A-polynomial, B-model, and S-duality

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Organizers:
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Conclusions

In this talk, I will present evidence for the following new results about Chern-Simons theory:

• Hidden symmetry
\[ G \rightarrow L \, G \quad \hbar \rightarrow L \, \hbar = -\frac{4\pi^2}{\hbar} \]

• A-polynomial = space of SUSY vacua in 3d N=2 gauge theory

• simple formula that turns classical curves \( A(x,y) = 0 \) into quantum operators
Chern-Simons gauge theory

\[ S = \int_M \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \]

- non-abelian interacting gauge theory
- has a long history ...
- has many applications ...
Chern-Simons gauge theory

\[ S = \int_\mathcal{M} \text{Tr} (A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \]

\[ \mathcal{M} = \text{3-manifold (possibly with boundary)} \]

\[ Z(\mathcal{M}) = \int e^{-\frac{S}{\hbar}} \mathcal{D}A \]

“quantum invariant” of \( \mathcal{M} \)

- depends on the choice of the gauge group
- depends on the “coupling constant” \( \hbar \)

\[ q = e^{\hbar} \]

[E.Witten]
Modular Form?

$Z(\includegraphics[width=0.1\textwidth]{example.png}) = -2 - q^{-1} + 1 - q + q^2$

Jones polynomial

$Z(\includegraphics[width=0.3\textwidth]{example.png}) = \ldots$

[E. Witten]
Modular Form?

\[ Z(\mathcal{L}) = q^{-2} - q^{-1} + 1 - q^1 + q^2 \]

Jones polynomial

- “experiments” with Don Zagier (circa 2007):
  analytically continue to complex \( q \)

leads to \( G \sim \rightarrow G_C \)  

[E.Witten]
Modular Form?

\[ Z(\begin{tikzpicture} \node (c) at (0,0) [circle,draw,thick] {}; \draw[thick] (-.5,-.5) to (0,0) to (.5,.5) to cycle; \end{tikzpicture}) = q^{-2} - q^{-1} + 1 - q^{1} + q^{2} \]

Jones polynomial

- “experiments” with Don Zagier (circa 2007): analytically continue to complex \( q \)

leads to \( G \xrightarrow{\sim} G_{\mathbb{C}} \)

\[ Z(\begin{tikzpicture} \node (c) at (0,0) [circle,draw,thick] {}; \draw[thick] (-.5,-.5) to (0,0) to (.5,.5) to cycle; \end{tikzpicture}) = 1 - (q^{-1} - 2 - q^{1}) + \ldots \]

\[ = \sum_{m=0}^{\infty} \prod_{n=1}^{m} (1 - q^n)(1 - q^{-n}) \]
Modular Form?

\[ Z \left( \begin{array}{c} \text{8} \\ \text{8} \end{array} \right) = q^{-2} - q^{-1} + 1 - q^1 + q^2 \]

Jones polynomial

• “experiments” (circa 2007): analytically continue to complex \( q \) leads to \( \sum_{m=0}^{\infty} \prod_{n=1}^{m} (1 - q^n)(1 - q^{-n}) \)

... let’s call it "quAntUm"
Realization in String Theory

6d five-brane theory on $\mathcal{M} \times S^3$

$\text{SL}(2) \text{ Chern-Simons}$ on 3-manifold $\mathcal{M}$

$3d \mathcal{N} = 2$ ``effective'' gauge theory on $S^3$

[J. Gauntlett, N. Kim]
[T. Dimofte, S. G., L. Hollands]
[S. Cecotti, A. Neitzke, C. Vafa]
[E. Witten]
[K. Hosomichi, S. Lee, J. Park]
[Y. Terashima, M. Yamazaki]
“Geometric Engineering”

6d five-brane theory on $\mathcal{M} \times S^3$

- (partial topological twist along $\mathcal{M}$)
- ($\Omega$-deformation along $S^3$)

$\text{SL}(2)$ Chern-Simons on 3-manifold $\mathcal{M}$

3d $\mathcal{N} = 2$ “effective” gauge theory on $S^3$
3d analog of AGT correspondence

2-manifold $\mathcal{C}$ $\sim \rightarrow$ 4d $\mathcal{N} = 2$ SUSY gauge theory on $\mathbb{R} \times S^3$

$\mathcal{H}_b^{\text{Liouv}}(C) = \mathcal{H}_{\epsilon_1,2}^\mathcal{N} = 2(S^3)$

[L.F. Alday, D. Gaiotto, Y. Tachikawa]

3-manifold $\mathcal{M}$ $\sim \rightarrow$ 3d $\mathcal{N} = 2$ "effective" gauge theory on $S^3$

$Z^{CS}(M; \hbar) = Z^{3d}(\epsilon_1/\epsilon_2)$
Reduction on \( \mathbb{M} = \mathbb{R} \times \mathbb{C} \)

6d five-brane theory

on \( \mathbb{M} \times \mathbb{S}^3 \)

\[
\mathcal{H}_b^{\text{Liouv}}(C) = \mathcal{H}_{\epsilon_{1,2}}^{\mathcal{N}=2}(\mathbb{S}^3)
\]

**SL(2)** Chern-Simons

on 3-manifold \( \mathbb{M} \)

3d \( \mathcal{N} = 2 \) “effective”

gauge theory on \( \mathbb{S}^3 \)
Reduction on $\mathcal{M} = \mathbb{R} \times C$

6d five-brane theory on $\mathbb{R} \times C \times S^3$

SL(2) Chern-Simons on 3-manifold $\mathbb{R} \times C$

4d $\mathcal{N} = 2$ SUSY gauge theory on $\mathbb{R} \times S^3$

$$\mathcal{H}^{CS}_\hbar (C) = \mathcal{H}^{Liouv}_b (C) = \mathcal{H}^{\mathcal{N}=2}_{\epsilon_1, \epsilon_2} (S^3)$$

$$\hbar = 2\pi i b^2 = 2\pi i \frac{\epsilon_1}{\epsilon_2}$$

[L.F. Alday, D. Gaiotto, Y. Tachikawa]  
[N. Nekrasov, E. Witten]
The semi-classical limit $\hbar = 0$

6d five-brane theory
on $\mathcal{M} \times S^3$

$SL(2)$ Chern-Simons
on 3-manifold $\mathcal{M}$

3d $\mathcal{N} = 2$ theory $T(\mathcal{M})$
on $S^3_b$
The semi-classical limit $\hbar \to 0$

5d $\mathcal{N}=2$ super-Yang-Mills

on $\mathcal{M} \times \mathbb{R}^2$

(partial topological twist along $\mathcal{M}$)

(Ω-deformation along $\mathbb{R}^2$)

SL(2) Chern-Simons on 3-manifold $\mathcal{M}$

3d $\mathcal{N}=2$ theory $T(\mathcal{M})$

on $\mathbb{R}^2 \times S^1$

under $SO(5)_{\mathcal{M}} \times SO(5)_{\mathbb{R}^2} \to SO(2)' \times U(1)^{\mathcal{M}} \times U(1)_{\mathbb{R}^2}$

bosons : $4, 4 \to 3^{(\pm1,\pm1)} \oplus 1^{(\pm1,\pm1)}$

fermions : $(4, 4) \to 3^{(\pm1,\pm1)} \oplus 1^{(\pm1,\pm1)}$

[T.Dimofte, S.G., L.Hollands]

[M.Blau, G.Thompson]
Flat connections = SUSY Moduli

$5d \mathcal{N}=2$ super-Yang-Mills on $\mathcal{M} \times \mathbb{R}^2$

(partial topological twist along $\mathcal{M}$)

SL(2) Chern-Simons on 3-manifold $\mathcal{M}$

3d $\mathcal{N}=2$ theory $T(M)$ on $\mathbb{R}^2 \times S^1$

(classical solutions: $dA + A \wedge A = 0$

SUSY vacua

[T.Dimofte, S.G., L.Hollands]
3d $\mathcal{N} = 2$ theory $\mathcal{T}(M)$: 

3-manifold $M$: 

$T1$ charged chiral $U(1)$ $k = -1/2$
3-manifold $\mathcal{M}$:

- $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

3d $\mathcal{N} = 2$ theory $T(\mathcal{M})$:

- $T_1$ charged chiral
  - $U(1)$ $k = -1/2$

- $T[SU(2)]$
  - $U(1)$ $k = -1$
  - $U(1)$ $k = +1$
• Notice, in all these examples $T(M)$ is an Abelian quiver gauge theory
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• Twisted superpotential has the following general form, cf. 2d $\mathcal{N} = 2$ SQED:

$$\mathcal{W} = \sum (\sigma_i - m_k) \left( \log \frac{\sigma_i - m_k}{\mu} - 1 \right)$$
• Notice, in all these examples $T(M)$ is an Abelian quiver gauge theory

• Twisted superpotential has the following general form, cf. 2d $\mathcal{N} = 2$ SQED:

$$\mathcal{W} = \sum (\sigma_i - m_k) \left( \log \frac{\sigma_i - m_k}{\mu} - 1 \right)$$

• Twisted superpotential in a theory on $\mathbb{R}^2 \times S^1_R$:

$$\mathcal{W} = \frac{i}{R} \sum \text{Li}_2 \left( e^{iR(\sigma_i - m_k)} \right)$$
Algebraic curves from 3d $\mathcal{N}=2$ theories

$\mathcal{W}(\sigma_i, \nu_k)$

1. Extremize w.r.t all dynamical fields $\sigma_i$

$$\frac{\partial \mathcal{W}}{\partial \sigma_i} = 0$$

2. Introduce “duals” of all non-dynamical parameters $\nu_k$ (twisted masses, FI terms, etc.)

$$u_k := \frac{\partial \mathcal{W}}{\partial \nu_k}$$

3. Expect something nice to happen ...
Algebraic curves from 3d $\mathcal{N} = 2$ theories

\[ C = T^2 \setminus \{p\} \]

4d $\mathcal{N} = 2^*$ theory

duality group:
\[ \Gamma(C) = PSL(2, \mathbb{Z}) \]

[Donagi, E. Witten]

$\mathcal{V} = \text{eigenvalue of } SL(2) \text{ holonomy around the puncture}$

mass of the adjoint matter multiplet
\[ m_{\text{adj}} \]
Algebraic curves from 3d $\mathcal{N} = 2$ theories

mapping cylinder

$$\varphi \in \Gamma(C)$$

Duality wall in 4d $\mathcal{N} = 2^*$ theory

$$\Gamma(C) = PSL(2, \mathbb{Z})$$

$\mathcal{V} = \text{eigenvalue of } SL(2) \text{ holonomy around the puncture}$

mass of the adjoint matter multiplet $m_{\text{adj}}$

[N.Drukker, D.Gaiotto, J.Gomis]
[K.Hosomichi, S.Lee, J.Park]
In Chern-Simons theory with a Wilson loop, the polynomial $A(x,y)$ is a topological invariant called the A-polynomial and plays a role similar to that of the Seiberg-Witten curve in $\mathcal{N} = 2$ gauge theory.

[S.G.] Strings'03
punctured torus
bundle $\mathcal{M}$

glue

$3d$ $\mathcal{N} = 2$ theory $T(M)$

$u \ := \ \frac{\partial \mathcal{W}_{\text{eff}}(v)}{\partial v}$

$x = e^u \quad y = e^v$

$A(x, y) = 0$

Example:

$\varphi = TST^{-1}S^{-1}$
punctured torus bundle $\mathcal{M}$

$3d$ $\mathcal{N} = 2$ theory $T(M)$

$$u := \frac{\partial \mathcal{W}_{\text{eff}}(v)}{\partial v}$$

$x = e^u, \quad y = e^v$

Example:

$\varphi = TST^{-1}S^{-1}$

$A(x, y) = x^4 - (1 - x^2 - 2x^4 - x^6 + x^8)y + x^4y^2$
One more example

\[ Z^{3d}(\epsilon_1/\epsilon_2) = s_b = \Phi_h \]  

"quantum dilogarithm"
Computing $\text{SL}(2)$ partition functions

$$Z^{CS}(M; \hbar) = \int_{C_p} \prod_{j=1}^{N} \Phi_{\hbar}(\Delta_j)^{\pm 1} \prod_{i=1}^{N-b_0(\Sigma)} \frac{dp_i}{\sqrt{4\pi \hbar}}$$

$$\hbar \to 0 \sim \exp \left( \frac{1}{\hbar} \mathcal{W} + \mathcal{O}(\log \hbar) \right)$$
Computing $\text{SL}(2)$ partition functions

\[ Z^{CS}(M; \hbar) = \int_{C_\alpha} dp \frac{\Phi_{\hbar}(p - u)}{\Phi_{\hbar}(-p - u)} e^{-\frac{2pu}{\hbar}} \]

\[ \hbar \to 0 \quad \exp \left( \frac{1}{\hbar} \mathcal{W} + \mathcal{O}(\log \hbar) \right) \]
Standard Model of 3-manifolds?

• We can build 3-manifolds from basic building blocks (tetrahedra, etc.)

• What are the corresponding building blocks of 3d $\mathcal{N} = 2$ SUSY theories? Is there a simple dictionary?

• YES! Work in progress with T.Dimofte and D.Gaiotto (see Davide’s talk)
The End