Exact results for loop operators in 4d gauge theories

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Vasily Pestun
Harvard University
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J. Gomis and T. Okuda
Loop operators in 4d gauge theories

Wilson loop

electric

\[ \exp \int_C i n A \]

EM duality

\[ \int_{S^2} F = 2\pi m \]

\( E \sim \frac{1}{r^2} \)

\( B \sim \frac{1}{r^2} \)

t’ Hooft loop

magnetic

strong/weak coupling

weak/strong coupling
S-duality for $\mathcal{N} = 4$ SYM

$$\mathcal{L} = \frac{1}{2g_{YM}^2} \text{Tr} F \wedge \star F + \frac{iv}{8\pi^2} \text{Tr} F \wedge F + \ldots$$

$$\tau = \frac{4\pi i}{g_{YM}^2} + \frac{\vartheta}{2\pi}$$

**EM duality**

$$\vartheta$$ periodicity

$$\tau \leftrightarrow -\frac{1}{\tau}$$

$$\tau \rightarrow \tau + 1$$

$SL(2, \mathbb{Z})$ symmetry

(十八条 theories)

Why modular group?
ADE (2,0) theory on arbitrary curve w/ punctures

\[ \mathcal{N} = 2 \text{ SYM in 4d} \]

Moore’s review talk

generic Langlands Kapustin, Witten

ADE 4d \( \mathcal{N} = 4 \) SYM with coupling \( \tau \)
Can we make a computation?

\[(G, \tau)\]

Wilson loop rep \( R \) of \( G \)
\[\langle W_R \rangle[\tau]\]

\[(L G, L \tau)\]

\('t\) Hooft loop rep \( R \) of \( G \)
\[\langle T_R \rangle[L \tau]\]

\[L \tau = -1/\tau\]
weak - strong
strong - weak
WANTED

exact computation
Localization

compute the integral

\[ Z_0 = \int [D\phi] e^{-S_0[\phi]} \]

given

\[ QS_0 = 0 \]

where \( Q \) is fermionic symmetry,

and \( Q^2 = R \) is bosonic symmetry
Deform the action

\[ S_t[\phi] = S_0[\phi] + t\{Q, V\} \]

assuming

\[ \{Q^2, V\} = 0 \]

Consider

\[ Z_t = \int [D\phi] e^{-S_0[\phi] - t\{Q, V[\phi]\}} \]

Integrate by parts

\[ \partial_t Z_t = 0 \]

Hence

\[ Z_0 = Z_\infty \]

Vasily Pestun

Exact results for loop operators
\[ Z = \int_X e^{-S} = \sum_{\alpha} \int_{Y_\alpha \subset X} e^{-S|_{Y_\alpha}} Z_{1\text{-loop}}[N_{Y_\alpha}] \]

\( X \) - the space of all fields

\( Y_\alpha \) - the zeroes of \( QV \)

Take \( V = (Q\Psi, \Psi) \)

then zeroes of \( QV \) are precisely zeroes of \( Q\Psi \)
Example

Compute

\[ Z(\beta) = \int_{S^2} \exp(\omega + \beta \cos \theta) \]

where

\[ \omega = \sin \theta d\theta \wedge d\varphi \]

is the volume form

Straight integration gives

\[ Z(\beta) = 2\pi \frac{\exp(\beta) - \exp(-\beta)}{\beta} \]
Localization exercise

Consider the operator

\[ Q = d - \beta i_v \]

\[ Q^2 = -\beta L_v \]

Notice that

\[ Z(\beta) = \int \exp S \]

with \( QS = 0 \)

Deform the action using

\[ V = \sin^2 \theta \, d\phi \]

Then

\[ Z(t) = \int \omega(1 + 2t \cos \theta) \exp(\beta \cos \theta - t \sin^2 \theta) \]

In the limit \( t \to \infty \) the integrand localizes

to \( \theta = 0 \) and \( \theta = \pi \)
\[ \mathcal{N} = 2 \text{ supersymmetry on } S^4 \]

\[ OSp(2|4) \subset SL(1|2, \mathbb{H}) \]

- 8 fermionic generators
- \( Sp(4) \cong SO(5) \): isometry of \( S^4 \)
- \( SO(2) \): R-symmetry

Seiberg’s talk
\[ \mathcal{N} = 2 \text{ vector multiplet} \]
\[ (A_\mu, \Phi_0, \Phi_9, \Psi, K_i) \]

\[ S_{\text{vect}} = \frac{1}{g_{\text{YM}}^2} \int \sqrt{g} d^4 x \left( \frac{1}{2} F^2 + (D\Phi)^2 + \frac{R}{6} \Phi^2 + \frac{1}{2} [\Phi_a, \Phi_b]^2 + K^2 + \Psi D\Psi \right) \]

\[ \mathcal{N} = 2 \text{ hypermultiplet} \]

To introduce hypermultiplet masses in \( OSP(2|4) \) theory on \( S^4 \):

1. gauge the flavor symmetry
2. give expectation value to \( \Phi_0 \) in the flavor vector multiplet
Choice of supercharge $Q$

$Q^2 = J + R + G_\phi$

$J$ – space-time rotation
$R$ – $R$-symmetry
$G_\phi$ – gauge transformation by

$\phi = i\Phi_0 - \cos \theta \Phi_9$

$\epsilon_1 = \epsilon_2 = 1/r$

North $\theta = 0$

South $\theta = \pi$
susy Wilson loop

\[ W_R(C) = \text{tr}_R \, \text{Pexp} \int_C (Adx + i\Phi_0 ds) \]

\[ R - \text{rep of } G \]

susy 't Hooft loop

\[ \mathbb{R}^3 \left( x_i \right) : F_A \rightarrow - \ast \frac{B}{2} \frac{dx}{x^2} \quad \Phi_9 \rightarrow \frac{B}{2} \frac{1}{x} \]

\[ B \in \mathfrak{g} - \text{weight of } \mathfrak{g}^\vee \]

\[ D_A \Phi_9 = \ast F_A \]
NOW LET’S LOCALIZE

Solution strategy

We need to localize the path integral of \( N = 2 \) gauge theory on \( S^4 \) in the presence of 't Hooft singularity.

1. Find the localization loci \( Y_\alpha \) (solve \( Q \Psi = 0 \)).
2. Compute \( \exp(-S|_{Y_\alpha}) \)
3. Compute the determinant \( Z_{1\text{-loop}}[N_{Y_\alpha}] \)
4. Integrate over \( Y_\alpha \) and sum over \( \alpha \)
Step 1. Find the localization loci $Y_\alpha$ (solve $Q\Psi = 0$)

$$Q\Psi = \frac{1}{2}F_{mn}\Gamma^{mn}\varepsilon - \frac{1}{2}\phi_a\Gamma^{a\mu}\nabla_\mu\varepsilon + iK_i\Gamma_{8i+4}\varepsilon$$

$\varepsilon$ – spinor on $S^4$ defining $Q$

\[
F^+ = 0, \quad D\Phi_9 = 0
\]

\[
D\Phi_9 = \ast F, \quad [D_{\tau}, \cdot] = 0
\]

\[
F^- = 0, \quad D\Phi_9 = 0
\]
Vanishing theorem

the only possible solutions to the susy equations smooth everywhere except at the ‘t Hooft loop singularity are given by

\[
\begin{align*}
A_\mu &= A_{\mu}^{bg} \\
\Phi_9 &= \Phi_9^{bg} \\
K_i &= 0, \quad i = 1, 2 \\
\Phi_0 &= \Phi_0^{bg} + a \\
K_3 &= -a
\end{align*}
\]

\[a \in \mathfrak{h}\]
The original infinite dimensional path integral localizes to finite (rank G) dimensional integral

\[ Z = \int [DA D\Phi \ldots] e^{-S} = \int \hbar [da] e^{-S(a)} Z_{1\text{-loop}}(a) + \text{non-perturbative corrections from other loci } Y_\alpha \]
Step 2. Compute $e^{-S|_{Y_0}}$

't Hooft loop:

$$S(a) = -\frac{8\pi^2}{g^2} \text{Tr} a^2 + \left( \frac{2\pi^2}{g^2} + \frac{g^2 \varphi^2}{32\pi^2} \right) \text{Tr} B^2$$

rewrite:

$$S(a) = -\pi i \tau \text{Tr} \hat{a}_N^2 + \pi i \bar{\tau} \text{Tr} \hat{a}_S^2$$

$$\hat{a}_N = i a - \frac{\epsilon}{2} B$$

$$\hat{a}_S = i a + \frac{\epsilon}{2} B$$

hence

$$e^{-S(a)} = e^{\pi i \tau \text{Tr} \hat{a}_N^2} \cdot e^{-\pi i \bar{\tau} \text{Tr} \hat{a}_S^2} = |e^{\pi i \tau \text{Tr} \hat{a}_N^2}|^2$$

$$Q^2 = J + R + G_{\phi}$$

$$\phi = i \Phi_0 - \cos \theta \Phi_9$$

$$\Phi_{9\text{bg}}(N) = \Phi_{9\text{bg}}(S) = \frac{\epsilon}{2} B$$
Step 3. Compute $Z_{1\text{-loop}}(a)$

organize fields in $Q$-multiplets

$$Q \cdot \varphi_{b,f} = \varphi'_{b,f}$$

$$Q \cdot \varphi'_{b,f} = R \cdot \varphi_{b,f}$$

so $Q^2 = R$ with $R = J + R + G_a$

$$Z_{1\text{-loop}}(a) = \int_{\mathcal{N}_Y} e^{tQV} = \frac{\det_{\text{coker}D} R}{\det_{\text{ker}D} R}, \quad t \to \infty$$

$D$ – transversally elliptic differential operator from the linearized equations

$F^+ = 0, \quad D\Phi_9 = 0$

$D\Phi_9 = \ast_3 F, \quad [D_\tau, \cdot] = 0$

$F^- = 0, \quad D\Phi_9 = 0$
\[ D : \Gamma(E_0) \rightarrow \Gamma(E_1) \]

\[ \Gamma(E_0) = \{ \phi_b \} \]
\[ \Gamma(E_1) = \{ \phi'_f \} \]

compute

\[ Z_{1\text{-loop}}(a) = \frac{\det_{\text{coker}D} \mathcal{R}}{\det_{\text{ker}D} \mathcal{R}} \]

from the index

\[ \text{ind } D = \text{tr}_{\text{Ker}D} e^\mathcal{R} - \text{tr}_{\text{Coker}D} e^\mathcal{R} \]

using the rule

\[ \sum m_j e^{w_j} \rightarrow \prod w_j^{m_j} \]

\( w_j \) -weights of \( \mathcal{R} \)

\( m_j \) -multiplicities
To compute $\text{ind} \, D$ we slice sphere into

- neighborhood of the north pole
- neighborhood of the equator
- neighborhood of the south pole

and use Atiyah-Singer index formula

$$\text{ind} \, D(\mathcal{R}) = \sum_{p \in F} \frac{\text{tr}_{E_0(p)} \mathcal{R} - \text{tr}_{E_1(p)} \mathcal{R}}{\det_{TM_p}(1 - \mathcal{R})}$$

$F$ - set of fixed points of $\mathcal{R}$ action on $M$
Each pole region gives

\[ Z_{1\text{-loop}}^{\text{pole}}(\hat{a}) = \frac{\prod_\alpha \left[ G\left(\frac{\alpha \cdot \hat{a}}{\epsilon}\right) G\left(2 + \frac{\alpha \cdot \hat{a}}{\epsilon}\right)\right]^{1/2}}{\prod_{f=1}^{N_F} \prod_{w \in R} \left[ G\left(1 + \frac{w \cdot \hat{a}}{\epsilon} - \frac{im_f}{\epsilon}\right) G\left(1 - \frac{w \cdot \hat{a}}{\epsilon} + \frac{im_f}{\epsilon}\right)\right]^{1/2}} \]

where

\[ \hat{a}_N = ia - \frac{\epsilon}{2}B \quad \hat{a}_S = ia + \frac{\epsilon}{2}B \]

Equator region gives

\[ Z_{1\text{-loop}}^{\text{eq}}(a) = \frac{\prod_{f=1}^{N_F} \prod_{w \in R} \left[ \sin \left(\pi w \cdot \left(\frac{ia}{\epsilon} + \frac{B}{2}\right) - \pi \frac{im_f}{\epsilon}\right)\right]|w \cdot B|/2}{\prod_{\alpha > 0} \left[ \sin \left(\pi \alpha \cdot \left(\frac{ia}{\epsilon} + \frac{B}{2}\right)\right)\right]|\alpha \cdot B|} \]

- \text{hypermultiplet rep}
- \text{Barnes G-function /related to q-dilog/}

\[ G(1 + z) = (2\pi)^{z/2}e^{-((1 + \gamma z^2) + z)/2} \prod_{n=1}^\infty \left(1 + \frac{z}{n}\right)^n e^{-z + \frac{z^2}{2n}} \]
$Y_0$ locus gives **exact result** up to nonperturbative corrections

$$\langle T_B \rangle \overset{Y_0}{=} \int da \left| Z_N(\imath a - \frac{\epsilon}{2} B) \right|^2 Z_{1\text{-}loop}^{eq}(a)$$

$$\langle W_R \rangle \overset{Y_0}{=} \int da \left| Z_N(\imath a) \right|^2 \text{tr}_R e^{2\pi \imath a}$$

where

$$Z_N(\hat{a}) = e^{\pi \imath \tau \hat{a}^2} Z_{1\text{-}loop}^N(\hat{a})$$

notice connection with Nekrasov’s $Z$

$$Z_N(\hat{a}) = \left[ Z_{\Omega^{-\text{bg}}}^{\mathbb{R}^4} (\hat{a}; \epsilon_1 = \frac{1}{r}, \epsilon_2 = \frac{1}{r}) \right]^{\text{pert}}$$
Step 4. Non-perturbative corrections
(from higher $Y_\alpha$ loci)

- point instantons at North pole
- point monopoles at the equator
- point anti-instantons at South pole
gauge theory in $\Omega$-background /Losev, Moore, Nekrasov, Shatashvili /
approximates $OSp(2|4)$ theory up to $O(x^2)$ at $x = 0$

point instantons at $x=0$
contribute in the same way
with instanton corrections the result is

\[ \langle T_B \rangle = \int da \left| Z_N \left( i \alpha - \frac{\epsilon}{2} B \right) \right|^2 Z_{\text{1-loop}}^{\text{eq}}(a) \]

\[ \langle W_R \rangle = \int da \left| Z_N(i \alpha) \right|^2 \text{tr} e^{2\pi i \alpha} \]

\[ Z_N(\hat{a}) = e^{\pi i \tau \hat{a}^2} Z_{\text{1-loop}}^N(\hat{a}) Z_{\text{inst}}(\hat{a} | \epsilon_1 = \epsilon_2 = r^{-1}; \text{im}) \]

if B is miniscule rep, the result is final

if not, expect screening corrections
**Monopole screening**

Non-abelian monopoles can screen the singularity and reduce effective magnetic charge seen at infinity.

\[ D_A \Phi = \star F_A \]

\[ \Phi = \frac{1}{2x} B, \quad x \to 0 \quad /\text{abelian}/ \]

\[ \Phi = \frac{1}{2x} B', \quad x \to \infty \quad /\text{abelian}/ \]

we need to localize on moduli space \( \mathcal{M}(B, B') \)

\( B' \subset \text{rep}(B) \)
The final exact result

\[ \langle T_B \rangle = \int da \sum_{B' \in \text{rep}(B)} \left| Z_N \left( ia - \frac{\epsilon}{2} B' \right) \right|^2 Z_{1\text{-loop}}^{\text{eq}}(a; B, B') \]

\[ \langle W_R \rangle = \int da \left| Z_N (ia) \right|^2 \sum_{R' \in \text{rep}(R)} e^{2\pi i R' \cdot a} \]

- S-duality for SU(2) N=2* checked /numerically/
- results agree with conjectures for AGT dual loop observables in Liouville (Toda) theories

Gomis, Okuda, V.P.’11

V.P.’07

Drukker, Gomis, Okuda, Teschner’09,
Alday, Gaiotto, Gukov, Tachikawa, Verlinde’09
Gomis, Floch’10.
Summary/Discussion

- Exact results for vevs of susy ‘t Hooft and Wilson loops in OSp(2|4) theories: all perturbative and non-perturbative corrections.
- Agreement with conjectured loop observables in AGT dual Liouville/Toda CFTs.
- Precision S-duality test
- similar localization techniques are used
  - more complicated loops in N=4 SYM V.P., Giombi
  - $S^3 \times S^1$, $S^3$, $S^2 \times S^1$ Seiberg’s review talk
  - gravity/black holes calculations OSV, Dabholkar-Gomes-Murthy-Sen
Thank you!