2d CFTs from 4d $\mathcal{N} = 2$ gauge theories

Yuji Tachikawa

based on the work in collaboration with

L. F. Alday and **D. Gaiotto**,

and on discussions with many others

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Message

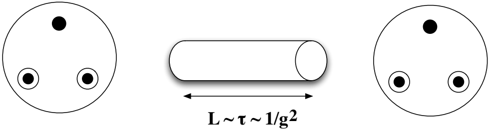
By wrapping N M5-branes on $X_4 \times C_2$,

Physical quantity calculated on X_4

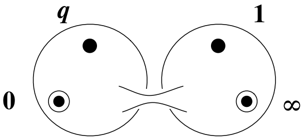
$= $ Physical quantity calculated on C_2

Ingredients and Combination

- Want to make a $\mathbf{SU}(N)$ with $\mathbf{2}N$ fundamentals?
- Take N fundamental hypers, gauge fields, another N fundamental hypers, constructed from M5-branes



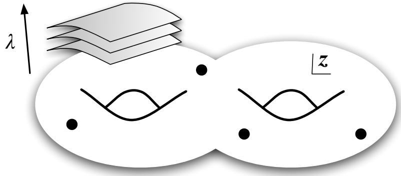
- Connect!



- $q \sim \exp(i\tau)$.

More precisely

- 11d spacetime is $\mathbb{R}^{3,1} \times T^*C \times \mathbb{R}^3$.
- Wrap N M5-branes at $\Sigma \subset T^*C$



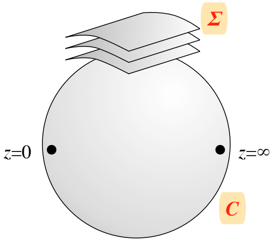
$$\Sigma : \lambda^N + \phi_2(z)\lambda^{N-2} + \cdots + \phi_N(z) = 0$$

- Degree-k form ϕ_k = worldvolume fields.
- One-form λ = position along T^*C = Seiberg-Witten differential

Pure SU(N)

$$\Lambda^N(z + \frac{1}{z}) = y^N + u_2 y^{N-2} + \cdots + u_N$$

[Martinec-Warner]



Let $\lambda = y dz/z$. Then

$$\lambda^N + \phi_2(z)\lambda^{N-2} + \cdots + \phi_N(z) = 0$$

where

$$\phi_k(z) = u_k \left(\frac{dz}{z}\right)^k, \qquad \phi_N(z) = (\Lambda^N z + u_N + \frac{\Lambda^N}{z}) \left(\frac{dz}{z}\right)^k.$$

4d gauge theory vs 2d CFT

- We now have a **map**

$$G_N : \text{ Riemann surface } C \text{ with punctures} \longrightarrow \text{ 4d field theory}$$

- G_N **behaves nicely under degenerations** of C
  i.e. if you split C, an $\mathbf{SU}(N)$ becomes weak,
  and the gauge theory is also split.

- Take whatever physical quantity Z calculable in 4d:

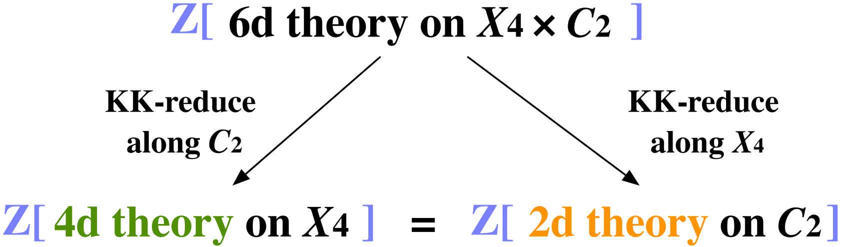
$$Z : \text{ 4d field theory} \longrightarrow \text{ number}$$

- Then, $Z(G_N(C))$ **factorizes under degenerations** of C,

- This morally means that $Z \circ G_N$ gives a **2d CFT**.

6d, 4d and 2d

- N M5-branes wrapped on $X_4 \times C_2$
- Consider a physical quantity Z of the 6d theory,
- Assume Z is protected in some sense, so that it can be evaluated by KK reduction.



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Z[4d theory on X_4] = Z[2d theory on C_2]

4d vs 2d

- **Nekrasov's instanton partition function**
  $\leftrightarrow$ the **Virasoro/W_N conformal block**.

- **Full partition function** on S^4
  $\leftrightarrow$ the **Liouville/Toda correlation function** at $b = 1$.

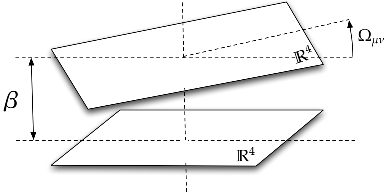
- **4d anomaly polynomial** $\leftrightarrow$ **2d anomaly polynomial**
  [Bonelli-Tanzini,Alday-Benni-YT]

- **Superconformal Index** $\leftrightarrow$ a **2d TQFT**
  [Gadde-Pomoni-Rastelli-Razamat]

Nekrasov's instanton partition function

$$Z(a;\epsilon_1,\epsilon_2) = \mathbf{exp}\left(\frac{\mathcal{F}(a)}{\epsilon_1\epsilon_2} + \cdots\right)$$

1. Take 5d version of the theory
2. Put it on a circle of length β.



3. Glue the two ends by

$$\begin{aligned} \text{4d rotation} \quad \Omega &= e^{\beta\epsilon_1 L_{12} + \beta\epsilon_2 L_{34}} \qquad \text{and} \\ \text{gauge rotation} \quad g &= \mathbf{diag}(e^{\beta a_1}, \ldots, e^{\beta a_N}) \end{aligned}$$

4. Take $\beta \to 0$.

Localization

$$Z(a) = \sum_k q^k \int_{\mathcal{M}_{N,k}} \text{(something complicated)}$$
$$= \sum_k q^k \sum_{\text{fixed point } p \text{ on } \mathcal{M}_{N,k}} \text{(contribution from } p\text{)}$$

- $\mathcal{M}_{N,k}$: k-instanton moduli of $\mathbf{U}(N)$ gauge theory.
- $\mathbf{SO}(4)$ and $\mathbf{SU}(N)$ act on $\mathcal{M}_{N,k}$.
- Fixed points on the moduli space =

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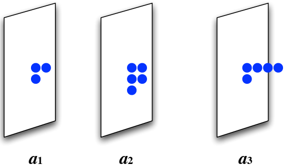


$$Z(a) = \sum_{\vec{Y}} q^{|\vec{Y}|} \cdot \text{(contribution from } \vec{Y}\text{)}$$

Pure SU(2)

$$Z(a) = \sum_{\vec{Y}} \Lambda^{4|\vec{Y}|} \frac{1}{G_a(\vec{Y})}$$

$G_a(\vec{Y})$: contribution from the **Gauge** multiplet.
Consider the vector space

$$V_a \ni |v\rangle\rangle = (v_{(0,0)};\ v_{(0,\square)},\ v_{(\square,0)};\ v_{(0,\square\square)},\ v_{(0,\begin{smallmatrix}\square\\\square\end{smallmatrix})},\ v_{(\square,\square)},\ \ldots)$$

Introduce the inner product via

$$\langle\langle v|w\rangle\rangle = \sum_{\vec{Y}} \frac{v_{\vec{Y}}\ w_{\vec{Y}}}{G_a(\vec{Y})}$$

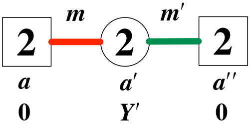
Consider this stupid vector

$$|w_a\rangle\rangle = (1;\ \Lambda^2,\ \Lambda^2;\ \Lambda^4,\ \Lambda^4,\ \ldots)$$

then

$$Z(a) = \langle\langle w_a|w_a\rangle\rangle$$

SU(2) with 4 flavors



$$Z = \sum q^{|\vec{Y}'|} \frac{H_a{}^m{}_{a'}(\vec{0},\vec{Y})\, H_{a'}{}^{m'}{}_{a''}(\vec{Y},\vec{0})}{G_a(\vec{0})\, G_{a'}(\vec{Y})\, G_{a''}(\vec{0})}$$

Let $H_m : V_{a'} \to V_a$ be such that for $v \in V_{a'}$

$$(H_m v)_{\vec{Y}} = \sum_{\vec{Y}'} \frac{H_a{}^m{}_{a'}(\vec{Y},\vec{Y}')\, v_{\vec{Y}'}}{G_{a'}(\vec{Y}')}$$

then

$$Z = \langle\!\langle a | H_m\, q^N\, H_{m'} | a'' \rangle\!\rangle$$

where N counts the number of boxes and

$$|a\rangle\!\rangle = (1; 0, 0; 0, 0, \ldots) \in V_a.$$

Summary so far & observation

	Pure $\mathbf{SU(2)}$	$\mathbf{SU(2)}$ with 4 flavors
Z	$\langle\langle w_a \| w_a \rangle\rangle$	$\langle\langle a \| H_m q^N H_{m'} \| a'' \rangle\rangle$

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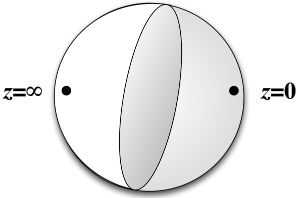
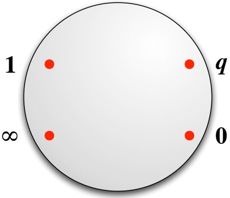
Gaiotto curve		

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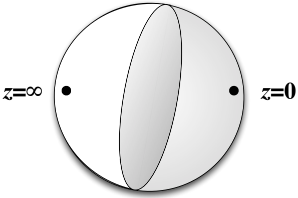
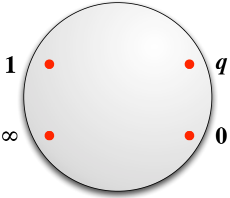
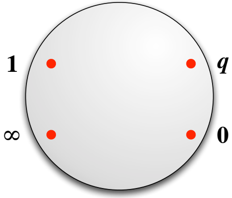
Gaiotto curve		
conformal block	$\langle w_a \vert w_a \rangle$	$\langle a \vert V_m(1) V_{m'}(q) \vert a'' \rangle$

Mapping: two M5-branes



- SW curve was $\lambda^2 = \phi_2(z)$.
- CFT has $T(z)$.
- Both have spin-2.

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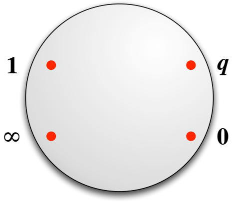

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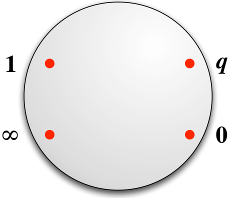
$$\langle T(z) \rangle dz^2 \to \phi_2(z) \quad \text{when} \quad \epsilon_{1,2} \to 0$$

where

$$c = 1 + 6Q^2, \qquad Q^2 = \frac{(\epsilon_1 + \epsilon_2)^2}{\epsilon_1 \epsilon_2}$$

SU(2) with four flavors



$$\Longleftrightarrow \quad \langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) V_{\alpha_3}(z_3) V_{\alpha_4}(z_4) \rangle$$

- What are the operators $V_\alpha(z)$?
- $\phi_2(z) \sim \dfrac{m_i^2 dz^2}{(z - z_i)^2} \quad \Longleftrightarrow \quad T(z) V_\alpha(z_i) \sim \dfrac{m_i^2}{(z - z_i)^2} V_\alpha(z_i)$

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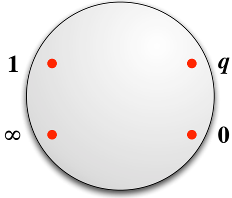

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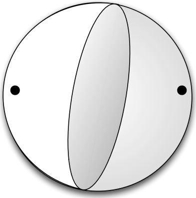
- $V_{\alpha_i}(z)$ is a **primary state** with dimension m_i^2.

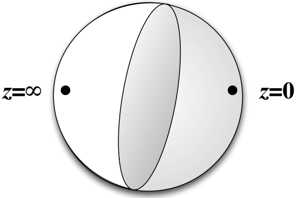
Pure SU(2)



$\Longleftrightarrow \quad \langle w_a | w_a \rangle$

- What's the state $|w_a\rangle$?
- $\lambda^2 = \phi_2(z)$ where $\phi_2(z) \sim (\frac{u}{z^2} + \frac{\Lambda^2}{z^3})dz^2$ around $z = 0$.
- Recall $\quad T(z)dz^2 \sim (\cdots + \frac{L_0}{z^2} + \frac{L_1}{z^3} + \cdots)dz^2.$

Pure SU(2)



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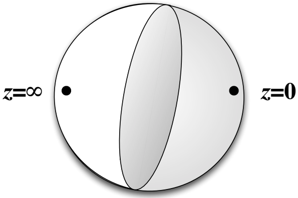
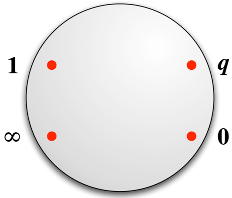
- Recall $\quad T(z)dz^2 \sim (\cdots + \frac{L_0}{z^2} + \frac{L_1}{z^3} + \cdots)dz^2$.

$$\longrightarrow L_0|w\rangle = (\frac{Q^2}{4} - a^2)|w\rangle, \quad L_1|w\rangle = \Lambda^2|w\rangle, \quad L_{n\geq 2}|w\rangle = 0.$$

- $|w_a\rangle$ is the **coherent state** of the Virasoro algebra !

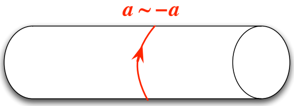
The Relation

	Pure $\mathbf{SU}(2)$	$\mathbf{SU}(2)$ with 4 flavors
Z	$\langle\!\langle w_a \| w_a \rangle\!\rangle$	$\langle\!\langle a \| H_m q^N H_{m'} \| a'' \rangle\!\rangle$
Gaiotto curve		
conformal block	$\langle w_a \| w_a \rangle$	$\langle a \| V_m(1) V_{m'}(q) \| a'' \rangle$

Z = **conformal block**

A word on Liouville

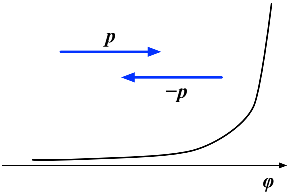
- Virasoro algebra emerged.
- What do we get from **2** M5-branes as the CFT?



- Each channel is labeled by one variable a with the identification $a \sim -a$.

- Such 2d CFT is bound to be Liouville [Teschner] [Pakman]
- Indeed, using [Pestun], we do get Liouville if we're on S^4

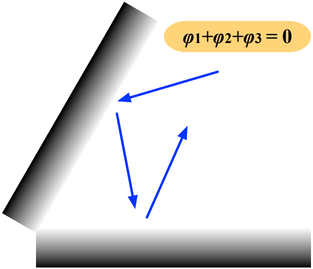
Liouville



- Reflection off an exponential wall $V(\varphi) = e^{2b\varphi}$
- $c = 1 + 6Q^2$ where the background charge is $Q = b + 1/b$

- $b^2 = \epsilon_1/\epsilon_2$.
- **DOZZ 3pt functions**
  $=$ **one-loop factors** in Nekrasov's partition function.
- Liouville reflection $p \sim -p \leftrightarrow$ Weyl reflection $a \sim -a$

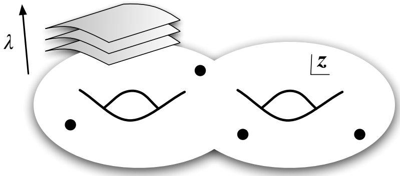
Toda



- Waves in $\varphi_1 + \varphi_2 + \cdots + \varphi_N = 0$.
- Reflection off exponential walls at $\varphi_i = \varphi_{i+1}$.
- Not just a CFT with $T(z)$, but has W_N symmetry.

$$W_2(z) = T(z), \quad W_3(z), \quad \ldots, \quad W_N(z)$$

Mapping: N M5-branes



- SW curve was $\lambda^N + \phi_2(z)\lambda^{N-2} + \cdots + \phi_N(z) = 0.$
- W-generators: $W_2(z) = T(z), W_3(z), \ldots, W_N(z)$

$$\langle W_k(z)\rangle dz^k \to \phi_k(z) \quad \text{when} \quad \epsilon_{1,2} \to 0$$

under vev $\vec{a} \leftrightarrow$ momentum $\vec{p}$, $\quad \frac{\epsilon_1}{\epsilon_2} \leftrightarrow b^2$

- Weyl reflection = Toda reflection

Status concerning conformal blocks

- Many checks for $\mathbf{SU(2)}$. [Marshakov,Mironov^2,Morozov^2], …
- [Fateev-Litvinov] **proved** the equality

  Z of SU(2) with massive adjoint

  = torus one-point conformal block

  by showing boths sides satisfy the same non-linear relation
  cf. [Poghossian], [Hadasz,Jaskólski,Suchanek]
- $\mathbf{SU}(N)$ relation is not as much studied.
  [Wyllard], [Mironov,Morozov], [Taki], … ,

How about U(1) ?

- Mathematicians [Carlsson-Okounkov] had done that in **2008**!
- They proved that

Nekrasov function of U(1) quiver

= correlator of a free boson

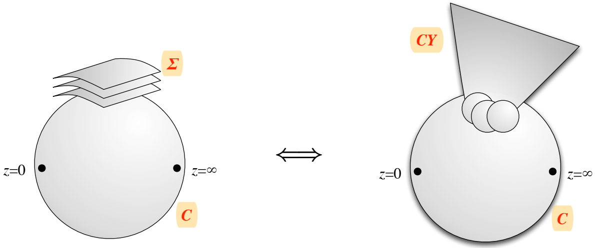
- We introduced the space

$$V_a \ni |v\rangle\rangle = (v_{(0,0)};\ v_{(0,\Box)},\ v_{(\Box,0)};\ v_{(0,\Box\Box)},\ v_{(0,\begin{smallmatrix}\Box\\\Box\end{smallmatrix})},\ v_{(\Box,\Box)},\ \ldots).$$

It's called the equivariant cohomology of the instanton moduli.

- Our observation **predicts** that
1) V_a is naturally the Verma module of W_N algebra, and
2) hypers should give the vertex operator.

Relation to topological strings/matrix models

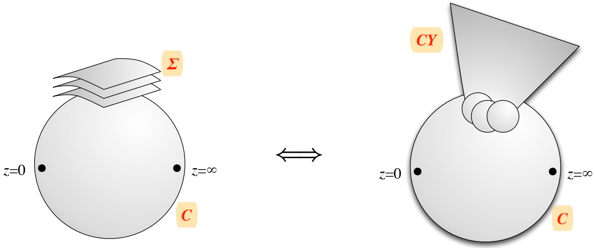


It was known ($\sim$ 2007) that when $\epsilon_1 + \epsilon_2 = 0$,

Nekrasov's Z

$= $ the part. function of the top. string on CY with $g_s = \epsilon_1$

Relation to topological strings/matrix models

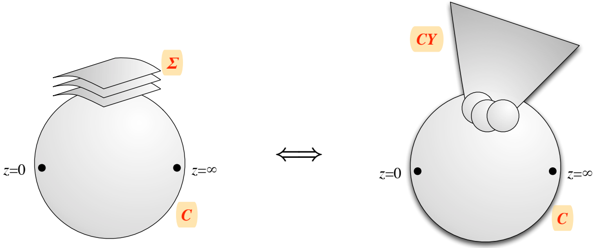


It was known (∼ 2007) that when $\epsilon_1 + \epsilon_2 = 0$,

Nekrasov's Z

$= $ the part. function of a 'free' chiral boson on Σ

Relation to topological strings/matrix models



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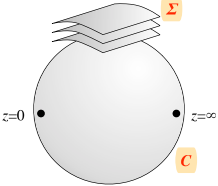
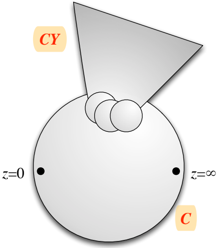
Relation to topological strings/matrix models



$\Longleftrightarrow$



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$=$ the part. function of N chiral boson on C

$=$ the part. function of a matrix model

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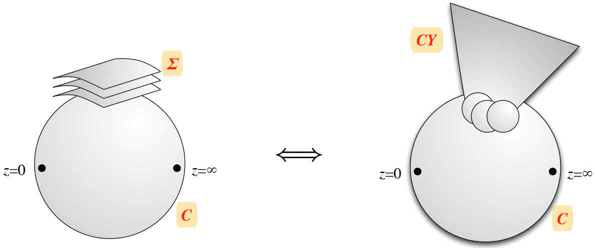

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$\epsilon_1 + \epsilon_2 \neq 0$ ⟶ N bosons into Liouville/Toda.

Vandermonde deformed. $\prod(\phi_i - \phi_j)^{-2b^2}$

[Dijkgraaf-Vafa],[Wyllard-Schiappa],[Eguchi-Maruyoshi],...

Summary

- Wrap N M5-branes on $X_4 \times C_2$.
- Physical quantity on X_4 gives physical quantity on C_2.
- **Nekrasov's instanton partition function**
  $\leftrightarrow$ the **Virasoro/W_N conformal block**.
- $\Sigma : \lambda^N + \phi_2(z)\lambda^{N-2} + \phi_3(z)\lambda^{N-3} + \cdots + \phi_N(z) = 0$
  $\leftrightarrow$ W_N generators $W_2(z) = T(z), W_3(z), \ldots, W_N(z)$
- Many things to be understood. e.g. taking $b \to \infty$,
  Liouville becomes classical. $e^{2b\varphi}$ should be visible.
  No one has done that.