Compact Open Topological String

Johannes Walcher
CERN

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Introduction

The Topological String is at the center of the landscape of topological quantum theories.

(a) a toy model for string dynamics: D-branes, non-perturbative effects, Open/Closed duality, S-duality, M-theory, . . .

(b) a tool for studying supersymmetric observables in (ordinary) string theory: (higher-derivative) $\mathcal{N} = 1, 2$ F-terms, string dualities, counting BPS states, . . .

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Most interesting connections arise when the target space is a Calabi-Yau threefold, and by combining A- and B-model through Mirror Symmetry.

A-model: Kähler structure
B-model: complex structure

Program

There exists a wealth of knowledge about topological string theory on local (i.e., non-compact) Calabi-Yau manifolds, and its relation with other "topological" theories through various dualities. In comparison, we do not have a very good understanding of topological strings with compact Calabi-Yau manifolds as targets, especially in the presence of background D-branes.

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There are several features that are generic in the compact case, but absent in the local case.

• Obstructions: The central holomorphic observable of supersymmetric theories with 4 supercharges is the “spacetime” superpotential $\mathcal{W}$. By definition, moduli are flat directions of the superpotential. $\rightsquigarrow$ $\mathcal{W}$ does not depend on them!?

• Tadpole cancellation: A- and B-model do not generically decouple in the presence of background D-branes. Imposing tadpole cancellation restricts the set of possible D-brane configurations. Open-closed duality can not be realized by making N large.

Outline

First part: Review some general features of mirror symmetry and topological strings on compact Calabi-Yau manifolds with background D-branes. In particular, obstructions and tadpole cancellation.

Second part: New calculations illustrating problems and some solutions. New phase transitions and exactly solvable model (non-compact!), the Real Topological Vertex.

Basic Setup

Topological String. Defined by topological twist of $\mathcal{N} = (2,2)$ superconformal field theory of $\hat{c} = 3$, such as obtained from supersymmetric sigma model on Calabi-Yau manifold X (A-model), or Y (B-model).

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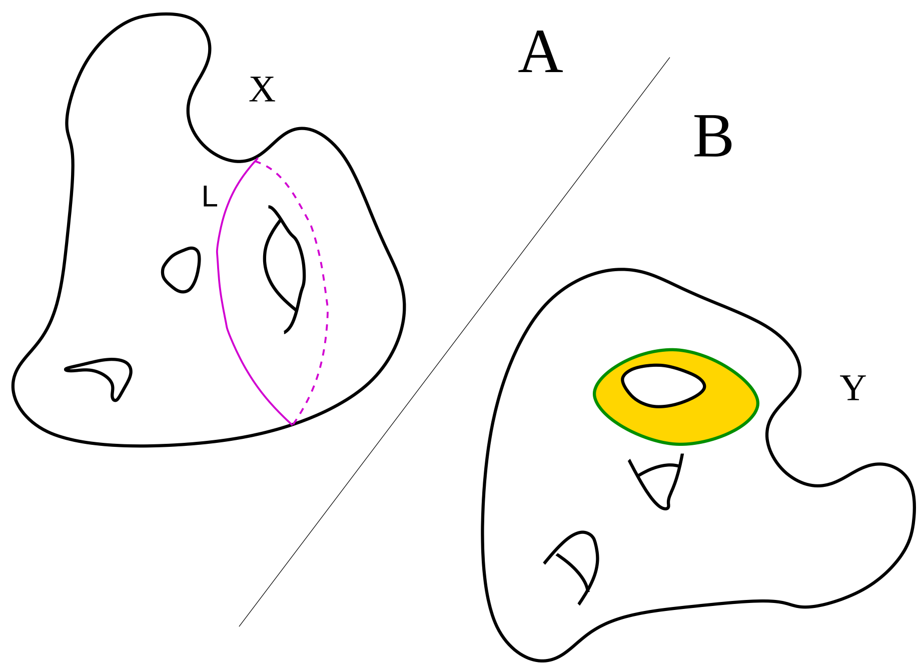
D-branes in Topological String. For sigma-model on (three-dimensional, simply-connected) Calabi-Yau:

A-branes: Lagrangian submanifolds with flat bundle. (More generally, objects of derived Fukaya category.)
B-branes: Complex submanifolds with holomorphic bundle. (More generally, objects of derived category of coherent sheaves.)

Open topological string at tree level computes superpotential of $\mathcal{N} = 1$ compactifications.



I. Formal Developments

I.1. Obstructions

In the A-model, superpotential vanishes classically, and comes entirely from worldsheet instanton effects—holomorphic disks with Lagrangian boundary conditions. Classical (tangent space to) moduli: $H^1(L)$. For compact Calabi-Yau and L, space of obstructions $H^2(L) \sim H^1(L)$.

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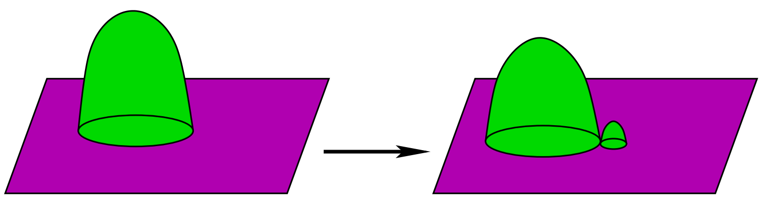
(Mathematical) problem of defining and computing open Gromov-Witten invariants is due to existence of boundaries of moduli space (Fukaya et al.). Bubbling of disks



u is modulus only at $t = \infty$!!

In the B-model, superpotential is given exactly by holomorphic Chern-Simons functional

$$\mathcal{W}_B = \int_Y \Omega \wedge \mathrm{Tr}\left(\frac{1}{2}A \wedge \bar{\partial} A + \frac{1}{3} A \wedge A \wedge A\right)$$

In the B-model, superpotential is given exactly by holomorphic Chern-Simons functional

$$\mathcal{W}_B = \int_Y \Omega \wedge \mathrm{Tr}\left(\frac{1}{2} A \wedge \bar{\partial} A + \frac{1}{3} A \wedge A \wedge A\right)$$

For mirror symmetry, need to make sense of equality

$$\mathcal{W}_A = \mathcal{W}_B$$

A priori, these are defined on different spaces. Even in cases in which on can argue convincingly for certain candidate mirror pairs, the A- and B-model deformation spaces typically differ (in dimension!) at the classical level.

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Example:

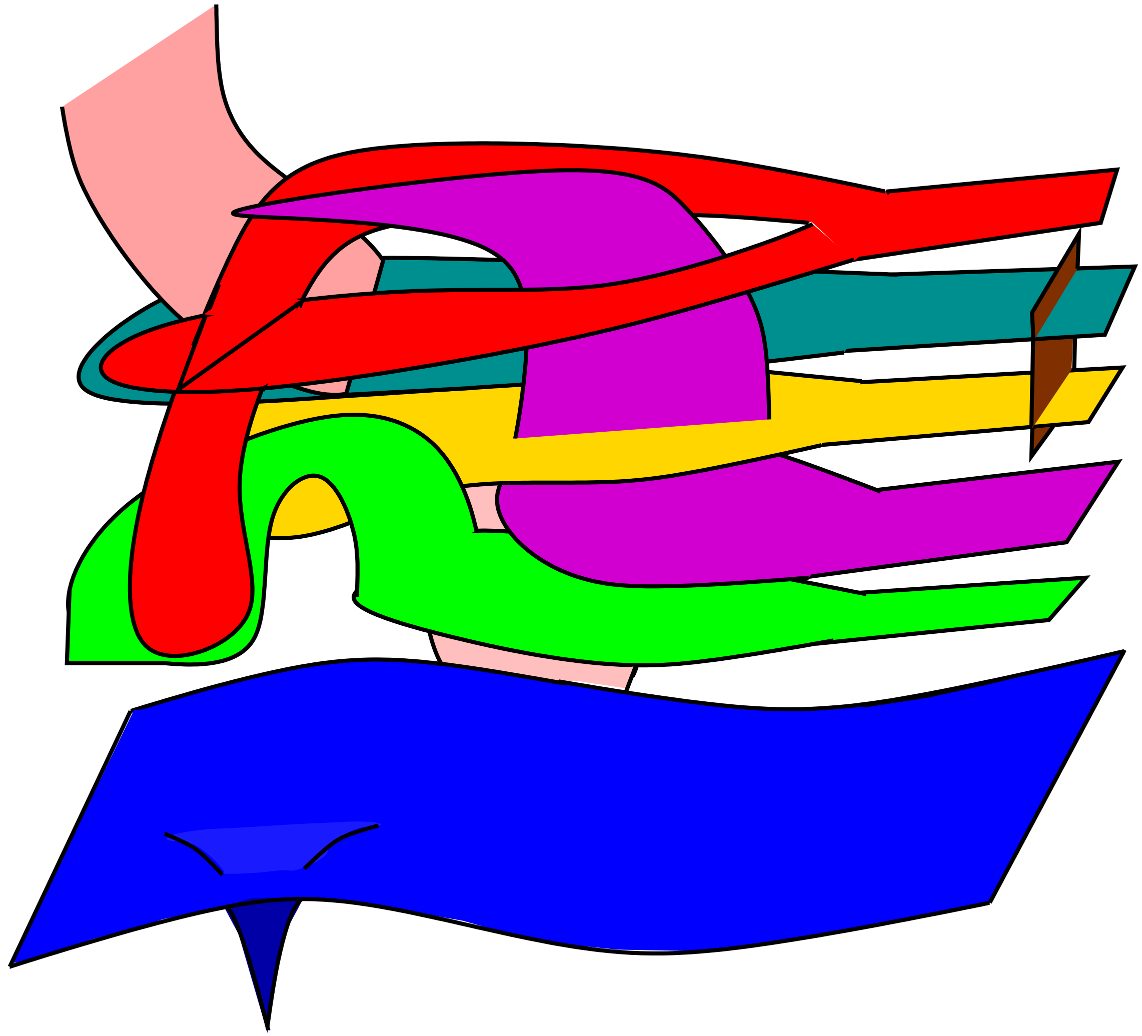
$$\mathcal{W}(u;z) = \frac{1}{3}u^3 - uz \qquad \begin{cases} u & \text{open string} \\ z & \text{closed string} \end{cases}$$

Without specifying Kähler potential, arbitrary redefinitions of u are allowed. (Note: Closed string moduli space comes with distinguished coordinates from special geometry). However,

$$\mathcal{W}|_{\partial_u \mathcal{W}=0} = \pm z^{3/2}$$

is independent of such redefinitions.

Open-closed string moduli space is fibered. In general, fibers (namely, open string moduli space for fixed closed string moduli) are highly complicated, fibration structure even more intricate. By above assumption, we are working on the simplest part of the open string moduli space (the isolated critical points). Cmp., Cachazo, Douglas, Seiberg, Witten (2003)



Example:

A likely familiar phenomenon of local models with D-branes is so-called framing ambiguity. (Related to framing of knots in a dual Chern-Simons gauge theory.)

Expressed in terms of superpotential, it is simply the ambiguity in the identification of open string moduli (in A-model)

$$u \to u + \nu \partial_u \mathcal{W} \qquad \nu \in \mathbb{Z}$$

This obviously disappears at the critical point, $\partial_u \mathcal{W} = 0$.

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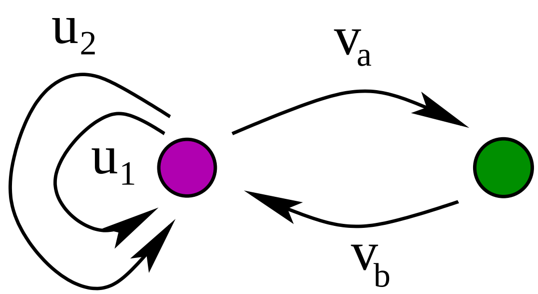
Example: What to do for:

$$\mathcal{W}(u_1, u_2; z) = u_1^2 u_2 - z u_2 = u_2(u_1^2 - z) \qquad \mathcal{W}|_{\partial_u \mathcal{W}=0} \equiv 0 \quad \text{!?!?}$$

Superpotential not enough to distinguish two distinct critical points $u_1 = \pm\sqrt{z}, u_2 = 0$

Algebraic K-theory explanation of coincidence of value of superpotential:

There exists another D-brane that fits in the quiver with superpotential



$$\mathcal{W} = \mathrm{Tr}\big(u_2(u_1^2 - z + v_b v_a)\big)$$

Enlarged theory has one-dimensional moduli space parameterized by vev of $v_a v_b$. Starting at $(u_1 = \sqrt{z}, v_a v_b = 0)$ we can encircle z, and come back at direct sum theory $(u_1 = -\sqrt{z}, v_a v_b = 0)$.

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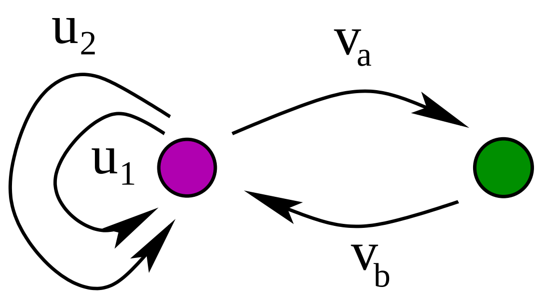

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Mathematical conjectures, applied to present siutation, imply that adding such a sector that allows for interpolation is always possible whenever $\mathcal{W}_+ = \mathcal{W}_-$ at two distinct critical points of the superpotential. The two vacua are "equivalent in algebraic K-theory". Moreover, the interpolation is rational (base space is $\mathbb{P}^1$), and hence stable under any additional (supersymmetric) perturbations.

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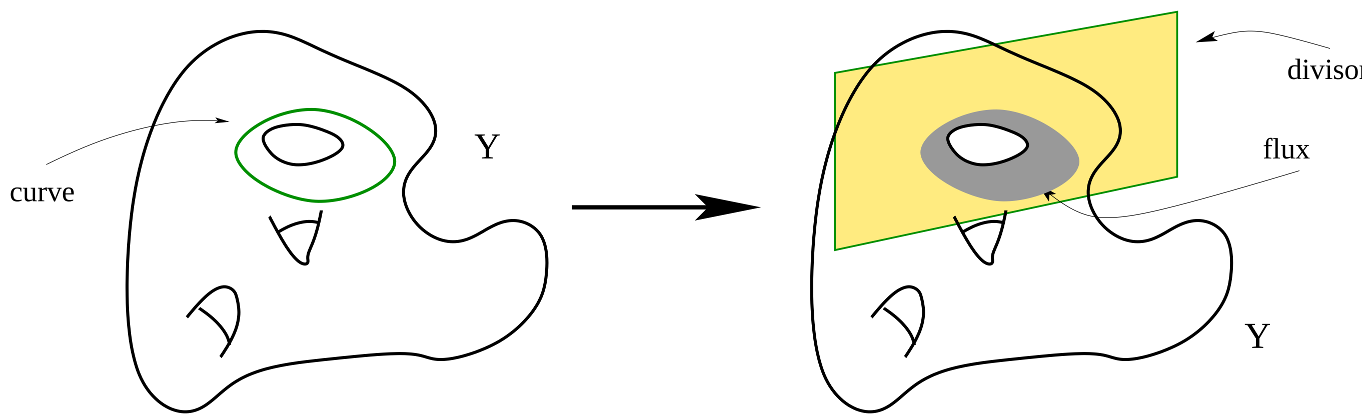
Popular, and successful, method, developped by Jockers et al. (2007–), following Lerche-Mayr-Warner (2002), more recent work by Alim-Hecht-Mayr-Mertens, Jockers-Soroush, Grimm-Ha-Klemm-Klevers, Aganagic-Beem, and several others:

Consider D5-branes wrapped on curves. Replace curve by gauge theory flux in D7-brane wrapped on 4-cycle.



Deformations of 7-brane provide distinguished holomorphic off-shell deformation space, with superpotential induced by flux (5-brane). Superpotentials essentially coincide.

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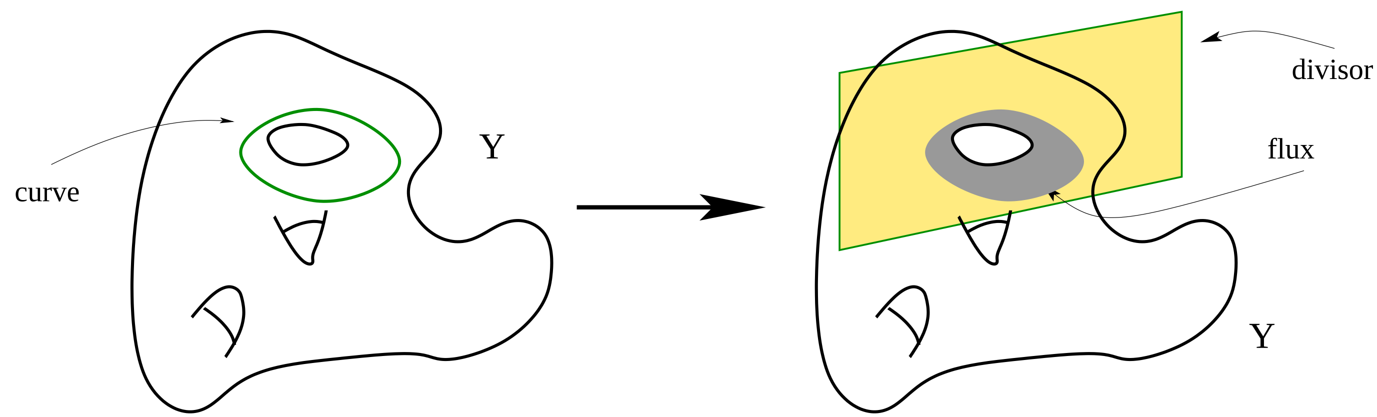

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This setup is suggestive of F-theory, and duality with heterotic string. (Jockers-Mayr-J.W., Grimm et al. (2009))

Checks:

• At critical points of superpotential, this method recovers the results obtained previously using on-shell methods.

• In large volume limit, one obtains non-trivial predictions for instanton corrections to superpotential (and also Kähler potential). Consistent interpretation in the context of heterotic/F-theory duality.

General question:

• This method exploits flat structure provided by a variation of mixed Hodge structure on the joint open/closed deformation space. ($\mathcal{N} = 1$ special geometry.) The A-model also appears to predict the existence of flat and integral structure on spaces of open strings, in large volume limit. *Can we recover/define this intrinsically, or in B-model?*

Summary

$*\mathcal{W}|_{\partial_u \mathcal{W}=0}$ is a useful numerical invariant to study topological D-brane landscape on compact Calabi-Yau manifolds.

$*$A complete understanding of open string mirror symmetry requires going off-shell (at least infinitesimally).

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Four Different Topological Models:

	BRST Q	anti-ghost b_0	moduli
A-model	$G^+ + \bar{G}^-$	$G^- + \bar{G}^+$	Kähler t
anti A-model	$G^- + \bar{G}^+$	$G^+ + \bar{G}^-$	$\bar{t}$
B-model	$G^+ + \bar{G}^+$	$G^- + \bar{G}^-$	Complex structure z
anti B-model	$G^- + \bar{G}^-$	$G^+ + \bar{G}^+$	$\bar{z}$

• Mirror Symmetry relates A-model with B-model (and anti A-model with anti B-model), in general changing the target space.

• Worldsheet CPT relates A-model with anti A-model, and B-model with anti B-model. $\rightsquigarrow$ non-trivial anti-ghost cohomology, holomorphic anomaly

• Non-triviality of "other model" (generic for compact target) $\rightsquigarrow$ mixed anti-ghost/BRST cohomology is non-empty.

Basic Fact

Topological charges of topological branes are naturally carried by the “other” model. (Ooguri-Oz-Yin, 1996)

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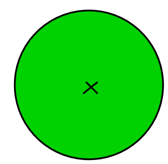
Example: (For simply connected CY) An A-brane is Lagrangian submanifold L, representing 3-cycle Γ. These naturally couple to three-forms, among which the complex structure deformations. Topological D-brane charge is measured by:

$$\mathrm{ch}(L) = \int_\Gamma (\text{3-form}) =$$



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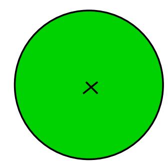
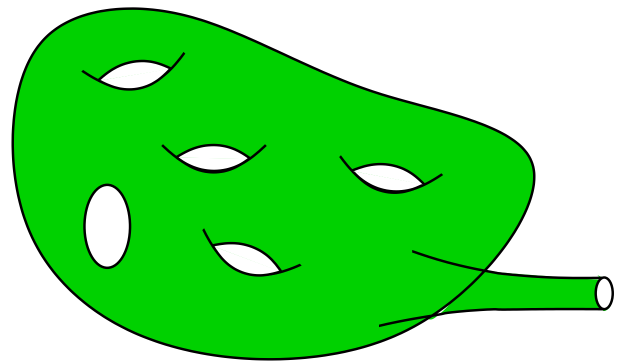


Analogy: Mixed BRST-anti-ghost cohomology of topological string $\leftrightarrow$ Compact RR-potentials of superstring compactification

But (why) should we cancel the tadpoles?

In the topological string, non-vanishing tadpoles are not quite as fatal as in the superstring. However, the non-trivial dependence of disk one-point functions on the "other" moduli means that if tadpoles are not cancelled, loop amplitudes will also depend on those wrong moduli.



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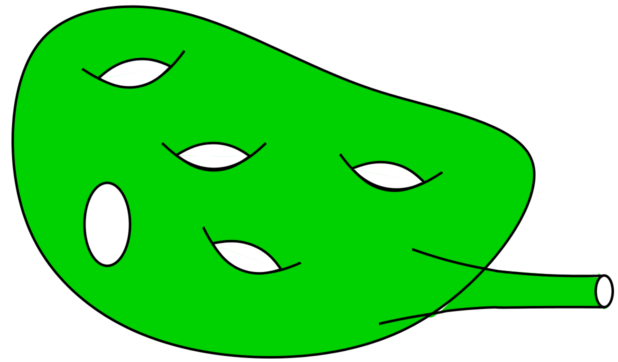

Tadpole cancellation in topological string: (J.W. 2007, Cook-Ooguri-Yang 2008, Bonelli-Prudenziati-Tanzini-Yang 2009) *A- and B-model decouple only for amplitudes with vanishing total D-brane charge.*

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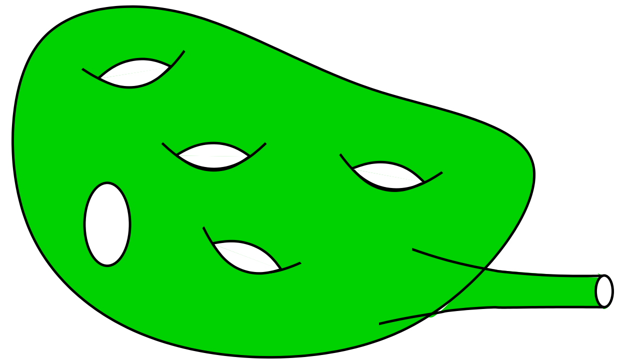

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Two ways to cancel tadpoles:

- Study dependence on open string moduli
  - ∗ Continuous moduli: Operator insertion on boundary
  - ∗ Discrete moduli: brane-anti-brane configuration
- Include orientifolds (preferred)

Comparison with spacetime tadpoles

Given a topological string background consisting (in A-model) of Calabi-Yau threefold plus D-branes on Lagrangians and, possibly, orientifolds, there are (at least) two different ways of embedding into type IIA superstring.

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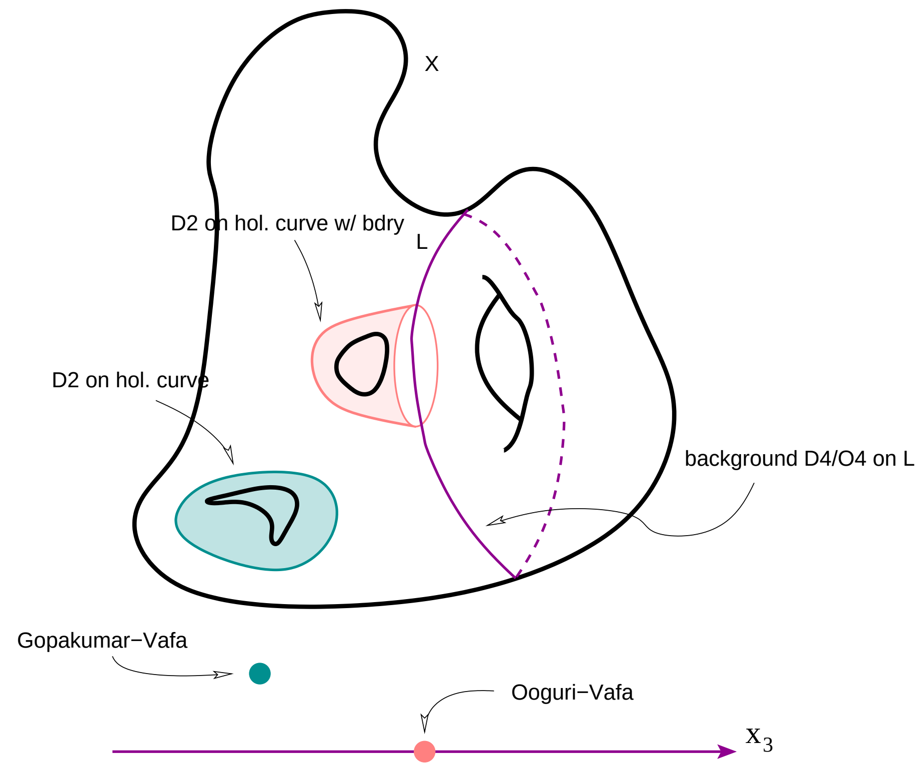
3. In topological string,

$$Q_{\text{top}}(\text{top. O-plane}) = 1$$

BPS interpretation; Ooguri-Vafa

→ Count degeneracy of BPS solitons in $1+1$-dimensional theory on D4-brane wrapped on L. Carry vectormultiplet charge, as well as topological charge asscociated with (discrete) open string moduli.

→ For compact X, topological amplitudes admit BPS interpretation only when tadpoles are cancelled using orientifolds. (J.W., 2007)



Summary

$*$Tadpoles of topological string are cancelled in Ooguri-Vafa setup precisely when O4/D4 charge cancels locally.

$*$Tadpoles of topological string are not cancelled when tadpoles of superstring are cancelled in O6/D6 "braneworld" setup.

$*$Note that "Ooguri-Vafa string" supporting the relevant BPS states is charged under axions in $\mathcal{N} = 2$ hypermultiplets. It appears that BPS state counting is only well defined when that axionic charge vanishes.

Speculations and open problems

$*$Supergravity derivation/interpretation of tadpole cancellation: Is there (Or why is there not) a BPS interpretation when tadpoles of topological string are not cancelled?

$*$What are implications of finite number of possible D-brane configurations for topological string?

II. Calculations

Subprogram 1: Construct (systematically!) D-branes on given closed string background. Study combined open-closed moduli space, interpret in mirror symmetry, zoom in to singularity, etc.

Subprogram 2: Connect progress with understanding of topological string on local Calabi-Yau. (A) topological vertex, (B) matrix models, etc.

II.1. New Phase Transitions (J.W., 2009)

Well-known: string (worldsheet) physics (gauged linear sigma model) allows smooth interpolation between topologically distinct target spaces. (Witten, Aspinwall-Greene-Morrison, 1992)

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Example: Flop of conifold: $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$

* Yukawa coupling for $t \gg 0$ ($q = e^{-t}$) $\kappa = \frac{q}{1-q}$
* Analytic continuation to $t \ll 0$: $\kappa = -1 - \frac{q^{-1}}{1-q^{-1}}$

The process can be understood from vacuum structure of $U(1)$ gauge theory with chiral fields of charge $(1, 1, -1, -1)$. D-term equation:

$$|\phi_1|^2 + |\phi_2|^2 - |\phi_3|^2 - |\phi_4|^2 = t$$

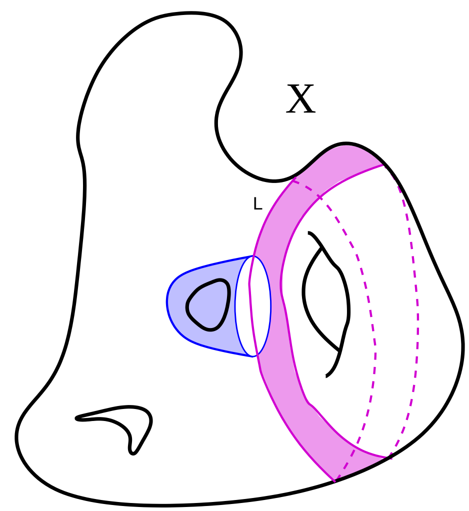
A different $\mathbb{P}^1$ has non-zero size for $t \gg 0$, $t \ll 0$

New: Open string analogue, on compact CY.

Recall: Open mirror symmetry at tree level is concerned with domain wall tensions, and domain wall degeneracy. Wrapped branes ending on branes

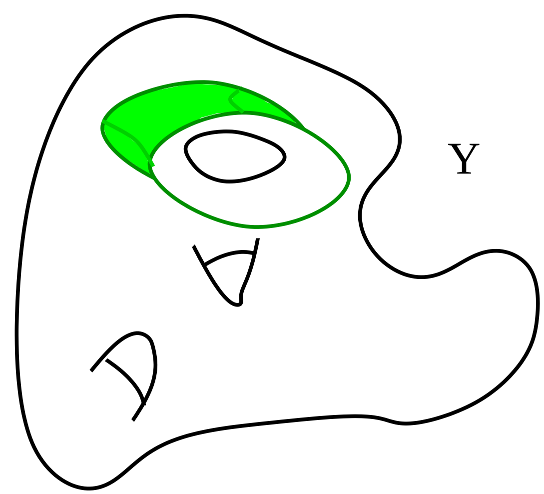
In A-model: four-chain interpolating between, or two-chain ending on, Lagrangian submf.

$$\mathcal{T}_A = \int_{C^{(4)}} \omega \wedge \omega + \int_{C^{(2)}} \omega + \text{hol. disks}$$



In B-model: three-chain interpolating between holomorphic curves

$$\mathcal{T}_B = \int_{C^{(3)}} \Omega$$



Example: Two-parameter model $\mathbb{P}^4_{11226}[12]$ (Candelas et al, 1992)

B-model:

$$\left(\frac{x_1^{12}}{12} + \frac{x_2^{12}}{12} + \frac{x_3^6}{6} + \frac{x_4^6}{6} + \frac{x_5^2}{2} - \psi x_1 x_2 x_3 x_4 x_5 - \frac{\phi}{6} x_1^6 x_2^6 = 0\right)/\Gamma$$

D-branes wrapped on curves: (J.W., 2009)

$$x_3 + (-2)^{-1/6} x_1^2 = 0 \,, \qquad x_4 + (-2)^{-1/6} x_2^2 = 0 \,, \qquad x_5 + \alpha x_1^3 x_2^3 = 0$$

with "open string modulus" restricted to

$$\alpha^2 - 2^{2/3} \psi \alpha - \frac{\phi}{3} = 0$$

(total of $2 \times 3 = 6$ vacua)

The computational tool is differential equation (inhomogeneous Picard-Fuchs) satisfied by $\mathcal{T}_B = \int_{C^{(3)}} \Omega$.

$$\begin{aligned}\Big(\theta_y^2(\theta_y - 2\theta_2) - 72y^3 z_2(2\theta_y+1)(2\theta_y+3)(2\theta_y+5)\Big)\mathcal{T}_B &= 3\,\tfrac{4y}{2(1-4y)^{3/2}}\\ \Big((\theta_y - 3\theta_2)^2 - 9z_2(2\theta_2-\theta_y)(2\theta_2-\theta_y+1)\Big)\mathcal{T}_B &= 3\,\tfrac{1}{(1-4y)^{1/2}}\end{aligned}$$

with $y = (z_1/z_2)^{1/3}$, $z_1 = \phi\psi^{-6}$, $z_2 = \phi^{-2}$, $\theta_z = z d/d\ln z$.

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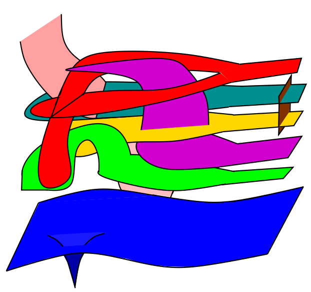
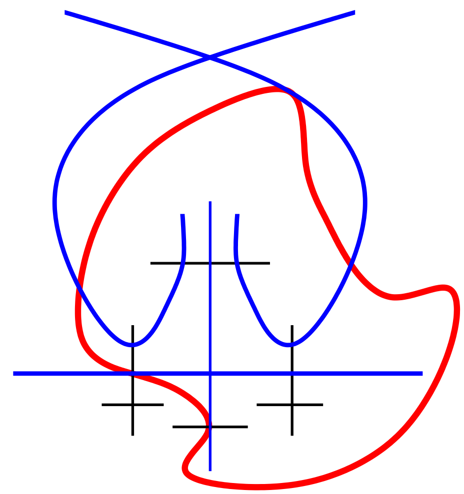
Global picture of moduli space.
Open string discriminant: $(4y)^3 = 1$.





Open mirror symmetry

The classical closed string mirror map identifies the two Kähler parameters of the A-model geometry as the periods with asymptotic behavior

$$t_1 \sim \log z_1 \ , \qquad\qquad t_2 \sim \log z_2$$

The Kähler cone is the subspace of t_1, t_2 real with $t_1 > 0, t_2 > 0$.

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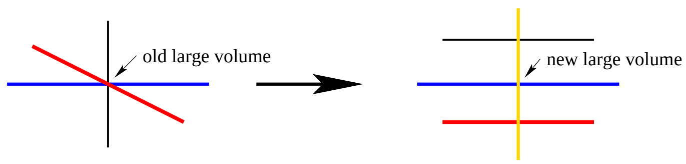
Solving inhomogeneous Picard-Fuchs, we deduce that the Lagrangian bounds holomorphic disks of area

$$\mathcal{T}_B^{(1)} = \frac{t_1 - t_2}{3} + \cdots$$

$\longrightarrow$ presence of D-brane introduces additional face in the Kähler cone, where $\frac{t_1 - t_2}{3} = 0$. We require further blowup



$\longrightarrow$ What happens across open string discriminant?

To study behavior accross the open string discriminant, we restrict Picard-Fuchs system to $z_2 = 0$

$$\theta_y^2 \mathcal{T}_B^{(2)} = \frac{1}{\sqrt{1-4y}}$$

with solution around $y \sim \exp\left(-\frac{t_1 - t_2}{3}\right) = 0$,

$$\mathcal{T}_B^{(2)} = \frac{(\log y)^2}{2} - \frac{\pi^2}{6} - \left(\log \frac{1+\sqrt{1-4y}}{2}\right) + 2\mathrm{Li}_2 \frac{1-\sqrt{1-4y}}{2} \sim \frac{(t_1-t_2)^2}{3} + \cdots$$

Scaling with t_i's implies this must be a domain wall represented by a 4-chain. (integral 4-cycles have tension $\sim 2(t_1^2 + t_1 t_2)$, t_1^2.)

Analytic continuation to $x = 1/y = 0$ yields

$$\mathcal{T}_B^{(2)} = i\pi \log x - \frac{1}{2}\left(\log \frac{\sqrt{x^2-4x}+x}{\sqrt{x^2-4x}-x}\right)^2 - 2\mathrm{Li}_2\left(\frac{\sqrt{x^2-4x}+x}{2}\right) \sim \frac{t_2 - t_1}{3} + \cdots$$

This domain wall must be represented by 2-chain.

$\Rightarrow$ Flop of disk accompagnied by 4-chain $\rightarrow$ 2-chain

For completeness: To extract the instanton corrections to the classical domain wall tension, we go back to the full inhomogeneous Picard-Fuchs system, and expand using Ooguri-Vafa multicover formula

$$\mathcal{W} = \frac{3}{2}\Big(\frac{t_1 - t_2}{3}\Big)^2 + \cdots + \sum_{(m,n)\neq(0,0),k>0} \frac{N_{m,n}}{k^2} p^{km} q^{kn}$$

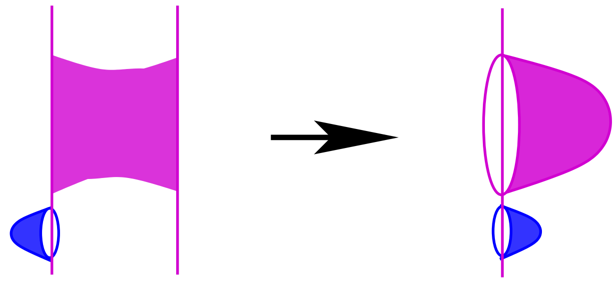
$$p = e^{2\pi i \frac{t_1 - t_2}{3}}\,, \qquad q = e^{2\pi i t_2}$$

Mirror Principle: *Expand fully quantum corrected superpotential in terms of classical domain wall tension corrected only by closed string instantons*

$m \setminus n$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	6	6	0	0	0	0	0
2	3	90	3	0	0	0	0
3	6	388	388	6	0	0	0
4	12	-258	2934	-258	12	0	0
5	30	-540	11016	11016	-540	30	0
6	75	-1388	67602	348 774	67602	-1388	75

Summary

∗New phase transition



Open problems

∗Identify precise A-model geometry.

∗Construct local model.

∗Look for more phase transitions at other points in moduli space. Lift to M-theory.

II.2. The Real Topological Vertex

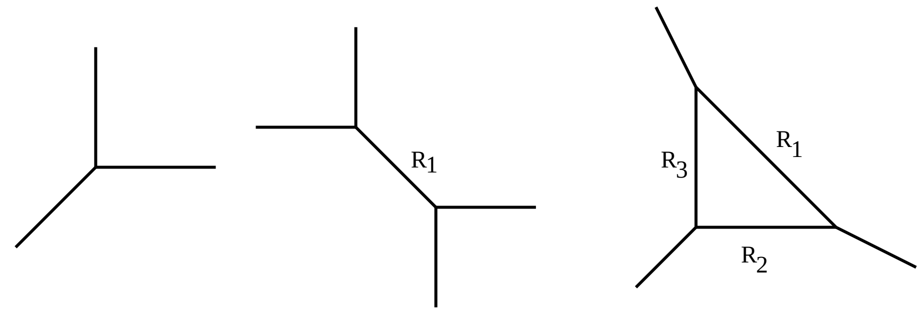
There is one situation in which loop amplitudes of topological string with D-branes and O-planes are essentially as well understood as familiar closed string amplitudes,

The Real Topological String: Orientifold action given by complex conjugation, D-brane wrapped on top of O-plane. Note tadpole cancellation is satisfied with gauge group "$O(1)$". (J.W., 2007)

To make contact with previous work, study the real toplogical string also on local Calabi-Yau manifolds, D.Krefl-J.W. (2008), Krefl-Pasquetti-J.W., (2009). In toric case $\rightsquigarrow$ Real Topological Vertex. (Note: real brane is not toric, in general.)

Topological Vertex (Aganagic, Klemm, Mariño, Vafa, 2003)

Action of T^3 on toric Calabi-Yau manifolds allows representation in terms of two-dimensional planar trivalent graph, Γ.



Here: $\mathbb{C}^3$ $\quad\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ $\quad\mathcal{O}_{\mathbb{P}^2}(-3)$.

Length of internal legs: Kähler parameters, t_i.

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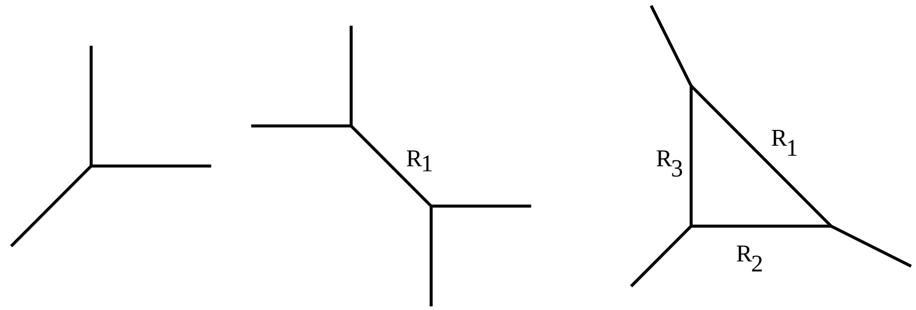


Here: $\mathbb{C}^3$ $\qquad \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \qquad \mathcal{O}_{\mathbb{P}^2}(-3)$.

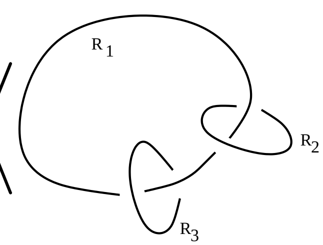
Length of internal legs: Kähler parameters, t_i.

All genus solution of topological string:

$$Z^{\text{top}}(X) = \text{F.R.}(\Gamma) = \sum_{(R_i)} \prod_{\text{legs}} \delta_{R_i R_j^T} e^{-\ell(R_i) t_i} \prod_{\text{vertices}} C_{R_i R_j R_k}$$

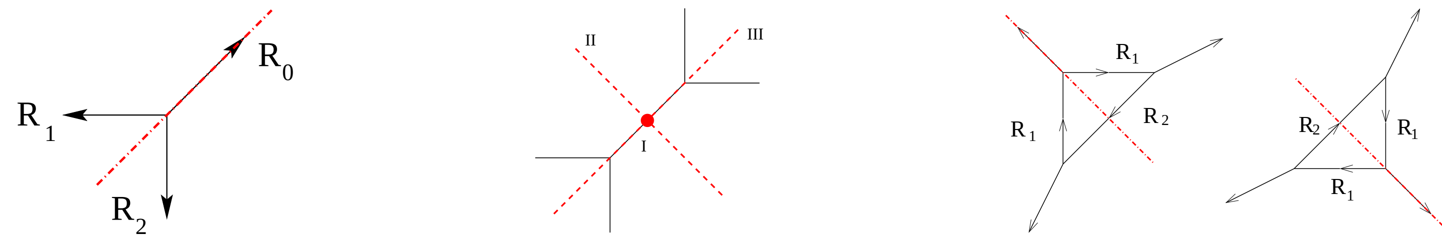
where R_i runs over all partitions (Young diagrams), and

$$C_{R_i R_j R_k}(g_s) = \lim_{g_s = 1/(N+k)} \Big\langle$$



$$\Big\rangle_{SU(N),\text{level } k}$$

Orientifolds of toric Calabi-Yau manifolds come from involutive symmetries of toric diagram.



The real vertex is essentially a squareroot of the ordinary vertex,

$$C_{R_j R_k R_i} = q^{(\kappa_{R_i}+\kappa_{R_j})/2} \sum_{Q_j,Q_k,Q} N_{QQ_k^t}^{R_k} N_{QQ_j}^{R_j^t} \frac{W_{R_i^t Q_k^t} W_{R_i Q_j}}{W_{R_i}} .$$

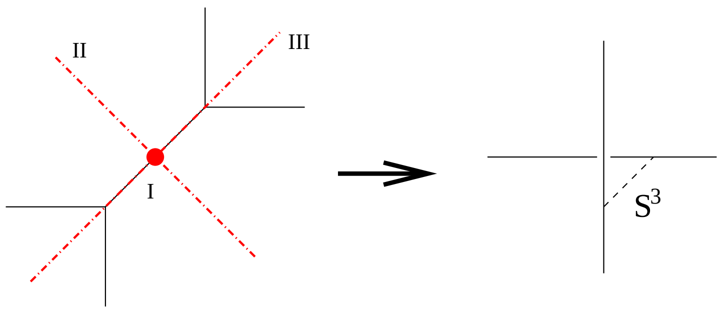
It comes in two varieties, which differ by a sign (of crosscap...)

$$C^{\rm real}_{R_j R_i} = q^{\kappa_{R_j}/4} \sum_{Q_j,Q} N_{QQ_j}^{R_j^t} \frac{W_{R_i Q_j}}{\sqrt{W_{R_i}}} , \qquad \tilde{C}^{\rm real}_{R_j R_i} = q^{\kappa_{R_j}/4} \sum_{Q_j,Q} (-1)^{\ell(Q)} N_{QQ_j}^{R_j^t} \frac{W_{R_i Q_j}}{\sqrt{W_{R_i}}} .$$

together with a (cumbersome) sign rule for combining O^+ and O^--planes.

Connections:

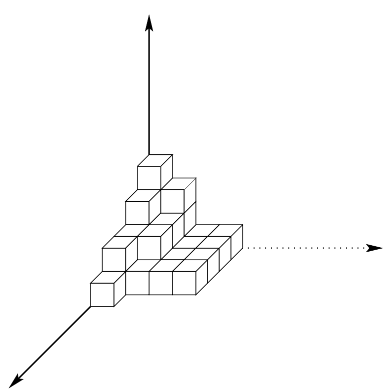
- Chern-Simons theory on orbifolds $S^3/\mathbb{Z}_2$. (Hořava 1990)

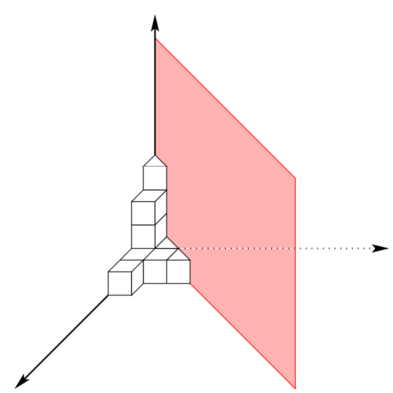


$$\begin{cases} \mathrm{I} & : SU \text{ on } \mathbb{RP}^3 \cong SO/Sp \text{ on } S^3 \\ \mathrm{II} & : SU \text{ on } S^3/\mathbb{Z}_2 \text{ with fixed locus } S^1 \\ \mathrm{III} & : idem \end{cases}$$

$$Z^{\mathrm{real}} = \exp\left(-\frac{1}{2} \sum_{k=1}^{\infty} \frac{Q^k}{k\left(q^{k/2} - q^{-k/2}\right)^2} - \sum_{k=1}^{\infty} \frac{(\pm Q)^{k/2}}{k\left(q^{k/2} - q^{-k/2}\right)} \right).$$

- Melting crystal (Okounkov-Reshetikhin-Vafa 2003)





• constant map contribution (real topological string on $\mathbb{C}^3$)

$$\log Z^{\rm top}(\mathbb{C}^3) \sim \tfrac{1}{g_s^2}\zeta(3) - \tfrac{1}{12}\log g_s + \textstyle\sum_{g=2}^{\infty} g_s^{2g-2}\frac{B_{2g}B_{2g-2}}{2g(2g-2)(2g-2)!}$$

$$Z^{\rm top} = \textstyle\prod_n \frac{1}{(1-q^n)^n} = M(q) \qquad\qquad q = e^{-g_s},\ M(q)\text{: MacMahon function.}$$

$$\log Z^{\rm real}/\sqrt{Z^{\rm top}} \sim \tfrac{3\zeta(2)}{4g_s} - \tfrac{1}{4}\log 2 + \textstyle\sum_{k,n=1}^{\infty}\frac{(-1)^k}{k}\left(e^{-4\pi^2 kn/g_s} - \frac{1}{2}e^{-2\pi^2 kn/g_s}\right)$$

$$Z^{\rm real} = \textstyle\prod_{n\ \rm even}\frac{1}{(1-q^n)^{[n/2]}}\prod_{n\ \rm odd}\frac{1}{(1-q^{n/2})}$$

No perturbative constant map contribution beyond one-loop, but non-trivial non-perturbative instanton sum.

• Via analytic continuation: Gap and new universality class at conifold. (Goshal-Vafa, Klemm-Huang, Krefl-J.W.)

$$\mathcal{F}^{(g)} \sim \frac{\Phi_g}{t_c^{2g-2}} + \mathcal{O}(t_c^0) \qquad \Phi_g = \frac{B_{2g}}{2g(2g-2)} = \left(-\tfrac{1}{12}, -\tfrac{1}{240}, \tfrac{1}{1008}, -\tfrac{1}{1440}, \ldots\right)$$

$$\mathcal{K}^{(g,0)} \sim \frac{\Psi_g}{t_c^{2g-2}} + \mathcal{O}(t_c^0) \qquad \Phi_g = \ ??? \ = \left(-\tfrac{1}{8}, -\tfrac{9}{128}, -\tfrac{81}{512}, -\tfrac{4239}{4096}, \ldots\right)$$

$\longrightarrow$ orientifold of $c = 1$ string @ self-dual radius?

Conclusions

We continue to make progress on extending knowledge of topological string to compact Calabi-Yau with background D-branes.

In this talk, I have illustrated some new features that appear with compact target spaces: obstructions from superpotential, and tadpole cancellation.