$$

$$\Omega(z) = \{e^{i\tilde{p}_1(z)}, e^{i\tilde{p}_2(z)}, e^{i\tilde{p}_3(z)}, e^{i\tilde{p}_4(z)}||e^{i\hat{p}_1(z)}, e^{i\hat{p}_2(z)}, e^{i\hat{p}_3(z)}, e^{i\hat{p}_4(z)}\}$$

V.K.,MarshakovMinahan,Zarembo
Beisert,V.K.,Sakai,Zarembo





- Adding small extra cuts (Bethe roots) allows to compute one-loop corrections to each finite gap classical solution.
- They contain all finite size corrections (wrapping effects)
- Can we test the Y-system with them?

Gromov,Vieira

Quasiclassical solution of AdS/CFT Y-system

Gromov
Gromov,V.K.,Tsuboi

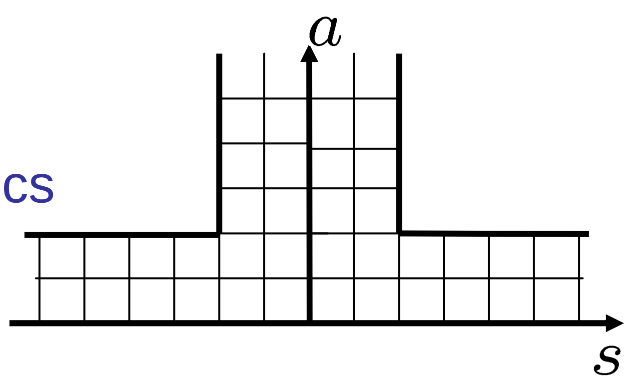
- Limit: large strings/operators , strong coupling, large momenta:

$$L \sim \sqrt{\lambda} \sim u \to \infty$$

- Explicit u-shift in Hirota eq. dropped (only slow parametric dependence)

$$T_{a,s}\left(u+\cancel{\frac{i}{2\sqrt{\lambda}}}\right) T_{a,s}\left(u-\cancel{\frac{i}{2\sqrt{\lambda}}}\right) = T_{a,s-1}(u)\, T_{a,s+1}(u) + T_{a+1,s}(u) T_{a-1,s}(u)$$

- (Quasi)classical solution - su(2,2|4) character !



- To fix 8 eigenvalues : compare a,|s| →∞ asymptotics with ABA, valid with accuracy $\sim \exp\left(-\text{const}\,\frac{L}{\sqrt{\lambda}}\right)$

- Result: eigenvalues ↔ quasimomenta :

$$x_j = e^{-i\tilde{p}_j},\; y_j = e^{-i\hat{p}_j} \qquad\qquad T_{a,s} = \text{Tr}_{a,s}\,\Omega$$

- Quasiclassical (1-loop) energy - same as from algebraic curve, e.g.

$$E_{wrap} = \int \frac{du}{2\pi i}\,\partial_u \epsilon_a(u) \log\left(1+Y_{a,0}(u)\right) = \oint \frac{dz}{2\pi i}\frac{z}{\sqrt{1-z^2}}\,\partial_z \log\left[\prod_{i=1,2;j=3,4}\frac{(1-y_i/x_j)(1-x_i/y_j)}{(1-x_i/x_j)(1-y_i/y_j)}\right]$$

Conclusions

- Y-system for exact spectrum of planar AdS/CFT is still only a conjecture but it passed many important checks.

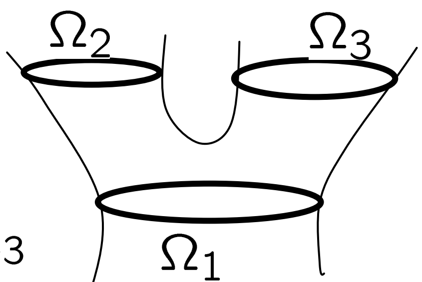
- A lot to do to understand where all this comes from.....

To do

- Building block for more complicated quantities: amplitudes, 1/N expansion etc.

classically

$$\Omega_1 = \Omega_2 \cdot \Omega_3$$



QM?

$$T_{a,s} = \langle \text{State} | \mathrm{Tr}\, \Omega_{a,s} | \text{State} \rangle$$

(natural !)

See also :
Mikhailov
Schaefer-Nameki

$$\langle \text{State} | \Omega_1 | \text{State} \rangle \overset{?}{=} \sum_S \langle \text{State} | \Omega_2 | S \rangle \cdot \langle S | \Omega_3 | \text{State} \rangle$$

- Y-system from SYM? A way to understand AdS/CFT correspondance?
- Finite set of equations out of integrable Y-system
- Regular weak and strong coupling expansions
- Common origin of spectrum integrability and of symmetries of *gluon amplitudes* (dual conformal sym., Yangian...)

Drummond,Henn,Korchemsky,Sokatchev
Beisert,Henn,McLoughlin,Plefka
Alday,Maldacena,Sever,Vieira,

- Lessons for QCD? BFKL via Y-system?
- New AdS/CFT's : new Y-systems?   CP3xAdS4 example works!

Zarembo

Gromov,V.K.,Vieira
Bombardelli,Fioravanti,Tateo
Gromov,Levkovich-Maslyuk