Strings 2011, Uppsala

K3, M_{24}, and Holographic Moonshines

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Introduction

The Moonshine Phenomenon

sporadic group $\longleftrightarrow$ ***modular objects***

$$J(\tau) = J(\tau+1) = J(-\tfrac{1}{\tau}) \qquad (q = e^{2\pi i \tau})$$

$$= q^{-1} + 196884\, q + 21493760\, q^2 + \dots$$

$$196884 = 1+196883 \qquad 21493760 = 1+196883+21296876$$

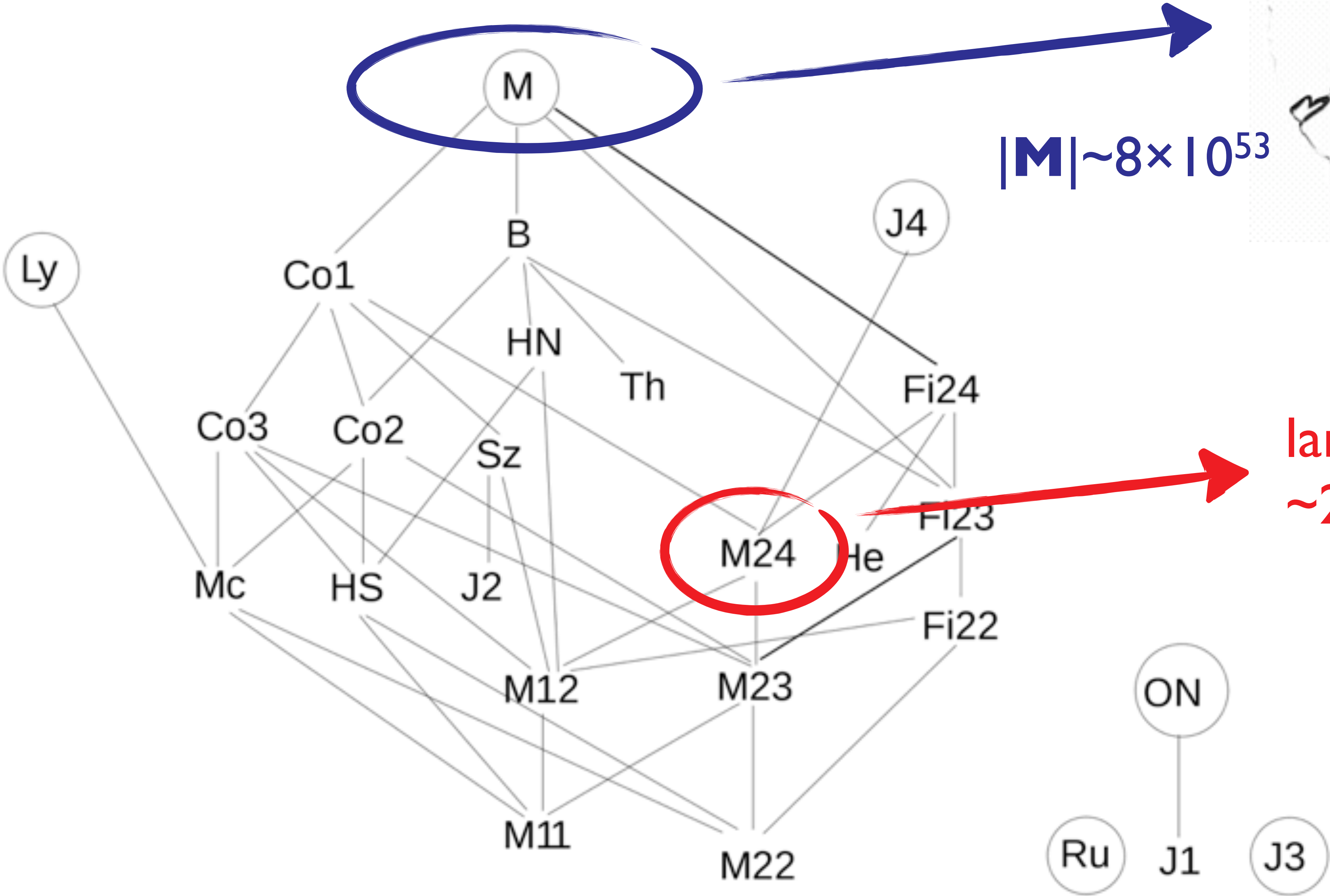
dim of irreps of Monster

[McKay late 70's]

The Largest Sporadic Group $\overset{??}{\underset{??}{\longleftrightarrow}}$ The Most Natural Modular Function

Sporadic Groups

The 26 finite simple groups that don't come in ∞-families.



Mathieu 24

It has a natural 24-dim representation, on which

$$M_{24} \subset S_{24}\,.$$

$$\begin{array}{c} N \\ \circlearrowright \\ M_{24} \end{array} \quad \text{: a 24-dim Niemeier lattice}$$

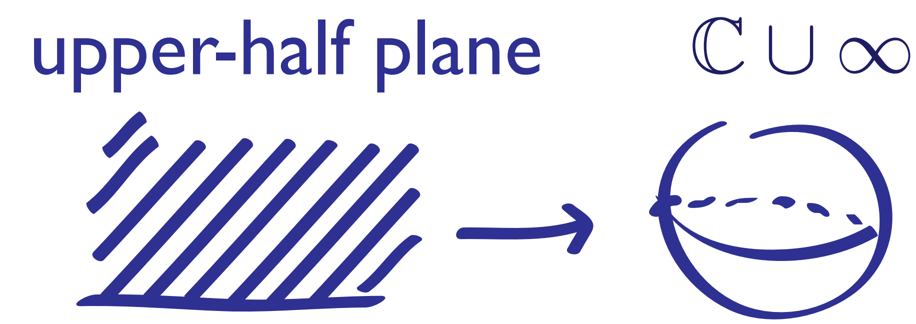
All symmetries G of $K3$ manifold preserving the hyper-Kähler structure satisfy

$$G \subset M_{23} \subset M_{24}.$$

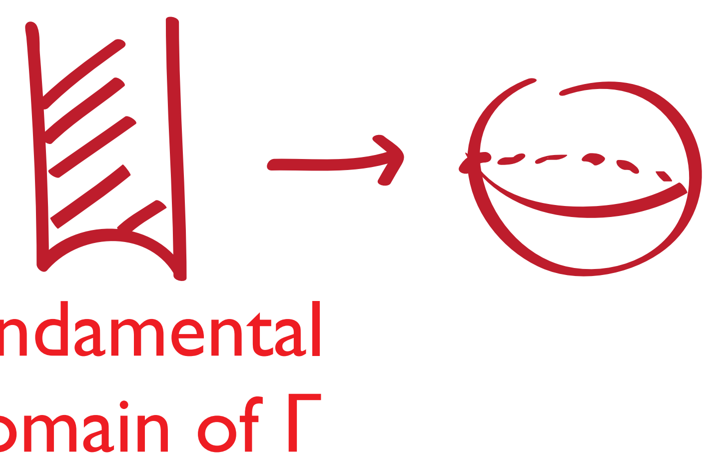
[Mukai '88, Kondo '98]

Modular$\subset$Automorphic Forms

A modular form $f(\tau)$ transforms "covariantly" under a subgroup Γ of $SL(2,R)$.



$f(\tau)$:

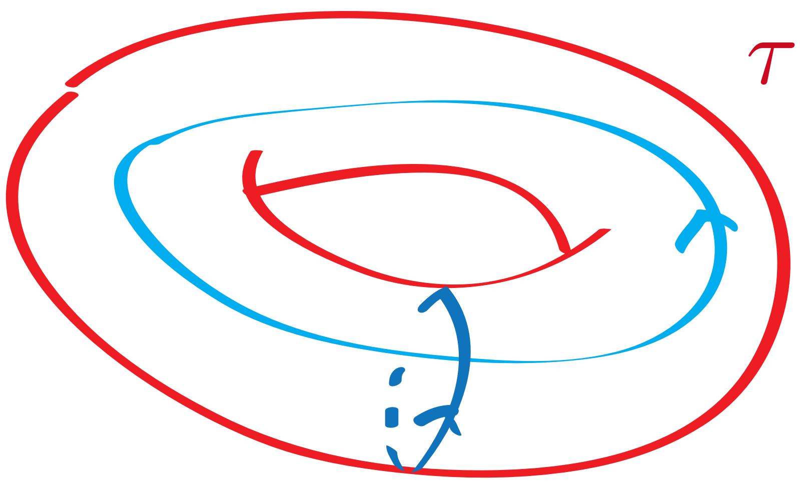


An "automorphic form" transforms under more general groups (a higher dimensional version of a modular form).
In physics, this "automorphism" reflects an underlying symmetry of the problem.

String theory is good at producing automorphic forms!

World-sheet Symmetries
(mapping class group of Σ)

eg. $Z_{\mathrm{2d\,CFT}}(\tau)$ transforms under $SL(2,\mathbb{Z})$.



Space-Time Symmetries
(Such as *T*-, *S*-dualities)

All symmetries have to be reflected in suitable partition functions.

String Theory and Moonshine

e.g. Monstrous moonshine (partially) explained by CFT and proven using bosonic string theory.

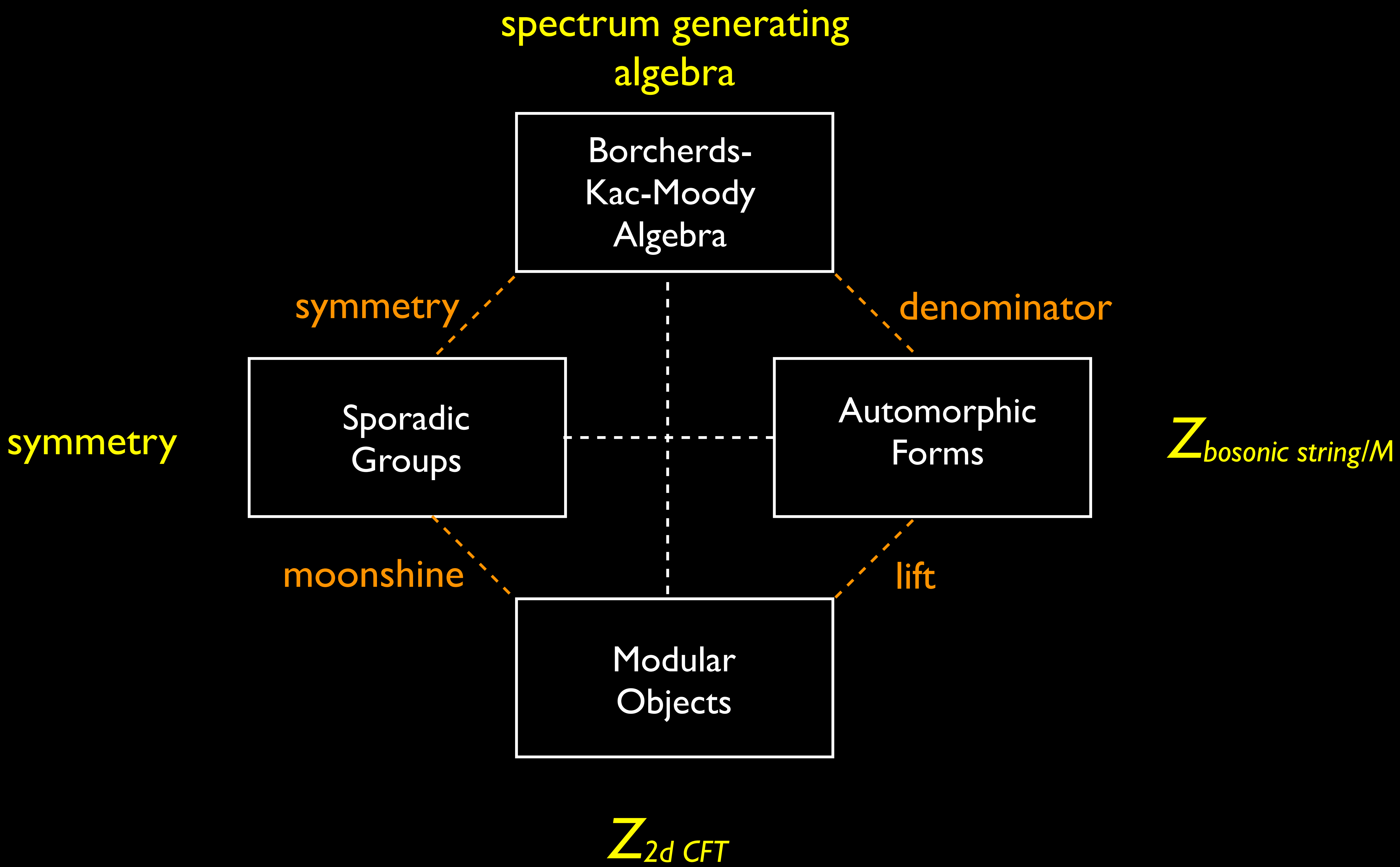
$$J(\tau) = q^{-1} + 196884\, q + 21493760\, q^2 + \dots$$
$$= \mathrm{Tr}_{\mathcal{H}} q^{L_0 - \frac{c}{24}}$$

(partition function of a c=24 2d chiral CFT with a Monster symmetry)

['88 Frenkel-Lepowsky-Meurmann /Dixon-Ginsparg-Harvey]

The "moonshine conjecture" was proven by introducing generalised Kac-Moody algebras and considering the lift of $J(\tau)$ into an automorphic form. [R. Borcherds '92]

Physically, this "lift" corresponds to considering the ***full 26-dim bosonic string***.


spectrum generating algebra
Borcherds-Kac-Moody Algebra
symmetry
denominator
symmetry
Sporadic Groups
Automorphic Forms
$Z_{bosonic\ string/M}$
moonshine
lift
Modular Objects
$Z_{2d\ CFT}$

K3 and M_{24}

Last year, Eguchi-Ooguri-Tachikawa made a striking observation on the BPS-sector of the *2d* SCFT with *K3* target:

$\mathcal{Z}(\tau, z; K3) = K3$ elliptic genus

[Eguchi-Ooguri-Taormina-Yang '89]

$$= \frac{\theta_1^2(\tau,z)}{\eta^3(\tau)} \Big(24\,\mu(\tau,z) + 2q^{-1/8}\left(-1 + 45q + 231q^2 + 770q^3 + \ldots\right)\Big)$$

number of massive *N=4* multiplets

also dimensions of irreps of *M_{24}*!!!

A New M_{24} Moonshine?

WHY?

WHAT?

- What precisely should we conjecture? In particular, what should be the moonshine diagram in this case?
- What are the consequences? Are they consistent with what we know?
- What are the evidences?

(First part)

A Long-Standing Puzzle

Along the way we were confronted with an old puzzle of a number theoretic nature (the genus zero property).

Surprisingly, considerations from AdS/CFT give a way out by providing natural explanations for some crucial properties of the modular groups appearing in all moonshines known so far.

$\rightarrow\rightarrow\rightarrow$

Holographic Modularity of Moonshines

(Second part)

$K3$ and M_{24}

K3 Sigma-model

In a 2d *N=(4,4)* SCFT, BPS states are counted by elliptic genus.
They are modular and are called Jacobi forms.

$$\mathcal{Z}(\tau, z; K3) = \mathrm{Tr}_{\mathcal{H}_{\mathrm{RR}}}\left((-1)^{J_0+\bar{J}_0} y^{J_0} q^{L_0-c/24} \bar{q}^{\bar{L}_0-c/24}\right) = \sum_{\substack{n\geq 0,\\ \ell\in\mathbb{Z}}} c(4n-\ell^2) q^n y^\ell$$

$$= \frac{\theta_1^2(\tau,z)}{\eta^3(\tau)} \left(24\,\mu(\tau,z) + 2q^{-1/8}\left(-1 + 45\,q + 231\,q^2 + 770\,q^3 + \dots\right)\right)$$

Counting the number of massive *N=4* multiplets:

$$H(\tau) = 2q^{-1/8}\left(-1 + 45q + 231q^2 + 770q^3 + \dots\right)$$

Mock Modular Form

[Zwegers '02/ Eguchi-Hikami'10]

1/4-BPS States in Type II/K3×T²

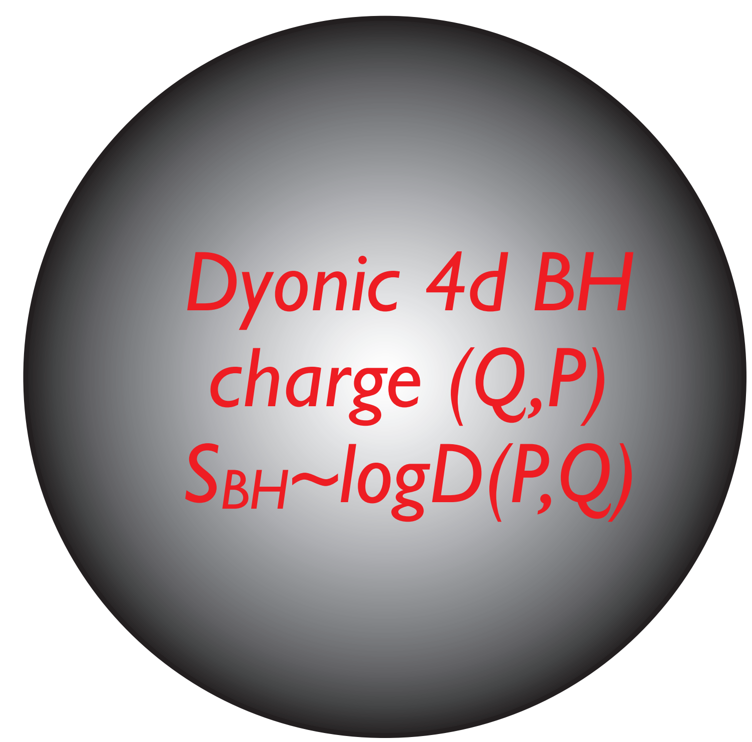
Further compactifying down to 4d is achieved by "*Borcherds-lifting*" the elliptic genus.

The resulting automorphic form Φ counts the 1/4-BPS states of the *N=4, d=4* theory: [Dijkgraaf-Verlinde² '97]

$$\Phi(\tau, z, \sigma) = \frac{1}{pqy} \prod_{n,m,\ell} \left(\frac{1}{1 - p^n q^m y^\ell} \right)^{c(4nm-\ell^2)}$$

Fourier coeff. of $\mathcal{Z}(\tau, z; K3)$

$$= \sum_{P,Q} D(P,Q) p^{Q\cdot Q/2} q^{P\cdot P/2} y^{P\cdot Q}$$



This automorphic form also defines an algebraic structure underlying the 1/4-BPS spectrum.

See A. Sen, Strings '07, '08, '10

$1/2$-BPS States in Type II/$K3\times T^2$

$\updownarrow$

$$\frac{1}{\eta^{24}(\tau)} = \sum d(Q) q^{\frac{Q\cdot Q}{2}} \Leftarrow \quad \text{heterotic}/T^6$$

$$= \frac{1}{q\prod_{n\geq 1}(1-q^n)^{24}}$$

[Dabholkar-Harvey '89]

$$= \text{partition function of 24 chiral bosons}$$

1/2-BPS Hilbert space of Type II/$K3\times T^2$ forms a M_{24}-representation!

M_{24} maps 1/2-BPS states to each other.

[G. Mason '85, Govindarajan-Krishna '09, M.C. '10]

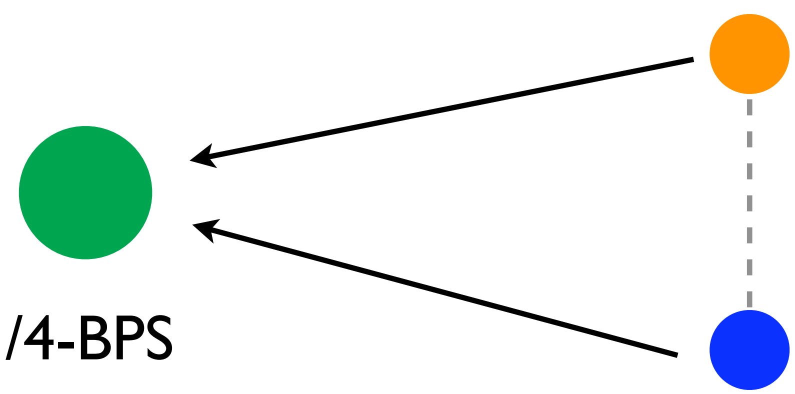
Dyonic Bound States in Type II/K3×T²

1/4-BPS spectrum has to know about the *1/2*-BPS spectrum too!

Two-Centered Bound States

$\updownarrow$

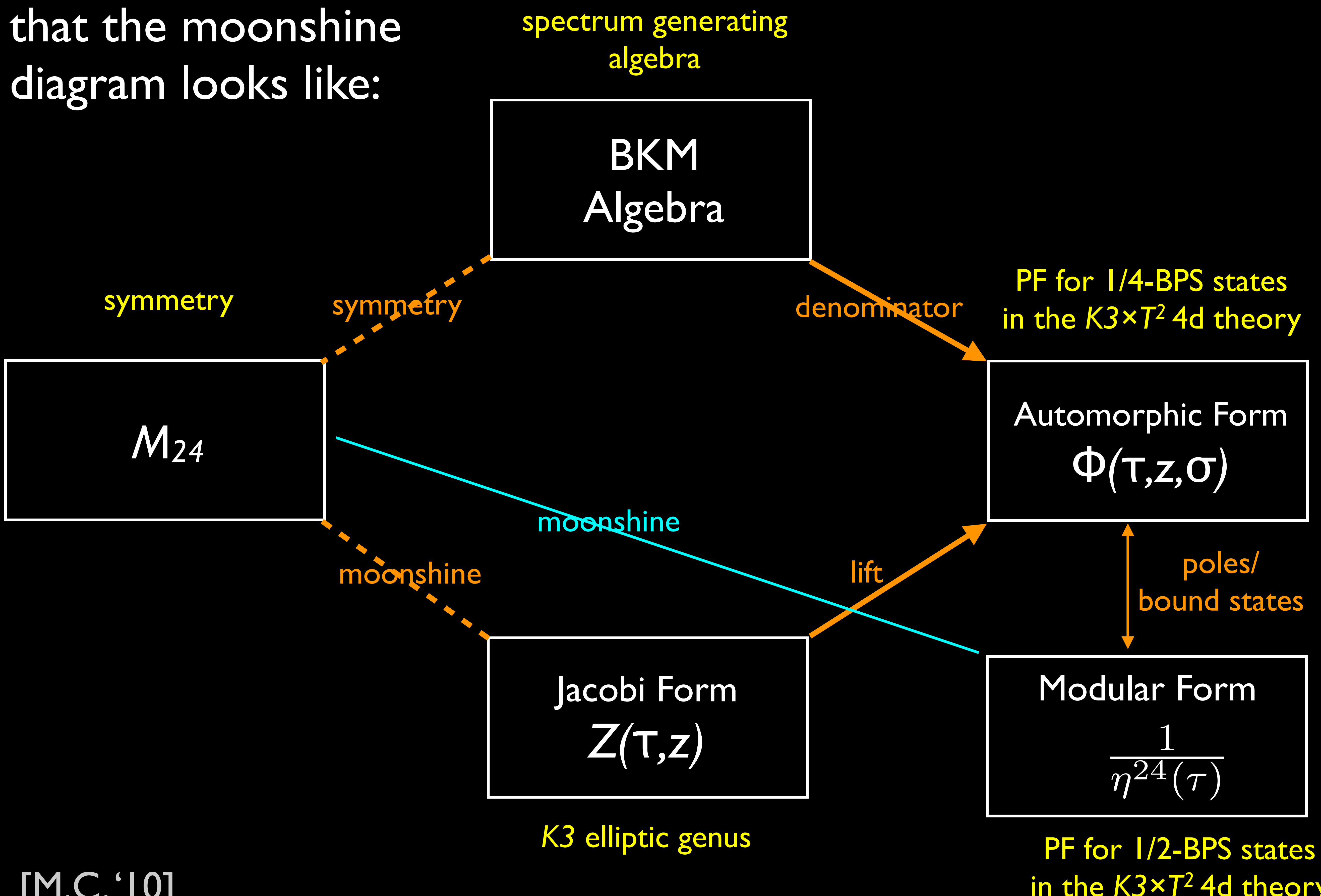
Poles in 1/4-BPS P.F.



$$\Phi(\tau,z,\sigma) \xrightarrow{z\to 0} \frac{1}{(2\pi i z)^2}\,\frac{1}{\eta^{24}(\tau)}\,\frac{1}{\eta^{24}(\sigma)}$$

[A. Sen/M.C.-Verlinde ’07, ’08]

Physics suggests that the moonshine diagram looks like:



[M.C.'10]

What exactly are we conjecturing?

The mock modular form containing the M_{24} information in $\mathcal{Z}(\tau, z; K3)$

$$H(\tau) = q^{-1/8}\Big(-2 + 90\,q + 462\,q^2 + 1540\,q^3 + \ldots\Big)$$

$$\stackrel{?}{=} q^{-1/8}\Big(-2 + \sum_{n=1}^{\infty} q^n \dim K_n\Big) \qquad K_n = M_{24}\text{-representation}$$

True. But meaningless.

Instead we need information about the vector (Hilbert) space!

Twisting the Symmetries

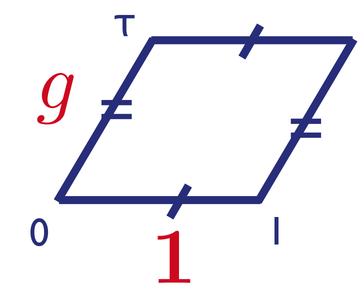
Hilbert space $\mathcal{H}$

$\circlearrowright$

$g \in G$

The ***twisted*** partition function $Z_g = Tr_{\mathcal{H}}(g.....)$
gives finer information about $\mathcal{H}$ than just the dimension.

e.g. 2d CFT $\qquad Z_g(\tau) = \mathrm{Tr}_{\mathcal{H}}(g\, q^{\hat{H}} \cdots)$



boundary condition: $\cancel{SL(2,Z)} \rightarrow \Gamma_g$

e.g. When there is supersymmetry, we can compute

$$\mathrm{Tr}_{\mathcal{H}}\left(q^{L_0}\right) \qquad \text{or} \qquad \mathrm{Tr}_{\mathcal{H}}\left((-1)^F q^{L_0}\right)$$

PF $\qquad$ index (twisted PF)

The invariance group has to preserve the spin structure.

A Concrete Conjecture

In particular, we want to compute the ***twisted*** *K3* elliptic genus $\mathcal{Z}_g(\tau, z; K3)$ and extract the interesting part $H_g(\tau)$.

Conjecture:

For all $g \in M_{24}$ It should be a (mock) modular form of a discrete subgroup $\Gamma_g \subset SL(2,R)$ and satisfy

$$H_g(\tau) = q^{-1/8}\Big(-2 + \sum_{n=1}^{\infty} q^n \mathrm{Tr}_{K_n} g \Big)$$

Status: The candidate (mock) modular forms $H_g(\tau)$ have been proposed for all $[g] \subset M_{24}$. [M.C. /Gaberdiel-Hohenneger-Volpato/Eguchi-Hikami '10]

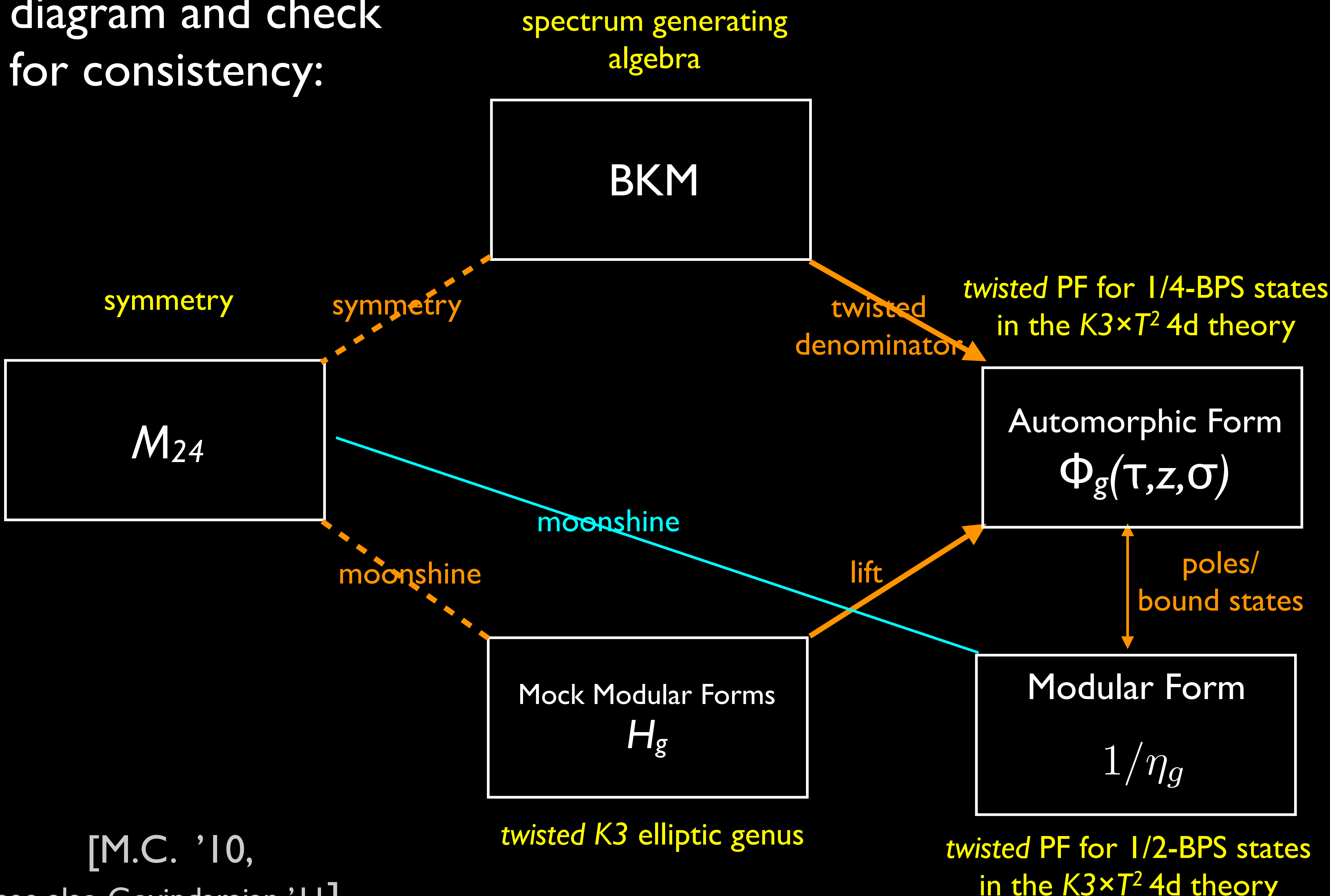
Checks:

1) $K_{n<1000}$ constructed! [Y. Tachikawa]

2) Coincides with geometric calculation when the latter is available.

[David-Jatkar-Sen '06]

Twist the moonshine diagram and check for consistency:

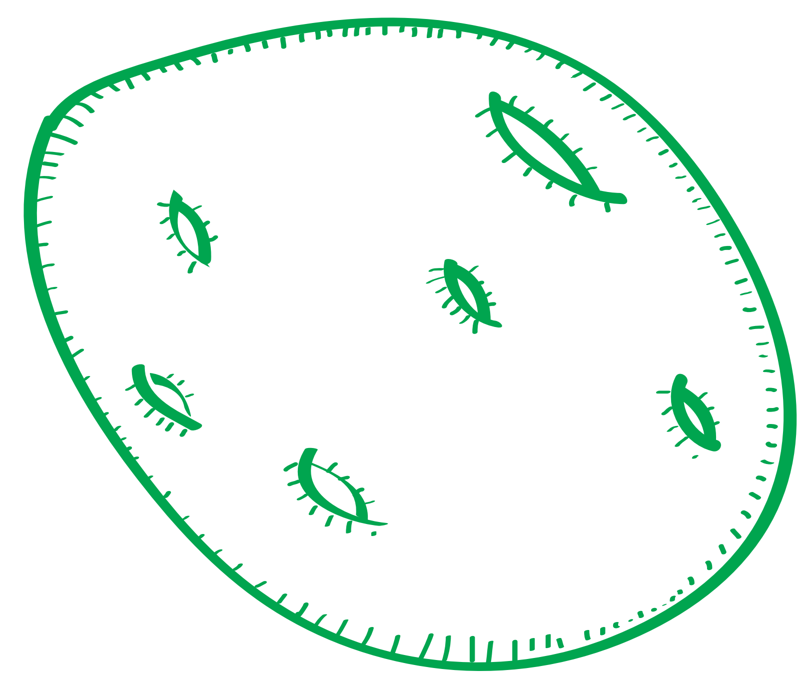


[M.C. '10,
see also Govindarajan '11]

Summary: M_{24} in K3 String Theories

We don't understand, but we are pretty sure!

Challenge:
Stringy *K3* Symmetries



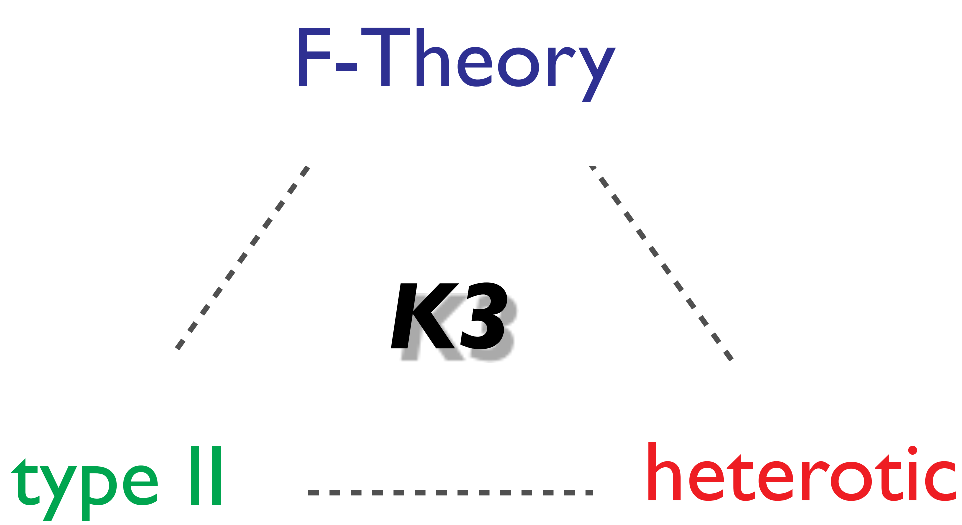
These results are clearly evidence for uncharted symmetries of the stringy geometries of *K3*. ***Note: Classical symmetries are not enough! Stringy symmetries are needed.***

[cf. Taormina-Wendland '10/Gaberdiel-Hohenegger-Volpato '11]

Summary: M_{24} in K3 String Theories

This moonshine is physically interesting.

- We have black holes.
- Since *K3* is ubiquitous, it is important to understand these new symmetries.
- We should also explore where else M_{24} shows its presence.



Sporadic Quantum
Black Holes

Holographic Modularity of the Moonshines

with **John Duncan** 1107.xxxx

A Long-Standing Mystery

For positive every integer *n* we can define a certain subgroup $\Gamma_0(n)^+ \subset SL(2,R)$.

Q: What are the prime numbers p such that $\Gamma_0(p)^+$ is ***genus zero***?

A: $\Gamma_0(p)^+$ is genus zero iff p divides $|\mathbf{M}|$!

[Ogg '73]

WHY??

WHY??

An "Explanation" from Monstrous Moonshine

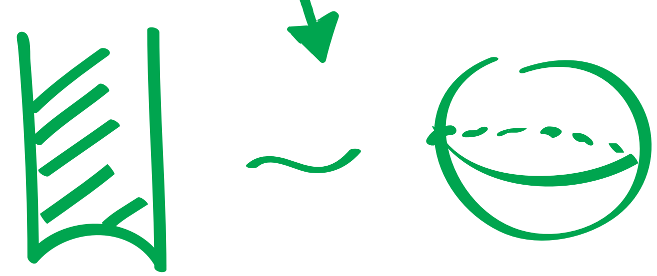
$$J(\tau) = q^{-1} + 196884\, q + 21493760\, q^2 + \ldots \qquad \begin{array}{c}\mathcal{H}\\ \circlearrowright \\ \mathbf{M}\end{array}$$

$$= \mathrm{Tr}_{\mathcal{H}} q^{L_0 - \frac{c}{24}}$$

As before, we consider the twisted partition function

$$J_g(\tau) = \mathrm{Tr}_{\mathcal{H}}(g q^{L_0 - \frac{c}{24}}) = \sum_{n \geq -1} c_g(n) q^n \qquad \text{for every } g \in \mathbf{M}$$

Moonshine Conjecture (Conway-Norton '79):
$J_g(\tau)$ is invariant under some genus zero $\Gamma_g \subset SL(2,R)$.



Genus Zero Property

Genus zero groups $\Gamma \subset SL(2,R)$ are rare.
But only genus zero groups seem to appear in moonshines.

Not just in Monstrous moonshine, this has also been extended to

- The "generalised moonshine"
- Groups other than the Monster

see for instance
Norton '84/Carnahan '08

Höhn '03/Duncan '05, '06

BUT WHY GENUS ZERO??

NO Genus Zero for the New M_{24} Moonshine

Heresy!

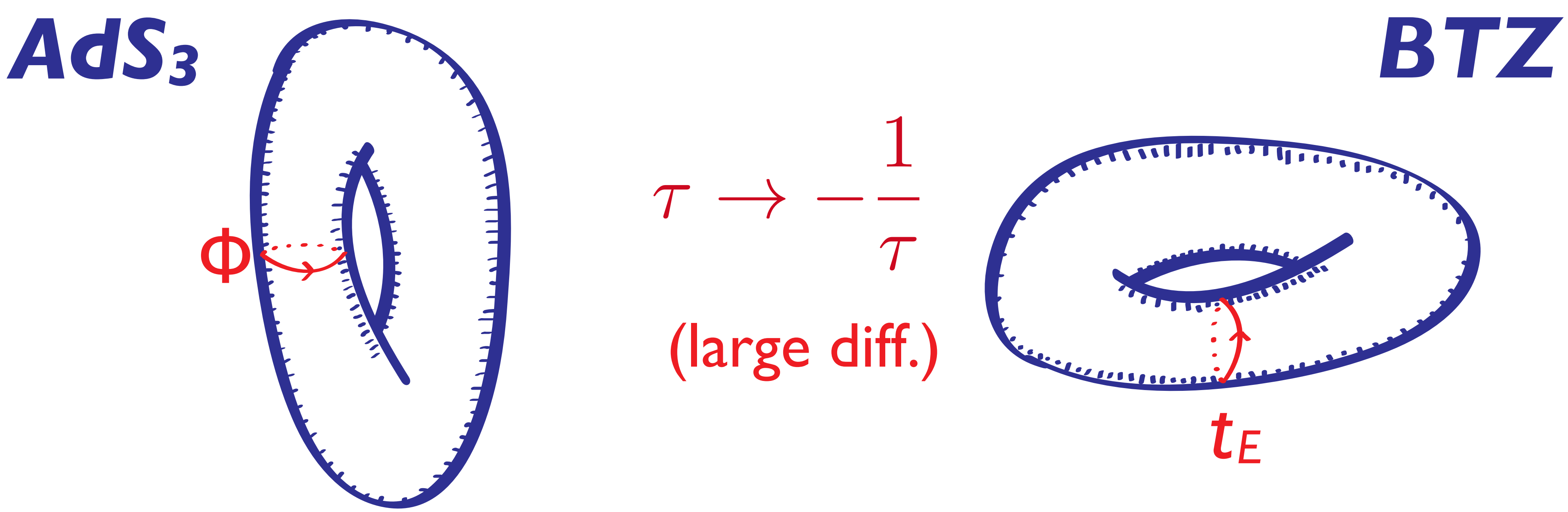
But true, by inspecting

$$H_g(\tau) = q^{-1/8}\Big(-2 + \sum_{n=1}^{\infty} c_{g,n}\, q^n \Big)$$

ℓ \ $[g]$	$1A$	$2A$	$2B$	$3A$	$3B$	$4A$	$4B$	$4C$	$5A$	$6A$	$6B$	$7A$	$\overline{7A}$	$8A$	$10A$	$11A$
1	90	-6	10	0	6	-6	2	2	0	0	-2	-1	-1	-2	0	2
2	462	14	-18	-6	0	-2	-2	6	2	2	0	0	0	-2	2	0
3	1540	-28	20	10	-14	4	-4	-4	0	2	2	0	0	0	0	0
4	4554	42	-38	0	12	-6	2	-6	-6	0	4	4	4	-2	2	0
5	11592	-56	72	-18	0	-8	8	0	2	-2	0	0	0	0	2	-2
6	27830	86	-90	20	-16	6	-2	6	0	-4	0	-2	-2	2	0	0
7	61686	-138	118	0	30	6	-10	-2	6	0	-2	2	2	-2	-2	-2
8	131100	188	-180	-30	0	-4	4	-12	0	2	0	-3	-3	0	0	2
9	265650	-238	258	42	-42	-14	10	10	-10	2	6	0	0	-2	-2	0

AdS_3/CFT_2

Recall: In a Euclidean 3d gravity, the only smooth solution with aAdS boundary conditions and torus boundary is the ***solid torus*** :



$\Rightarrow$ Saddle points of the gravity path integral are labeled by the ***contractible cycle.***

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash SL(2,\mathbb{Z}) \quad (\tau \sim \tau + 1)$$

[Maldacena-Strominger '98]

Twisting the boundary condition: $\cancel{SL(2,Z)} \to \Gamma_g$.

AdS_3/CFT_2

Assuming a CFT has a dual description given by **semi-classical-like AdS gravity**

⇒ The (twisted) partition function $Z_g(\tau)$ can also be computed from the gravity side by summing over saddle point contributions

$$\sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma_g} f\left(\frac{a\tau + b}{c\tau + d}\right)$$

⇒ $Z_g(\tau)$ has to be **Rademacher-summable!**

See also the "Farey Tail" papers:
Dijkgraaf-Maldacena-Moore-Verlinde '00,
Kraus-Larsen/Dijkgraaf-de Boer-M.C.-Manschot-Verlinde/Denef-Moore/Manschot-Moore '06

Rademacher-Summability

$$Z_g(\tau) = \mathrm{Reg}\Big[\sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma_g} f\Big(\frac{a\tau + b}{c\tau + d}\Big) \Big]$$

$\boxed{\text{convergent, anomaly-free}}$

$$e.g. \quad J(\tau) = e(-\tau) + \lim_{K\to\infty} \sum_{\substack{0<c<K \\ -K^2<d<K^2}} e\Big(-\frac{a\tau+b}{c\tau+d}\Big) - e\Big(-\frac{a}{c}\Big) - 12 \quad , \quad e(x) = e^{2\pi i x}$$

$$= \lim_{K\to\infty} \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in (\Gamma_\infty \backslash SL(2,\mathbb{Z}))_{<K}} \Big(q^{-1}\big|_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\cdot\tau} - q^{-1}\big|_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\cdot i\infty} \Big) - 12$$

$$= \mathrm{Reg}\Big[\sum_{\gamma \in (\Gamma_\infty \backslash SL(2,\mathbb{Z}))} q^{-1}\big|_{\gamma\cdot\tau} \Big]$$

[Rademacher 1939]

Rademacher-Summability

puts stringent constraints on Γ_g.

e.g. If we want to have a **modular function** (weight 0, weakly holomorphic),

$$\boxed{\begin{array}{c}\text{Rademacher-summability}\\ \Leftrightarrow \\ \Gamma_g \text{ is genus zero}\end{array}}$$

[Duncan-Frenkel '09]

In particular, $J_g(\tau) = \mathrm{Reg}\left[\sum_{\gamma\in(\Gamma_\infty\backslash\Gamma_g)} q^{-1}\big|_{\gamma\cdot\tau}\right]$ for all g∈**M**!

Note: this holds for higher central charge cases *(q^{-n})* as well.

Rademacher-Summability of the M_{24} Moonshine

Here we have $H_g(\tau)$ = weight 1/2 Mock modular form.

Rademacher-summability $\cancel{\Longleftrightarrow}$ Γ_g is genus zero

Instead, with the help of previous results of Bringmann-Ono ('06) and Eguchi-Hikami ('09), we show

$$H_g(\tau) = -2 \,\mathrm{Reg} \left[\sum_{\gamma\in(\Gamma_\infty\backslash\Gamma_g)} q^{-1/8}\Big|_{\gamma\cdot\tau} \right] \qquad \text{for all } [g]\subset M_{24}\,.$$

To Summarise

We propose a new pattern
Rademacher Summability

$Z_g(\tau)$ = modular functions
e.g. Monster moonshine

g=0

$Z_g(\tau)$ = *mock* modular forms
for the new M_{24} moonshine

verified

Motivation for the new pattern:
All known theories of moonshine have a CFT interpretation.

Assuming the existence of a good dual description
(for a higher c version of the sporadic CFT)
$\Rightarrow$ All twisted PF have to be Rademacher-summable.

A glass of whiskey for physicists?

AdS/CFT →→AdS/CMT, AdS/QCD,
AdS/Hydrodynamics..... +AdS/NT??

Thank You!

To Summarise

Our result raises many physical questions:

- For M_{24} moonshine the dual situation is (more or less) clear: $AdS_3 \times S^3 \times K3$. How about other sporadic CFT's? Our results can be interpreted as an evidence that there exists 2d chiral CFT's with sporadic symmetries with semi-classical-like (saddle-dominated) AdS_3 duals.

- Relatedly, this brings back the question: What are the criteria for a CFT to have a (weakly coupled) AdS_3 duals?

[see for instance, Heemskerk-Penedones-Polchinski-Sully '09, El-Showk-Papadodimas '11]

- How to understand the twisting from a bulk viewpoint in general? [some progress reported in A. Sen Strings 2011]